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Special Algorithms for the Simulation of Homogeneous Random Fields

G. A. Babicheva[1*], N. A. Kargapolova[1,2], and V. A. Ogorodnikov[1,2***]**

[1] *Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia*

[2] *Institute of Computational Mathematics and Mathematical Geophysics, Siberian Branch, Russian Academy of Sciences, pr. Akad. Lavrent'eva 6, Novosibirsk, 630090 Russia*

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Abstract—In this paper, two new algorithms for the simulation of homogeneous random fields are proposed. Both algorithms are based on the well-known algorithm of rows and columns for the simulation of Gaussian fields with special correlation functions. The algorithms make it possible to efficiently simulate homogeneous random fields with nonconvex correlation functions.

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1. INTRODUCTION

When solving various applied problems (for instance, ones of statistical meteorology, oceanology, hydrology, and population biology) by stochastic simulation methods one should often take into account the correlation structure of the real processes and fields under study. In many cases the corresponding correlation functions needed for the simulation are obtained by approximating typical correlation functions, which have been estimated on the basis of observations, by using special positive definite functions of a continuous argument. For instance, to describe the correlation structure of spatial two-dimensional fields of near-surface air temperature and geopotential over a given terrain, the following approximating functions are used if the assumption that the field has isotropic character agrees well with the real data:

$$r(\rho) = \exp\bigl(-\alpha\rho^2\bigr),$$

where $\rho = \sqrt{x^2 + y^2}$ is the distance between two points of the field. In cases where the assumption of field homogeneity agrees well with the data, functions of a more general form are used:

$$r(x, y) = \exp\Bigl(- \bigl(\alpha x^2 + \beta xy + \gamma y^2\bigr)^\theta\Bigr).$$

The existing algorithms for the simulation of random fields, for instance, the vector algorithms, are based on the method of conditional distributions, on the vector models of autoregression, moving average, the mixed models of autoregression and moving average, etc. They make it possible to construct realizations of Gaussian fields of discrete arguments with a given correlation structure [10]. However, they have one common shortcoming: the simulation of a large number of realizations of a random field requires much calculation time and computer memory. In addition, for the correlation functions of the above type the standard simulation algorithms are computationally unstable, since the corresponding correlation matrices have a large condition number and eigenvalues that are close to zero. When simulating random fields, the calculation errors may accumulate and, as a result, some eigenvalues of the correlation matrix may become negative. For matrices with negative eigenvalues, the real Cholesky factorization, as well

[*] E-mail: galinab2811@gmail.com
[**] E-mail: nkargapolova@gmail.com
[***] E-mail: ova@osmf.sscc.ru

as the algorithms based on the method of conditional distributions cannot be used in the simulation. To eliminate these shortcomings, more economical and stable algorithms are being developed for various classes of correlation functions [7–10].

For correlation functions of the form $r(\rho) = \exp\bigl(-\alpha\rho^2\bigr)$ and $r(x,y) = \exp\bigl(-\bigl(\alpha x^2 + \gamma y^2\bigr)\bigr)$, there exists an efficient algorithm for the simulation of Gaussian fields on regular grids. It is called an algorithm of rows and columns [3, 4, 10]. To implement it the correlation matrices are specified at individual horizontal and vertical cross sections of the field. This algorithm is based on the fact that the correlation matrix of the field can be represented as a direct product of correlation matrices of its horizontal and vertical cross sections. An advantage of this method is that it implies successive use of algorithms for the simulation of Gaussian vectors with Toeplitz correlation matrices specified by functions of the form $r(x) = \exp(-\alpha x^2)$. For them algorithms based on the method of conditional distributions are implemented with high accuracy.

The algorithm of rows and columns has a specific feature, which can be illustrated as follows. Let us simulate a homogeneous Gaussian field on a uniform rectangular $m \times n$ grid (for simplicity, a grid spacing of 1 is used). The elements of the correlation matrices of horizontal ($R^{(1)} = \bigl(r_{ij}^{(1)}\bigr)$, $i,j = 1,2,\ldots,n$) and vertical ($R^{(2)} = \bigl(r_{kp}^{(2)}\bigr)$, $k,p = 1,2,\ldots,m$) cross sections of the field are given by the functions $r_{ij}^{(1)} = \exp\bigl(-\alpha_1|i-j|^2\bigr)$ and $r_{kp}^{(2)} = \exp\bigl(-\alpha_2|k-p|^2\bigr)$, respectively. The correlation function of the homogeneous random field constructed by the algorithm of rows and columns has the form

$$r(i,j,k,p) = \exp\bigl(-\alpha_1(i-j)^2 - \alpha_2(k-p)^2\bigr).$$

In this case the section, by a horizontal plane, of the correlation function surface between the field value at an arbitrary fixed node and the values at all the other nodes is an ellipse whose main axes are oriented along the axes Ox and Oy.

However, in applied problems, there are frequent correlation functions of random fields whose isolines are ellipses with main axes turned through some angle φ with the positive directions of the axes Ox and Oy. For instance, for the correlation function of the geopotential field at a level of 500 millibar the rotation angle $\varphi \approx 45^\circ$ [2].

In the present paper, two modifications of the algorithm of rows and columns are proposed. With one of them, one can simulate fields for which the rotation angle $\varphi \neq 0^\circ$. The other modification greatly extends the class of correlation functions for which the algorithm can be applied.

2. MODIFIED ALGORITHM OF ROWS AND COLUMNS

Here we consider a modification of the algorithm of rows and columns that makes it possible to numerically implement homogeneous random fields for which the rotation angle φ described in Section 1 differs from 0°.

2.1. Algorithm for the Simulation of a Random Field

Let us formulate an algorithm for the simulation of a two-dimensional Gaussian homogeneous random field $\bigl\{\eta_j^i\bigr\}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, on a uniform rectangular $m \times n$ grid (for convenience, the grid spacing is 1).

Step 1. Simulate a two-dimensional random field $\bigl\{\varphi_j^i\bigr\}$ as an array of standard Gaussian independent random quantities

$$\begin{pmatrix} \varphi_1^1 & \cdots & \varphi_n^1 \\ \vdots & \ddots & \vdots \\ \varphi_1^m & \cdots & \varphi_n^m \end{pmatrix},$$

where $\vec{\varphi}^{\,(i)\top} \sim N(0, I)$, $\vec{\varphi}^{\,(i)\top} = \left(\varphi_1^i, \varphi_2^i, \ldots, \varphi_n^i\right)$ is a row vector, $i = 1, \ldots, m$, and I is a unit matrix.

Step 2. Transforming

$$\vec{\xi}^{\,(i)\top} = \vec{\varphi}^{\,(i)\top} L^{(1)\top},$$

for every $i = 1, \ldots, m$, we obtain m independent row vectors

$$\vec{\xi}^{\,(i)\top} = \left(\xi_1^i, \xi_2^i, \ldots, \xi_n^i\right).$$

Here

$$L^{(1)} = \begin{pmatrix} l_{1,1}^{(1)} & 0 & \cdots & 0 \\ l_{2,1}^{(1)} & l_{2,2}^{(1)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_{n,1}^{(1)} & l_{n,2}^{(1)} & \cdots & l_{n,n}^{(1)} \end{pmatrix}$$

is a lower triangular matrix such that $L^{(1)} L^{(1)\top} = R^{(1)}$, and the matrix

$$R^{(1)} = \begin{pmatrix} r_{1,1}^{(1)} & \cdots & r_{1,n}^{(1)} \\ \vdots & \ddots & \vdots \\ r_{n,1}^{(1)} & \cdots & r_{n,n}^{(1)} \end{pmatrix}$$

is a given correlation matrix of a horizontal cross section of the field. From the row vectors $\vec{\xi}^{\,(i)\top}$ we form a field $\left\{\xi_j^i\right\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$:

$$\left\{\xi_j^i\right\} = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^m & \cdots & \xi_n^m \end{pmatrix}$$

whose row vectors $\vec{\xi}^{\,(i)\top} \sim N\left(0, R^{(1)}\right)$ are independent of each other.

Step 3. Form p vectors $\vec{\zeta}_{(k)}$ from random quantities ξ_j^i "lying" on straight lines inclined with respect to the horizontal direction of the grid, that is,

$$\vec{\zeta}_{(k)} = \left(\xi_k^1, \xi_{k+d}^2, \xi_{k+2d}^3, \ldots, \xi_{k+(m-1)d}^m\right)^\top,$$

where $k = 1, \ldots, n - (m-1)d$, $d = 0, 1, 2, \ldots$, is a parameter of shift in every horizontal layer of the grid. Figure 1 illustrates this method of constructing the vectors $\vec{\zeta}_{(k)}$. Since the row vectors $\vec{\xi}^{\,(i)\top} = \left(\xi_1^i, \xi_2^i, \ldots, \xi_n^i\right)$ of the field $\left\{\xi_j^i\right\}$ are independent of each other, the vectors $\vec{\zeta}_{(k)} \sim N(0, I)$.

Step 3 in the modified algorithm of rows and columns differs from that in the standard algorithm of rows and columns as follows: in the latter after Step 2 the random quantities are connected with each other at grid nodes located on vertical layers.

Step 4. Finally, transforming

$$\vec{\nu}_{(k)} = L^{(2)} \vec{\zeta}_{(k)},$$

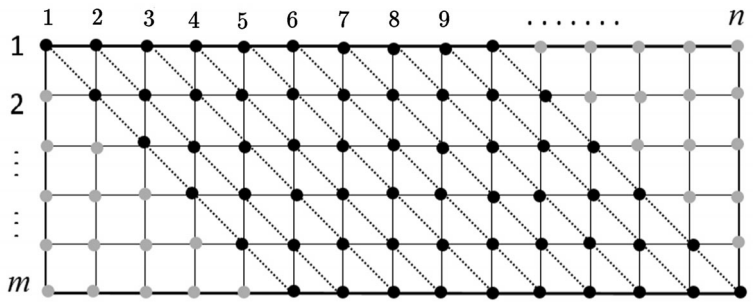

Fig. 1. Scheme of construction of vectors $\vec{\zeta}_{(k)}$ at Step 3 of the modified algorithm of rows and columns with $d = 1$. Random quantities making the vectors $\vec{\zeta}_{(k)}$, $k = 1, \ldots, n - (m-1)d$, are shown by black color.

we obtain the vectors $\vec{\nu}_{(k)} = \left(\nu_k^1, \nu_{k+d}^2, \nu_{k+2d}^3, \ldots, \nu_{k+(m-1)d}^m\right)^\top$, $k = 1, \ldots, n - (m-1)d$. Here the matrix

$$L^{(2)} = \begin{pmatrix} l_{1,1}^{(2)} & 0 & \cdots & 0 \\ l_{2,1}^{(2)} & l_{2,2}^{(2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_{n,1}^{(2)} & l_{n,2}^{(2)} & \cdots & l_{n,n}^{(2)} \end{pmatrix}$$

is a lower triangular matrix such that $L^{(2)} L^{(2)\top} = R^{(2)}$, and the matrix

$$R^{(2)} = \begin{pmatrix} r_{1,1}^{(1)} & \cdots & r_{1,m}^{(1)} \\ \vdots & \ddots & \vdots \\ r_{m,1}^{(1)} & \cdots & r_{m,m}^{(1)} \end{pmatrix}$$

is the given correlation matrix of the vertical cross section of the field. The random field $\{\nu_j^i\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, is formed from the vectors constructed. For $d = 1$ the field has the following form:

$$\begin{pmatrix} \nu_1^1 & \nu_2^1 & \cdots & \nu_{m-1}^1 & \nu_m^1 & \cdots & \nu_{n-(m-1)}^1 & \xi_{n-(m-1)+1}^1 & \cdots & \xi_n^1 \\ \xi_1^2 & \nu_2^2 & \cdots & \nu_{m-1}^2 & \nu_m^2 & \cdots & \nu_{n-(m-1)}^2 & \nu_{n-(m-1)+1}^2 & \cdots & \xi_n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_1^{m-1} & \xi_2^{m-1} & \cdots & \nu_{m-1}^{m-1} & \nu_m^{m-1} & \cdots & \nu_{n-(m-1)}^{m-1} & \nu_{n-(m-1)+1}^{m-1} & \cdots & \xi_n^{m-1} \\ \xi_1^m & \xi_2^m & \cdots & \xi_{m-1}^m & \nu_m^m & \cdots & \nu_{n-(m-1)}^m & \nu_{n-(m-1)+1}^m & \cdots & \nu_n^m \end{pmatrix}.$$

Since in the case under consideration a rectangular grid is used for the simulation, we eliminate the $d(m-1)$ row vectors at the sides of this random array. We obtain the final random field $\{\eta_j^i\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n - 2d(m-1)$:

$$\{\eta_j^i\} = \begin{pmatrix} \nu_m^1 & \nu_{m+1}^1 & \cdots & \nu_{n-(m-1)}^1 \\ \nu_m^2 & \nu_{m+1}^2 & \cdots & \nu_{n-(m-1)}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \nu_m^m & \nu_{m+1}^m & \cdots & \nu_{n-(m-1)}^m \end{pmatrix} = \begin{pmatrix} \eta_1^1 & \eta_2^1 & \cdots & \eta_{n-2(m-1)}^1 \\ \eta_1^2 & \eta_2^2 & \cdots & \eta_{n-2(m-1)}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \eta_1^m & \eta_2^m & \cdots & \eta_{n-2(m-1)}^m \end{pmatrix}.$$

Notice that here the angle θ of inclination of the straight lines by which the random quantities are connected belongs to the interval $[-\pi/4, 0)$. In the general case $\theta \in [-\pi/4, \pi/4]/\{0\}$, and since the grid is discrete, the inclination angle θ of the straight lines varies also discretely. We have

$$\tan\theta = \begin{cases} -1/d, & \theta \in [-\pi/4, 0)\,, \\ 1/d, & \theta \in (0, \pi/4]\,, \end{cases}$$

where d is the integer shift parameter. When $\theta \in (0, \pi/4]$, at Step 3 the vectors $\vec{\zeta}_{(k)}$, $k = 1, \ldots, n-(m-1)d$, are formed from the random quantities ξ_j^i, $i = 1, \ldots, m$, $j = 1, \ldots, n$, as follows:

$$\vec{\zeta}_{(k)} = \left(\xi^1_{k+(m-1)d}, \ldots, \xi^{m-2}_{k+2d}, \xi^{m-1}_{k+d}, \xi^m_k\right)^\top.$$

In the discussion that follows $\theta \in [-\pi/4, 0)$, unless otherwise specified.

Notice that the number p of vectors $\vec{\zeta}_{(k)}$, $k = 1, \ldots, p$, namely, $p = n-(m-1)d$, depends on the shift parameter d at a fixed dimension of $(m \times n)$ of the grid. This, in turn, decreases the dimension of the resulting field $\{\eta_j^i\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n-2d(m-1)$. If exactly p vectors $\vec{\zeta}_{(k)}$ must be constructed with a given shift d and a fixed number of nodes m in a vertical cross section of the grid, the needed number of nodes in the horizontal is calculated by the formula $n = p + d(m-1)$.

Remark 1. In some applications, it is convenient to simulate random fields not on a rectangular grid, but, for instance, on a parallelogram one. In such cases one need not eliminate the $d(m-1)$ row vectors of the array, but may use the thus obtained field of this shape.

2.2. Correlation Structure of the Field

Let us investigate the correlation structure of the Gaussian random field $\{\eta_j^i\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n-2d(m-1)$, constructed using the above-described modified algorithm of rows and columns. The correlation matrix elements $R^{(1)} = \left(r^{(1)}(i,j)\right)$ and $R^{(2)} = \left(r^{(2)}(k,p)\right)$ are calculated in the following way:

$$r^{(1)}(i,j) = \exp\left(-\alpha_1|i-j|^2\right), \qquad i,j = 1, \ldots, n,$$
$$r^{(2)}(k,p) = \exp\left(-\alpha_2|k-p|^2\right), \qquad k,p = 1, \ldots, m.$$

Here α_1 and α_2 are arbitrary parameters. Since the random quantities η_j^i have a zero mean and unit variance, the correlations between them have the form

$$M\eta_i^k\eta_j^p = r(i,j,k,p) = r^{(2)}(k,p)r^{(1)}\big(i-d(k-1), j-d(p-1)\big),$$

where $i,j = 1, \ldots, n-2d(m-1)$, $k,p = 1, \ldots, m$. Since the grid spacing is 1, we rewrite the correlation structure of the field $\{\eta_j^i\}$ in Cartesian coordinates Oxy:

$$\begin{aligned} M\eta_i^k\eta_j^p &= \exp\left(-\alpha_2|k-p|^2\right)\exp\left(-\alpha_1|i-d(k-1)-j+d(p-1)|^2\right) \\ &= \exp\left(-\alpha_2 y^2\right)\exp\left(-\alpha_1|x-d\,y|^2\right) = \exp\left(-\alpha_1 x^2 + 2d\alpha_1 xy - \left(d^2\alpha_1+\alpha_2\right)y^2\right). \end{aligned}$$

Thus, the correlation function of the field $\{\eta_j^i\}$ constructed has the form

$$r(x,y) = \exp\left(-\alpha_1 x^2 + 2d\alpha_1 xy - \left(d^2\alpha_1+\alpha_2\right)y^2\right).$$

The isolines of this correlation function can be described by the equations

$$f(x,y) = -\alpha_1 x^2 + 2d\alpha_1 xy - \left(d^2\alpha_1+\alpha_2\right)y^2 + C = 0.$$

They are second-order curves (namely, ellipses) whose main axes are turned through some angle φ with respect to the coordinate axes Ox and Oy. These are shown in Fig. 2.

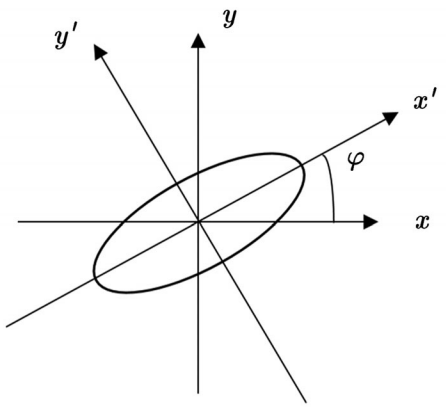

Fig. 2. Example of an isoline of $f(x, y)$.

The rotation angle φ is calculated by the formula

$$\tan 2\varphi = \frac{2d\alpha_1}{\alpha_2 + \alpha_1 (d^2 - 1)}.$$

It follows from this formula that the rotation angle φ of correlation function isolines can be controlled by varying the coefficients α_1 and α_2 that specify the correlation matrices $R^{(1)}$ and $R^{(2)}$, respectively.

Remark 2. If $\theta \in (0, \pi/4]$, the correlations between the random quantities of the field $\{\eta_j^i\}$ are equal,

$$M\eta_i^k \eta_j^p = r(i, j, k, p) = r^{(2)}(k, p) r^{(1)}(i + d(k - 1), j + d(p - 1)),$$

and the equation for isolines takes the form

$$f(x, y) = -\alpha_1 x^2 - 2d\alpha_1 xy - y^2 \left(\alpha_2 + d^2\alpha_1\right) + C = 0.$$

The rotation angle φ can be found from the equation

$$\tan 2\varphi = \frac{-2d\alpha_1}{\alpha_2 + \alpha_1 (d^2 - 1)}.$$

Remark 3. There exist some limitations on the parameters α_1, α_2 and the rotation angle φ. First, $\alpha_1, \alpha_2 > 0$. Second, the following condition must be satisfied:

$$\tan 2\varphi \in \left(0, \frac{2d}{d^2 - 1}\right).$$

That is, the range of the rotation angle φ is related to the shift parameter d. There also exists a relation between the rates of decrease of the correlation function of the field along the coordinate axes. Actually, at $y = 0$ the correlation function of the field $\{\eta_j^i\}$ has the form $r(x) = \exp\left(-\alpha_1 x^2\right)$, Hence, the rate of decrease of the function along the axis Ox depends only on the parameter α_1. At $x = 0$ the correlation function is described by the expression $r(y) = \exp\left(-\left(d^2\alpha_1 + \alpha_2\right) y^2\right)$, and the rate of decrease of the correlation function of the field along the axis Oy depends on the sum $\left(d^2\alpha_1 + \alpha_2\right)$. Thus, as the parameter α_1 changes, the rate of decrease of the correlation function changes not only along the axis Ox, but also along Oy.

Remark 4. In the main axes $Ox'y'$ the equation of isolines of the correlation function has the form

$$f\left(x', y'\right) = Ax'^2 + By'^2 + C = 0,$$

where

$$A = -\alpha_1 \left(d^2 \sin^2 \varphi - d \sin 2\varphi + \cos^2 \varphi\right) - \alpha_2 \sin^2 \varphi,$$

$$B = -\alpha_1\left(d^2\cos^2\varphi + d\sin 2\varphi + \sin^2\varphi\right) - \alpha_2\cos^2\varphi.$$

In the new system of coordinates $Ox'y'$, the correlation function of the field can be rewritten as follows:

$$r\left(x', y'\right) = \exp\left(Ax'^2 + By'^2\right).$$

At $y' = 0$ the correlation function is described by the equation $r\left(x'\right) = \exp\left(Ax'^2\right)$, and at $x' = 0$ by the equation $r\left(y'\right) = \exp\left(By'^2\right)$, respectively.

In addition, with $d = 1$ the following constraints in the form of the inequalities

$$0 < A < \alpha_1 + \frac{\alpha_2}{2}, \qquad 1 < B < 2\alpha_1 + \frac{\alpha_2}{2}$$

are imposed on the parameters A and B.

The following statement is valid for the correlation functions of the form $r(x, y)$.

Statement . *The function obtained by cutting the surface $r(x, y)$ by an arbitrary plane $ax + by = 0$ has inflection points.*

Rewrite the function $r(x, y)$ as follows:

$$r(x, y) = \exp\left(-\left(\alpha_1\left(x - dy\right)^2 + \alpha_2 y^2\right)\right).$$

It is evident that $\alpha_1(x - dy)^2 + \alpha_2 y^2 \geq 0$ and $\alpha_1(x - dy)^2 + \alpha_2 y^2 = 0$ if and only if $x = y = 0$. The function obtained by the plane $y = kx$, $k \in \mathrm{R}$ is

$$\widetilde{r}(x) = \exp\left(-\left(\alpha_1(1 - dk)^2 + \alpha_2 k^2\right)x^2\right) = \exp\left(-D_1 x^2\right), \quad D_1 > 0,$$

and by the plane $x = 0$,

$$\widehat{r}(y) = \exp\left(-y^2\left(\alpha_1 d^2 + \alpha_2\right)y^2\right) = \exp\left(-D_2 y^2\right), \quad D_2 > 0.$$

The existence of inflection points follows from the fact that the functions $\partial^2\tilde{r}/\partial x^2$, $\partial^2\tilde{r}/\partial y^2$ become 0 and change sign at points $\pm 1/\sqrt{2D_1}$ and $\pm 1/\sqrt{2D_2}$, respectively.

It is easy to verify by direct substitution that all the inflection points, regardless of the choice of the cutting plane $ax + by = 0$, belong to the plane $z = \exp(-0.5)$.

Figure 3 provides examples of isolines of the correlation function $r(x, y)$ for various values of the parameters α_1, α_2, φ, and d.

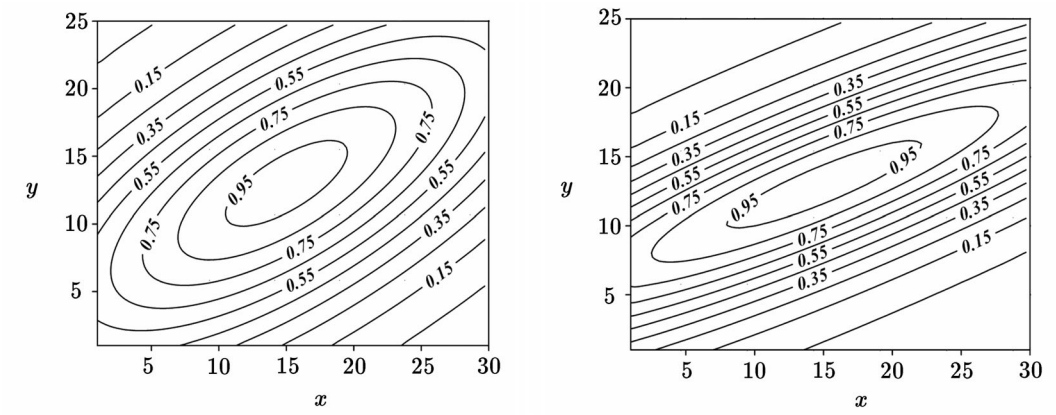

Fig. 3. Isolines of the correlation function $r(x, y)$: left: $\alpha_1 = \alpha_2 = 0.005$, $\varphi = 31.7$, $d = 1$; right: $\alpha_1 = \alpha_2 = 0.005$, $\varphi = 22.5$, $d = 2$.

3. RANDOMIZED ALGORITHM OF ROWS AND COLUMNS

In this section we consider further modification of the algorithm of rows and columns. By analogy with [6, 7], a sufficiently wide class of correlation functions applicable for the simulation algorithm proposed in the present paper can be obtained by randomizing the values of one of the parameters α_1 or α_2.

Let the rotation angle of the main axes of the ellipse be fixed, that is,

$$\tan 2\varphi = c,$$

where $c \in \left(0, \frac{2d}{d^2-1}\right)$ is a constant.

The parameter α_2 can be expressed in terms of α_1:

$$\alpha_2 = \left(\frac{2d}{c} - d^2 + 1\right)\alpha_1.$$

Substituting this expression for α_2 into the formula for the correlation function of the field, we have

$$r(x,y) = \exp\left(-\alpha_1\Big((x-dy)^2 + \Big(\frac{2d}{c} - d^2 + 1\Big)y^2\Big)\right) = \exp(-D\alpha_1),$$

where $D = D(x,y) = (x-dy)^2 + \left(\frac{2d}{c} - d^2 + 1\right)y^2 \geq 0$ (the equality $D = 0$ is valid only if $x = y = 0$).

Now let α_1 be a random quantity with one-dimensional distribution density $f(z)$, $z \in (a,b)$, $0 < a < b$. Assume that $f(z) > 0$, $z \in (a,b)$. Consider some examples of correlation functions of the form

$$R(x,y) = \int\limits_a^b f(z)\exp(-Dz)\,dz.$$

All the examples are presented for a shift $d = 1$.

Example . Let $f(z)$ be the density of a uniform distribution on the interval (a,b). Then

$$\begin{aligned} R(x,y) &= \int\limits_a^b \frac{1}{b-a}\exp(-Dz)\,dz = \frac{\exp(-aD) - \exp(-bD)}{(b-a)D} \\ &= \frac{\exp\left(-a\Big((x-dy)^2 + \left(\frac{2d}{c}-d^2+1\right)y^2\Big)\right) - \exp\left(-b\Big((x-dy)^2 + \left(\frac{2d}{c}-d^2+1\right)y^2\Big)\right)}{(b-a)\left((x-dy)^2 + \left(\frac{2d}{c} - d^2 + 1\right)y^2\right)}, \\ &\qquad (x,y) \in \mathrm{R}^2\backslash(0,0), \\ R(0,0) &= 1. \end{aligned}$$

If $a = 0$, $b = 1$, we have

$$R(x,y) = \frac{1 - \exp\left(-\left((x-dy)^2 + \left(\frac{2d}{c} - d^2 + 1\right)y^2\right)\right)}{\left((x-dy)^2 + \left(\frac{2d}{c} - d^2 + 1\right)y^2\right)}, \quad (x,y) \in \mathrm{R}^2\backslash(0,0).$$

Example 2. If $f(z)$ is the density of an exponential distribution with a parameter $\lambda > 0$,

$$R(x,y) = \frac{\lambda}{\lambda + D} = \frac{\lambda}{\lambda + (x-dy)^2 + \left(\frac{2d}{c} - d^2 + 1\right)y^2}, \quad (x,y) \in \mathrm{R}^2.$$

Example 3. If $f(z)$ is the density of a triangular (Simpson) distribution, that is,

$$f(z) = \begin{cases} \dfrac{4(z-a)}{(b-a)^2}, & z \in \left(a, \dfrac{a+b}{2}\right], \\ \dfrac{4(b-z)}{(b-a)^2}, & z \in \left(\dfrac{a+b}{2}, b\right), \end{cases}$$

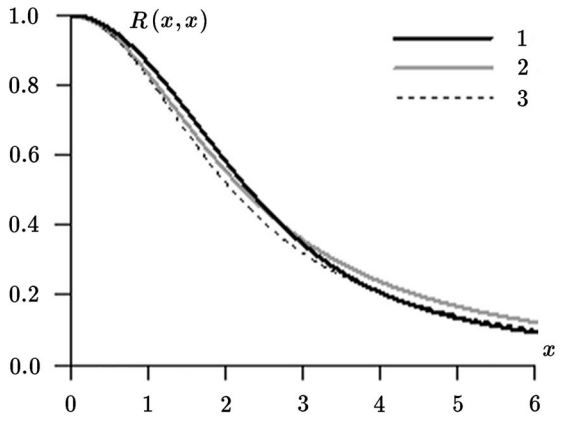

Fig. 4. Sections of $R(x,y)$ by the plane $y=x$. Curve 1: $f(z)$, uniform distribution density on interval $(0,0.3)$; curve 2: $f(z)$, exponential distribution density with $\lambda=5$; curve 3: $f(z)$, exponential distribution density with $\lambda=3.7$ truncated on interval $(0,0.7)$.

we have

$$R(x,y) = \frac{4}{(b-a)^2}\,\frac{\left(\exp(-aD)-2\exp\left(-\frac{a+b}{2}D\right)+\exp(-bD)\right)}{D^2}, \quad (x,y)\in \mathrm{R}^2\backslash(0,0),$$
$$R(0,0) = 1.$$

Figure 4 shows sections of the surface $R(x,y)$ by the plane $y=x$ for $x>0$ and various densities $f(z)$.

Notice that correlation functions of a random field that decrease in various ways can be obtained by changing the densities $f(z)$.

The functions presented in Fig. 4 have one common property: they all have inflection points. It turns out that regardless of the density $f(z)$ and cross section plane $Ax+By=0$ the function always has inflection points. This can be formulated as follows.

Statement 2. *If $0<a<b<+\infty$ and the distribution density $f(z)$ is continuous on $[a,b]$ and $f(z)>0$, $z\in[a,b]$, the function that is the cross section by the plane $Ax+By=0$ of the surface*

$$R(x,y)=\int_a^b f(z)\exp(-Dz)\,dz,$$

where $D=D(x,y)=(x-dy)^2+\left(\frac{2d}{c}-d^2+1\right)y^2$ has inflection points.

Proof. Consider a section plane with equation $y=kx$, $k\in\mathrm{R}$. In this case the following chain of equalities is valid:

$$D=D(x,kx)=\left((1-dk)^2+\left(\frac{2d}{c}-d^2+1\right)k^2\right)x^2=mx^2, \quad m>0.$$

As a result of simple calculations we have

$$\frac{\partial^2 R(x,kx)}{\partial x^2}=2m\int_a^b f(z)\left(2mx^2z^2-z\right)\exp\left(-mzx^2\right)dz.$$

Now consider the function

$$I(x)=\frac{1}{2m}\,\frac{\partial^2 R}{\partial x^2}=\int_a^b f(z)\left(2mx^2z^2-z\right)\exp\left(-mzx^2\right)dz.$$

Let us show that under the conditions of this statement the function

$$G(x,z) = f(z)\left(2mx^2z^2 - z\right)\exp\left(-mzx^2\right)$$

as $x \to M$, tends to the function,

$$g(z) = f(z)\left(2mM^2z^2 - z\right)\exp\left(-mzM^2\right)$$

uniformly in z on $[a;b]$.

For this, let us prove that for any $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that for all $z \in [a;b]$ the inequality

$$|G(x_1,z) - G(x_2,z)| < \varepsilon_1$$

is valid if $|x_1 - M| < \delta_1$ and $|x_2 - M| < \delta_1$. The function $G(x,z)$ is continuous in a bounded closed domain, $[-M,M] \times [a,b]$. Hence, according to Cantor's theorem it is uniformly continuous in this domain. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$|G(x,z) - G(x_0,z_0)| < \varepsilon$$

is valid for all points (x,z), (x_0,z_0) simultaneously if $|x - x_0| < \delta$ $|z - z_0| < \delta$. Let $z = z_0$. Then the inequality $|z - z_0| < \delta$ is satisfied automatically. Let us write the definition of uniform continuity twice if $x_0 = M$:

$$\forall\,\frac{\varepsilon_1}{2}\;\exists\,\delta' \;:\; |G(x_1,z) - G(M,z)| < \frac{\varepsilon_1}{2} \text{ if } |x_1 - M| < \delta',$$
$$\forall\,\frac{\varepsilon_1}{2}\;\exists\,\delta'' \;:\; |G(x_2,z) - G(M,z)| < \frac{\varepsilon_1}{2} \text{ if } |x_2 - M| < \delta''.$$

Hence, the chain of inequalities

$$|G(x_1,z) - G(x_2,z)| \le |G(x_1,z) - G(M,z)| + |G(M,z) - G(x_2,z)| < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1$$

is valid simultaneously for all z if $|x_1 - M| < \delta_1$ and $|x_2 - M| < \delta_1$, where $\delta_1 = \min\{\delta', \delta''\}$. Thus, the function $G(x,z)$, as $x \to M$, tends to the function $f(z)\left(2mM^2z^2 - z\right)\exp\left(-mzM^2\right)$ uniformly in z on $[a;b]$.

Taking into account the above result and using the theorem on taking limits under the integral sign, we have

$$\lim_{x\to M} I(x) = \int_a^b f(z)\left(2mM^2z^2 - z\right)\exp\left(-mzM^2\right)dz.$$

On the other hand, according to the theorem on proper integral continuity

$$\lim_{x\to M} I(x) = I(M).$$

Hence,

$$I(M) = \int_a^b f(z)\left(2mM^2z^2 - z\right)\exp\left(-mzM^2\right)dz.$$

Let us show that there exist such M_1, M_2 that $I(M_1) > 0$, $I(M_2) < 0$. Since $f(z) > 0$, $a > 0$, $\exp\left(-mzM^2\right) > 0$, we have $I(M) > 0$ if $2mz^2M^2 - z > 0$, that is, if $M^2 > \frac{1}{2mz}$. Since $m\frac{1}{2ma} > m\frac{1}{2mz}$, the inequality $2mz^2M^2 - z > 0$ is valid if $M^2 > \frac{1}{2ma}$. Hence, one can take any value that is larger than $m\frac{1}{\sqrt{2ma}}$ as M_1. Similarly, if any value smaller than $\frac{1}{\sqrt{2mb}}$ is taken as M_2, the inequalities $2mz^2M_2^2 - z < 0$ and $I(M_2) < 0$ are valid.

The function $I(x)$ is continuous, $\exists\, M_1, M_2$: $I(M_1) > 0$, $I(M_2) < 0$. Hence, there exists a point M_0: $I(M_0) = 0$. This means that the function $\frac{\partial^2 R(x,kx)}{\partial x^2}$ vanishes at point M_0 and changes its sign. That is, the function $R(x,kx)$ has an inflection point.

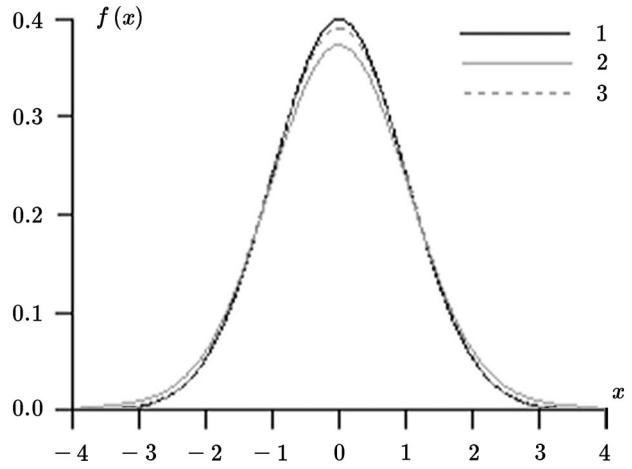

Fig. 5. One-dimensional distributions. Curve 1: standard normal distribution density; curves 2, 3: empirical densities estimated at two different points of the field.

In the case where the surface $R(x, y)$ is cut by the plane $x = 0$, the existence of an inflection point is proved in a similar way as above when the variable x is replaced by y and the constant m is replaced by $m_1 = \frac{2d}{c} + 1$. □

Remark 5. If $f(z) \geq 0$, $z \in [a, b]$, the above statement is valid, with points M_1, M_2 chosen so that $f(M_1) \neq 0$, $f(M_2) \neq 0$. In the case where $f(z)$ is a distribution density on the interval $[a, +\infty)$, the above proof becomes more complicated.

Notice that the randomized algorithm (in contrast to the standard algorithm of rows and columns) is approximate in that the one-dimensional distributions of the above-constructed field are not standard normal ones, but close to them. The one-dimensional distributions depend on the distribution density of the randomized parameter and on the coordinates of the field point at which the distribution is defined. This is illustrated in Fig. 5, which presents the density of a standard normal distribution and estimates of the densities of one-dimensional distributions at two points of the simulated field located at a maximal distance from each other. The density of the randomized parameter is assumed to be exponential with a parameter equal to 2. The densities have been estimated with 20,000,000 realizations.

4. CONCLUSIONS

The algorithms proposed in this paper can be used to simulate meteorological fields. For instance, according to [1, 2] the correlation structure of the fields of monthly average surface temperature of the air can be described by correlation functions with inflection points. Paper [5] presents an example of a correlation function that describes well the structure of the real field of monthly average temperature of the Lake Baikal region. This algorithm used to simulate a field with this correlation function, decreased the calculation time by approximately a factor of 100, in comparison to the method of conditional distributions. In the estimation of the probabilities of rare events when a large number of model realizations of the field is needed, this is a considerable decrease in the calculation time.

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