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Numerical Modeling of Fluid Flow in Anisotropic Fractured Porous Media

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Abstract—A model of double porosity in the case of an anisotropic fractured porous medium is considered (Dmitriev, Maksimov; 2007). A function of fluid exchange between the fractures and porous blocks depending on flow direction is given. The flow function is based on the difference between the pressure gradients. This feature enables one to take into account anisotropic properties of filtration in a more general form. The results of numerical solving a model two-dimensional problem are presented. The computational algorithm is based on a finite-element space approximation and explicit-implicit time approximations.

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Most of the oil-bearing beds in the world are carbonate reservoirs. A fractured porous medium is considered as a system of porous blocks separated by a system of fractures. The fluid saturates both porous blocks and fractures. The linear sizes of fracture openings are much greater than the typical sizes of pores in the blocks. Thus, the permeability of fractures is greater than that of pore blocks. On the other hand, the void space of fractures is much less than that of pore blocks. Therefore, the coefficient of medium's capacitance (porosity) of fractures is much less than that of pore blocks. The bulk of the fluid is contained in the pore blocks and transferred through the fractures. These properties of the void space structure of a fractured porous medium are often attributed to a medium of double porosity described in [1] and [2]. The mechanism of fluid exchange between the fractures and pore blocks is governed by the pressure difference. In the classical model of double porosity, the system of fractures is assumed to be isotropic, although (as it is well known) fractured reservoirs usually have an anisotropic structure. This fact is analyzed in [3–5].

The present study is based on a model of double porosity proposed in [6] for the case of an anisotropic fractured-porous medium. Its major feature is specifying a direction-dependent function of fluid exchange between the fractures and porous blocks. A model two-dimensional unsteady boundary value problem for a system of two parabolic equations for pressures in pores and fractures is formulated. The equations are coupled by means of a nonlinear flow function. The numerical solution to the problem is based on a finite-element space approximation. In time, we use explicit-implicit approximations when the flow function is taken from the lower time layer. The results of calculations illustrate the capabilities of the proposed algorithm and the effects of the porous medium's parameters on fluid filtration in the pores and fractures.

1. DOUBLE-POROSITY FILTRATION MODEL

To describe fluid filtration in a fractured-porous medium, Barenblatt et al. [1] developed a model based on the interrelation between the filtration flows in the fractures and those in the pore blocks. The mathematical model, called a double porosity model, can be written in tensor form as follows:

$$w_i^\alpha = -\frac{k_{ij}^\alpha}{\mu}\,\nabla_j p^\alpha, \tag{1}$$

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$$\frac{\partial m^\alpha \rho}{\partial t} + \nabla_j \rho w_j^\alpha = q^\alpha, \tag{2}$$

$$q^\alpha = q^\alpha(p^\alpha, \rho, \mu), \quad \rho^\alpha = \rho(p^\alpha), \quad m^\alpha = m(p^\alpha), \quad \alpha = 1, 2. \tag{3}$$

Here w_i^α are the vector components of the filtration velocity, k_{ij}^α are the components of the permeability tensors, m^α is the porosity, ρ is the fluid density, μ is the fluid viscosity, q^α is the fluid exchange between the fractures and pore blocks (α is equal to 1 and 2, respectively) such that $q_1 = -q_2 = q$. Equations (1) describe Darcy's law, (2) are continuity equations, and (3) specify the flow function and equations of state.

The classical model [1] considers the case of an isotropic space of the pore blocks and fractures. Nevertheless, the known investigations of gas- and oil-bearing formations have shown that fractured media usually have considerable anisotropy. In most cases, the anisotropic properties of fractured and porous media can be described by introducing a permeability tensor to Darcy's law. On the other hand, it is evident that fluid exchange between the fractures and pore blocks under the conditions of anisotropy depends on flow direction. For this reason, the scalar function of fluid exchange, q^α, must be replaced by a function of a vector argument. This situation is studied in detail in [7].

Consider the case of a directed permeability, which is a scalar function defined as $k(n_i) = k_{ij} n_i n_j$ (that is, it depends on the vector argument n_i). The sought-for relation follows from Darcy's law:

$$Q^\alpha = w_i^\alpha n_i^\alpha = -\frac{k_{ij}^\alpha n_i^\alpha n_j^\alpha}{\mu} |\nabla_j p^\alpha|.$$

Here Q^α is the velocity vector projection onto the pressure gradient axis, and $|\nabla_j p^\alpha|$ is the pressure gradient modulus, $\nabla_j p^\alpha = |\nabla_j p^\alpha| n_j^\alpha$. Then the directed permeability is defined as follows:

$$k^\alpha(n_i) = k_{ij}^\alpha n_i^\alpha n_j^\alpha = -\frac{\mu w_i^\alpha n_i^\alpha}{|\nabla_j p^\alpha|}.$$

A flow function can be constructed using the same reasoning.

2. FLOW FUNCTION IN AN ANISOTROPIC MEDIUM WITH DOUBLE POROSITY

Assuming that the flow function is similar to the flux w_i^α in Darcy's law and multiplying (scalarly) this flux by $\rho S n_i$ (where S is the surface through which the fluid flows from the pore blocks into fractures), we obtain

$$\tilde{q^\alpha} = \rho S n_i w_i^\alpha = -\rho S n_i \frac{k_{ij}^2}{\mu} \nabla_j p^\alpha. \tag{4}$$

It should be noted that, as shown in (4), the smallest permeability, k_{ij}^2, is responsible for the fluid transfer from the pore blocks into fractures. In the original model described in [1] and [8], the difference in the pressures of fractures and pore blocks is the driving force of the fluid exchange between the fractures and pore blocks. To define a flow function for anisotropic media (see [6]), we introduce the gradient of the pressure difference:

$$\tilde{q^1} - \tilde{q^2} = \rho S n_i \frac{k_{ij}^2}{\mu} \nabla_j p^2 - \rho S n_i \frac{k_{ij}^2}{\mu} \nabla_j p^1. \tag{5}$$

With (5), we write the following expression for a unit surface:

$$q = \frac{\rho}{\mu} n_i q_{ij} \nabla_j \left(p^2 - p^1\right), \tag{6}$$

where q_{ij} is a tensor consisting of the coefficients that define the flow function $\left(q = \left(\tilde{q^1} - \tilde{q^2}\right)/S\right)$. Formula (6) can be written as

$$q = \frac{\rho}{\mu} q_{ij} n_i n_j \left|\nabla_j \left(p^2 - p^1\right)\right|,$$

where $\left|\nabla_j \left(p^2 - p^1\right)\right|$ is the modulus of the gradient of the pressure difference between the pore blocks and fractures.

3. MODEL PROBLEM

Consider a simple case of a two-dimensional anisotropic model of double porosity with a flow function of the type (6). The system of equations of this model is written in nondimensional form as follows:

$$c\frac{\partial u_1}{\partial t} - \mathrm{div} K_1 \mathrm{grad}\, u_1 + r\,|K\mathrm{grad}(u_2 - u_1)| = 0, \tag{7}$$

$$\frac{\partial u_2}{\partial t} - \mathrm{div} K_2 \mathrm{grad}\, u_2 - r\,|K\mathrm{grad}(u_2 - u_1)| = 0. \tag{8}$$

Here K is a second-order tensor expressed in terms of a 2×2 matrix, and K_α is a permeability tensor specifying permeability in the fractures and pore blocks, respectively:

$$K_1 = K, \quad K_2 = dK. \tag{9}$$

Consider the two-dimensional problem in the domain Ω shown in Fig. 1. This domain geometry is chosen to demonstrate the effect of anisotropy on the results of calculations. The presence of a *step* on the left boundary of the domain (in contrast to typical rectangular geometry) makes it possible to visually discern one kind of anisotropy from another. In this case ($\Gamma_D = \Gamma_1 \cup \Gamma_3$, $\Gamma_N = \Gamma_2 \cup \Gamma_4$) we take the following boundary conditions:

$$u_\alpha(\boldsymbol{x}, t) = 1 - \exp(-\delta t), \quad \boldsymbol{x} \in \Gamma_1; \qquad u_\alpha(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Gamma_3; \tag{10}$$

$$\boldsymbol{v}_\alpha \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \Gamma_2 \cup \Gamma_4; \quad t \in (0, T], \tag{11}$$

where $\boldsymbol{v}_\alpha = K_\alpha \mathrm{grad}\, u_\alpha$. The initial state of the system is defined by the conditions

$$u_\alpha(\boldsymbol{x}, 0) = 0, \quad x \in \Omega, \quad \alpha = 1, 2. \tag{12}$$

It follows from the statement of the problem (7)–(12) that the pressure on the boundary Γ_1 increases from 0 to 1. The dynamics of the pressure increase depends on the parameter δ. The boundary conditions on Γ_1 simulate conditions on an injection well.

According to a generally accepted assumption in the double-porosity models, most filtration flows are in the fractures and the bulk of the fluid is in the pore blocks. Thereby we consider a type of fracturing in which the linear sizes of fracture openings are much greater than the pore diameters (the parameter $d = 0.01$) and the void volume of the fractures is much less than that of the pore blocks (the parameter $c = 0.01$). The following three major forms of the tensor K are considered for the model problem:

1. Isotropic filtration:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

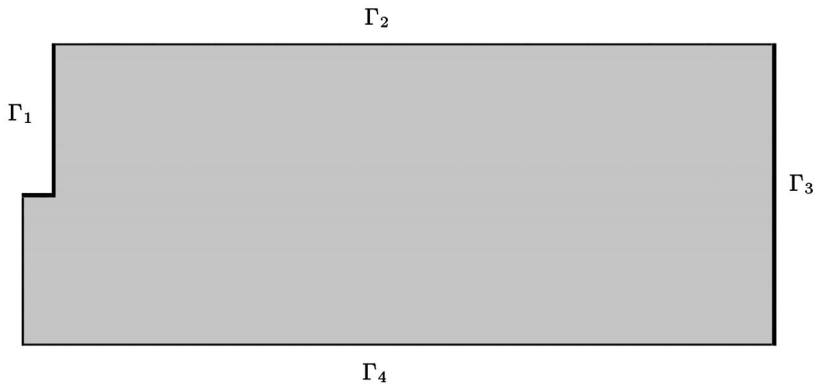

Fig. 1. Domain Ω.

2. Dominant permeability in one of the coordinates:

$$K = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

3. Dominant filtration in the diagonal:

$$K = \begin{pmatrix} 0.6 & -0.4 \\ -0.4 & 0.6 \end{pmatrix}. \tag{15}$$

4. COMPUTATIONAL ALGORITHM

The model problem is solved numerically by a finite element method (see [9] and [10]). The programs are written in a computer language `Python` using a finite element library `FEniCS` [11] and [12]. To approximately solve the unsteady-state filtration problem, we introduce, for simplicity, a uniform time grid with step τ:

$$\overline{\omega}_\tau = \omega_\tau \cup \{T\} = \{t^n = n\tau, \quad n = 0, 1, \dots, N, \quad \tau N = T\},$$

$y^n = y(t^n)$, $t^n = n\tau$. A finite element approximation in space consisting of standard second-order Lagrangian finite elements is used. A triangulation (with a software package METIS) is made in the domain Ω. A finite-dimensional finite-element space $V \subset H^1(\Omega)$ is defined on this grid. Here $H^1(\Omega)$ is the Sobolev space of functions v such that v^2 and $|\nabla v|^2$ have a finite integral in Ω and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \Gamma_D\}$. The calculation grid is refined in regions with large gradients of the solution.

Consider the following time scheme:

$$c\frac{y_1^{k+1} - y_1^k}{\tau} - \operatorname{div} K \operatorname{grad} y_1^{k+1} + r\left|K \operatorname{grad}(y_2^k - y_1^k)\right| = 0, \tag{16}$$

$$\frac{y_2^{k+1} - y_2^k}{\tau} - d \operatorname{div} K \operatorname{grad} y_2^{k+1} - r\left|K \operatorname{grad}(y_2^k - y_1^k)\right| = 0. \tag{17}$$

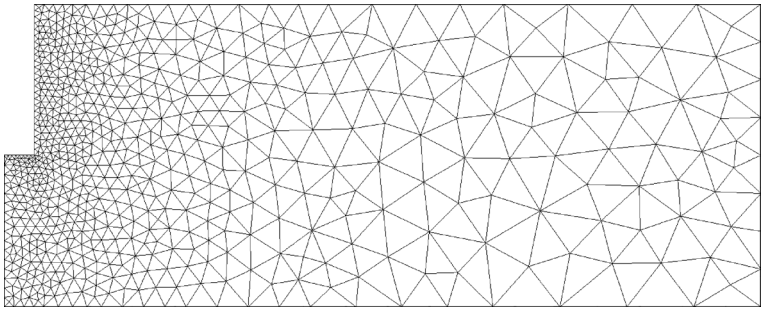

Fig. 2. Calculation grid 1: 700 nodes, 1294 elements.

Table. Time step for various values of r and various grids

Grid	$r = 0.1$	$r = 0.5$	$r = 1$
1 (1294 elements)	0.148	0.041	0.022
2 (5068 elements)	0.143	0.041	0.022
3 (19664 elements)	0.142	0.041	0.022

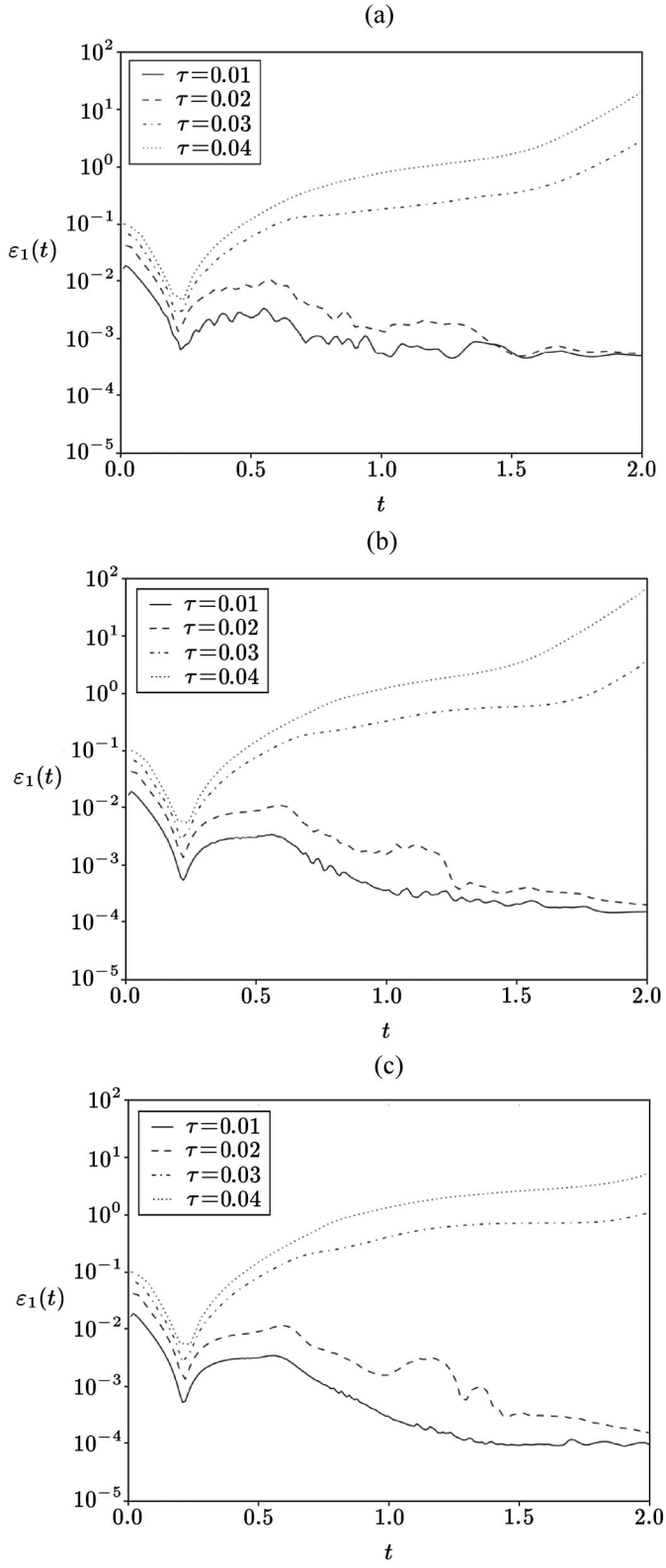

Fig. 3. $L_2(\Omega)$ norm of solution error of u_1: (a) coarse grid (1294 elements); (b) fine grid (5068 elements); (c) finest grid (19664 elements).

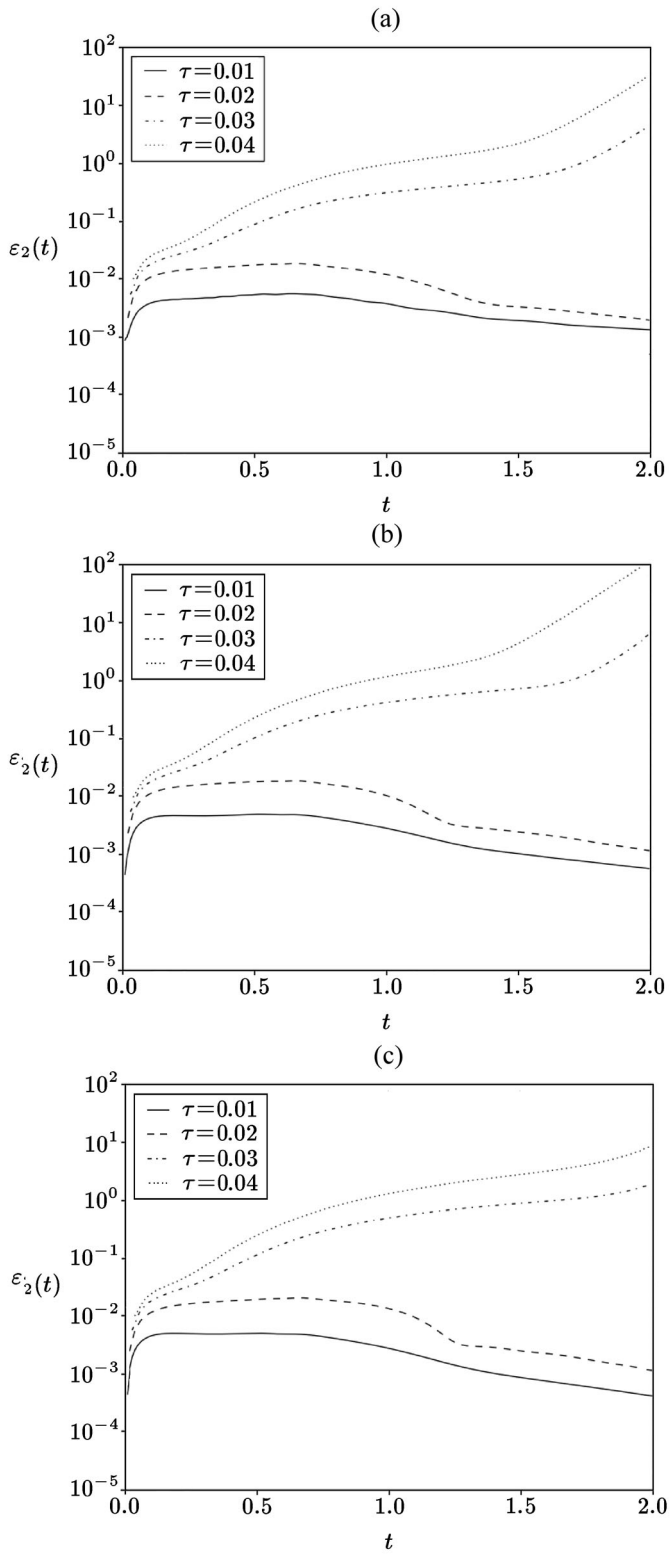

Fig. 4. $L_2(\Omega)$ norm of solution error of u_2: (a) coarse grid (1294 elements); (b) fine grid (5068 elements); (c) finest grid (19664 elements).

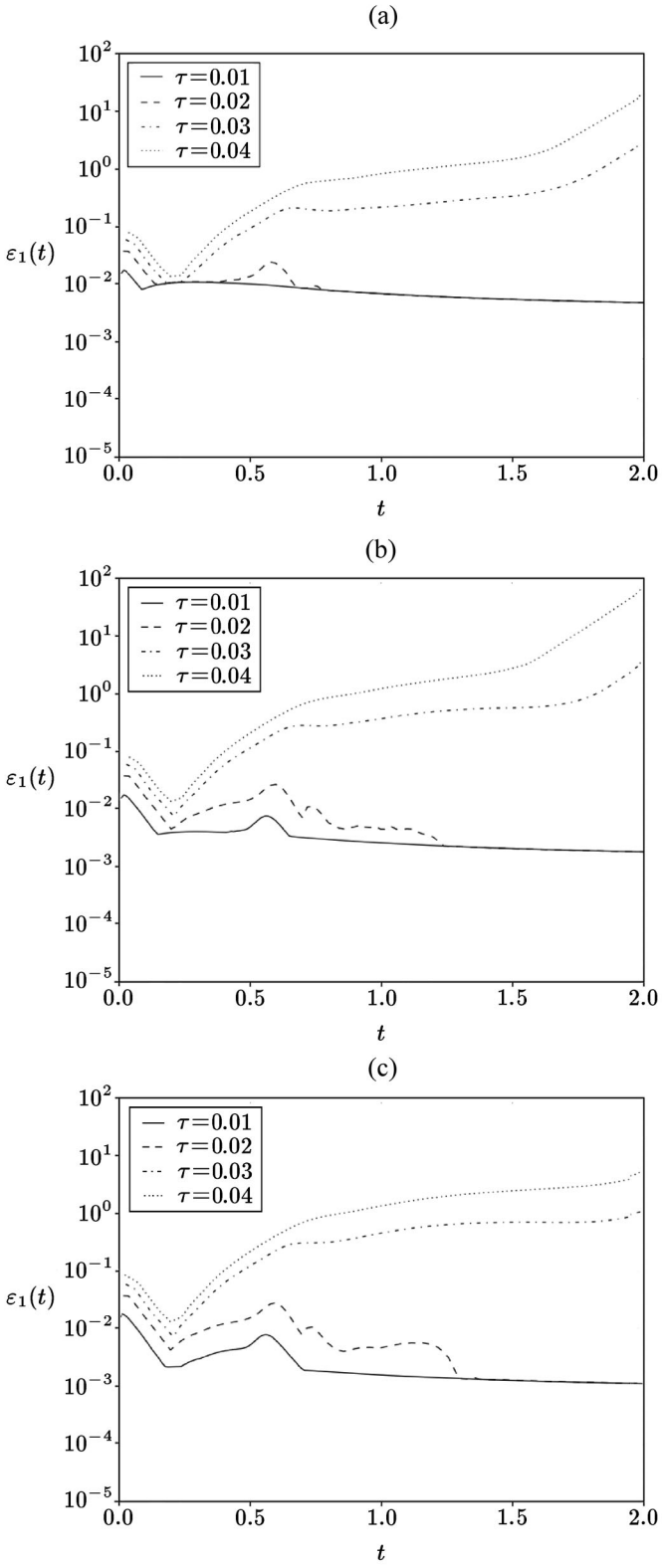

Fig. 5. $C(\Omega)$ norm of solution error of u_1: (a) coarse grid (1294 elements); (b) fine grid (5068 elements); (c) finest grid (19664 elements).

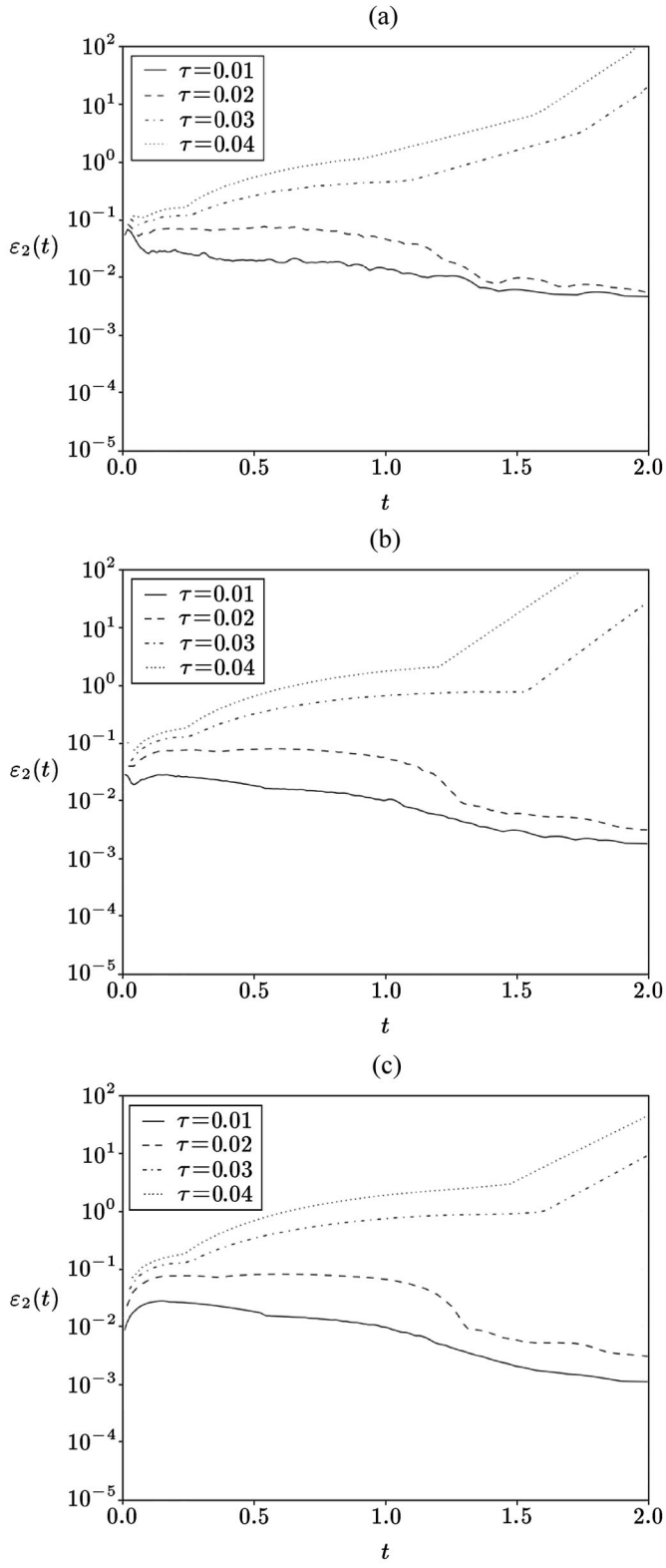

Fig. 6. $C(\Omega)$ norm of solution error of u_2: (a) coarse grid (1294 elements); (b) fine grid (5068 elements); (c) finest grid (19664 elements).

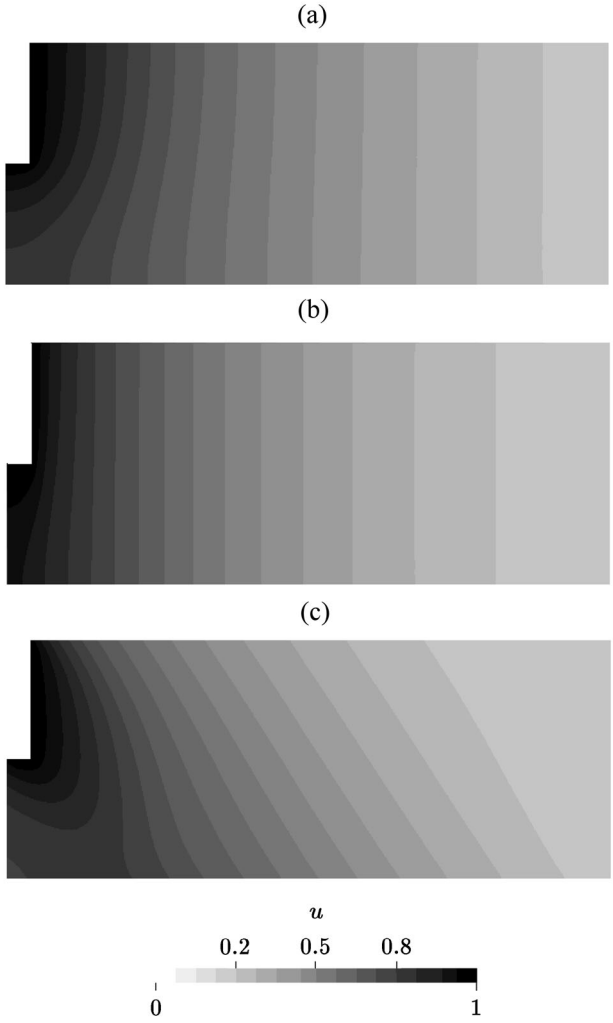

Fig. 7. Pressure distribution in fractures u_1 at time $t = 1$: (a) tensor K in the form (13); (b) in the form (14); (c) in the form (15).

Thereby, we use explicit-implicit approximations in time: the nonlinear exchange terms are taken from the previous time level.

The problem (16), (17) is presented in a variational form: Each equation is multiplied by a corresponding test function and integrated over the domain Ω using the formula of integration by parts. We obtain the following variational formulation of the problem:

$$c\int\limits_{\Omega}\frac{y_1^{k+1}-y_1^k}{\tau}v_1\,d\boldsymbol{x}+\int\limits_{\Omega}K\operatorname{grad}y_1^{k+1}\operatorname{grad}v_1\,d\boldsymbol{x}+r\int\limits_{\Omega}\left|K\operatorname{grad}(y_2^k-y_1^k)\right|v_1\,d\boldsymbol{x}=0,$$

$$\int\limits_{\Omega}\frac{y_2^{k+1}-y_2^k}{\tau}v_2\,d\boldsymbol{x}+d\int\limits_{\Omega}K\operatorname{grad}y_2^{k+1}\operatorname{grad}v_2\,d\boldsymbol{x}-r\int\limits_{\Omega}\left|K\operatorname{grad}(y_2^k-y_1^k)\right|v_2\,d\boldsymbol{x}=0$$

with allowance for the boundary conditions on Γ_D. Next, we assemble the global finite-element matrix and the solution vector. As a result, we obtain a SLAE solved by the (iterative) generalized minimal residual method (GMRES), which converges in a finite number of iterations.

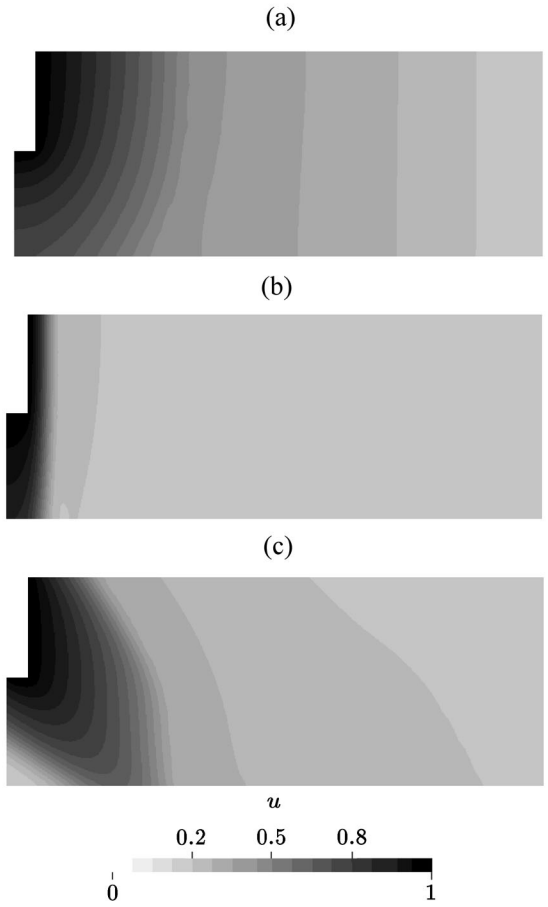

Fig. 8. Pressure distribution in pores u_2 at time $t = 1$: (a) tensor K in the form (13); (b) in the form (14); (c) in the form (15).

5. RESULTS OF CALCULATIONS

Here we present the results of numerical calculations performed on rather fine grids (1294, 5068, 19664, and 77870 elements). The coarse grid is shown in Fig. 2.

For this problem, we define a reference solution $\overline{y}^k(\boldsymbol{x})$, which is the numerical solution on the finest grid (77870 elements, $\tau = 10^{-3}$). The numerical solution is controlled by comparing it with this reference solution. The simulations were performed with the following parameter values: $c = 0.01$, $d = 0.01$, $r = 1.0$, and $\delta = 10.0$. Figures 3–6 show solution errors versus time in comparison to the reference solution in the norms of $L_2(\Omega)$ and $C(\Omega)$.

The table represents the time steps needed to achieve an error $\varepsilon = 0.01$ (at the final time) for various grids and values of the parameter r, where

$$\varepsilon = \left\| \overline{y}^k - y^k \right\|_{L_2(\Omega)}.$$

The explicit-implicit scheme imposes certain constraints on the time step.

From the results we can conclude that:

- the double-porosity model makes it possible to take into account the anisotropic properties of both media and the exchange flow;
- the numerical solution converges to the reference solution at a sufficiently small time step;
- the pressure in the fractures (u_1) reaches the steady state faster than the pressure in the pores (u_2);

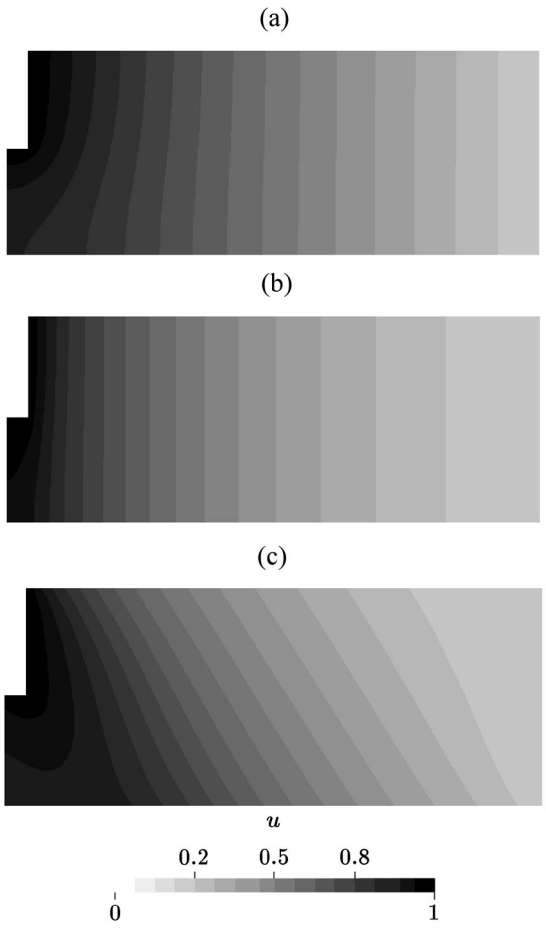

Fig. 9. Pressure distribution in fractures u_1 at time $t = 2$: (a) tensor K in the form (13); (b) in the form (14); (c) in the form (15).

- the error is almost independent of the space grids;
- the maximum time step is inversely proportional to the flow parameter r.

The results of calculations presented below were obtained on the grid shown in Fig. 2. The calculations were performed with $r = 1$. The pressure distributions in the fractures and pores are depicted in Figs. 7–9.

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REFERENCES

1. Barenblatt, G.I., Zheltov, Yu.P., and Kochina, I.N., Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks, *Prikl. Mat. Mekh.*, 1960, vol. 24, no. 5, pp. 852–864.
2. Maryška, J., Sever'yn, O., Tauchman, M., and Tondr, D., Modeling of Processes in Fractured Rock Using FEM/FVM on Multidimensional Domains, *J. Comput. Appl. Math.*, 2008, vol. 215, no. 2, pp. 495–502.
3. Loret, B. and Rizzi, E., Strain Localization in Fluid-Saturated Anisotropic Elastic-Plastic Porous Media with Double Porosity, *J. Mech. Phys. Solids*, 1999, vol. 47, no. 3, pp. 503–530.
4. Bera, P. and Khalili, A., Double-Diffusive Natural Convection in an Anisotropic Porous Cavity with Opposing Buoyancy Forces: Multi-Solutions and Oscillations, *Int. J. Heat Mass Transfer*, 2002, vol. 45, no. 15, pp. 3205–3222.

5. Schoenberg, M. and Sayers, C.M., Seismic Anisotropy of Fractured Rock, *Geophys.*, 1995, vol. 60, no. 1, pp. 204–211.
6. Dmitriev, N.M. and Maksimov, V.M., Models of Flow through Fractured-Porous Anisotropic Media, *Fluid Dyn.*, 2007, vol. 42, no. 6, pp. 937–942.
7. Basniev, K.S., Dmitriev, N.M., and Maksimov, V.M., *Podzemnaya gidromekhanika* (Subsurface Hydromechanics), Moscow: Nedra, 1993.
8. Barenblatt, G.I., Entov, V.M., and Ryzhik, V.M., *Theory of Fluid Flows Through Natural Rocks*, Springer, 2010.
9. Quarteroni, A. and Valli, A., *Numerical Approximation of Partial Differential Equations*, Springer, 2008.
10. Brenner, S.C. and Scott, L.R., *The Mathematical Theory of Finite Element Methods*, Springer, 2008.
11. Dupont, T., Hoffman, J., Johnson, C., et al., *The FEniCS Project*, Göteborg, Sweden: Chalmers Finite Element Centre, Chalmers Univ. of Technology, 2003, Technical rep. no. 2003-21.
12. Langtangen, H.P., A FEniCS Tutorial, in *Automated Solution of Differential Equations by the Finite Element Method*, Springer, 2012, pp. 1–73.