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An Inverse Eigenvalue Problem for a Class of Second- and Third-Order Matrices

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Abstract—A method for solving an inverse eigenvalue problem for a product of second- and third-order matrices is proposed. Necessary and sufficient conditions for the existence of a solution to the problem have been obtained.

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1. INTRODUCTION

Inverse eigenvalue problems may be found in physics, mechanics, and control theory [1]. Generally, the inverse eigenvalue problem may be formulated as follows: Given a set of square matrices M of order n (as a rule, this is a set of matrices with real elements), find a matrix $A \in M$ with a given set of eigenvalues $\Lambda = \{\lambda_1; \lambda_2; \ldots; \lambda_n\}$. In the case of real matrices, the complex numbers λ_i are included in the set Λ as conjugate pairs.

There exist additive, multiplicative, parameterized, and other inverse eigenvalue problems [1]. In the present paper, a problem for a product of matrices is considered. This problem is known in automatic control theory as the problem of synthesis of a periodic feedback for a linear discrete system [2].

Despite the simplicity of its statement, the problem under consideration is rather complicated from a computational perspective. In the present paper, necessary conditions for the existence of a solution to the problem are formulated in the general case of matrices of arbitrary order. For the particular case of second- and third-order matrices, necessary and sufficient conditions for existence of a solution are obtained, and an algorithm for solving the problem is described. The idea of the algorithm is to successively handle two subproblems: solving a system of linear algebraic equations and finding the roots of a polynomial.

Some examples of solving the problem on the basis of this approach are presented. The calculations were performed with the MATLAB system of scientific and engineering calculations. The results of the calculations are presented in approximate form, up to five significant digits.

2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

Let A, b, and c be, respectively, $n \times n$, $n \times 1$, and $1 \times n$ real matrices, and let $\Lambda = \{\lambda_1; \lambda_2; \ldots; \lambda_n\}$ be a set of complex numbers. The complex numbers are included in Λ as conjugate pairs. Find real numbers $F_n = \{f_1; f_2; \ldots; f_n\}$ such that the spectrum of the matrix

$$\Phi_n = (A + bcf_1)(A + bcf_2) \cdots (A + bcf_n)$$

coincides with Λ.

Consider a polynomial

$$q(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + q_1 \lambda^{n-1} + \cdots + q_n.$$

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An equivalent problem statement is in finding an F_n at which the characteristic polynomial Φ_n coincides with $q(\lambda)$.

Remark 1. In [2], a similar problem is solved for $n+1$ number of $f_1, f_2, \ldots, f_{n+1}$, that is, for the matrix

$$(A + bcf_1)(A + bcf_2)\cdots(A + bcf_{n+1}).$$

Here A, b, and c are $n \times n$, $n \times 1$, and $1 \times n$ matrices, respectively. In this case the algorithm to calculate $f_1, f_2, \ldots, f_{n+1}$ described in [2] differs from the algorithm proposed in the present paper.

Let us introduce square matrices:

$$X_n(A, b) = [b,\ Ab,\ \ldots,\ A^{n-1}b], \qquad Y_n(A, c) = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}.$$

The following properties of the matrices $X_n(A, b)$ and $Y_n(A, c)$ are well known in the mathematical theory of linear control systems [3]:

Lemma 1. *The matrix $X_n(A, b)$ is nonsingular, that is,* $\det X_n(A, b) \neq 0$*, if and only if* $\operatorname{rank}[\lambda E - A,\ b] = n$ *for any eigenvalue λ of the matrix A.*

Lemma 2. *The matrix $Y_n(A, c)$ is nonsingular, that is,* $\det Y_n(A, c) \neq 0$*, if and only if*

$$\operatorname{rank} \begin{bmatrix} \lambda E - A \\ c \end{bmatrix} = n$$

for any eigenvalue λ of the matrix A.

The next two lemmas are obtained from Lemmas 1 and 2.

Lemma 3. *If* $\det X_n(A, b) = 0$*, there exists an eigenvalue of the matrix Φ_n that is invariant with respect to the choice of F_n, and this eigenvalue is an eigenvalue of the matrix A^n.*

Proof. Let $\det X_n(A, b) = 0$. According to Lemma 1, $\operatorname{rank}\left[\lambda E - A,\ b\right] <$ n for some eigenvalue λ of the matrix A. This means that there exists a vector v such that $v^\top \left[\lambda E - A,\ b\right] = 0$. Hence, $v^\top A = \lambda v^\top$, $v^\top b = 0$. Note that the vector v is a left eigenvector of the matrix A corresponding to the eigenvalue λ. With this taken into account, we obtain

$$v^\top (A + bcf_1)(A + bcf_2)\cdots(A + bcf_n) = \lambda v^\top (A + bcf_2)\cdots(A + bcf_n) = \lambda^n v^\top.$$

Thus, λ^n is an eigenvalue of the matrix Φ_n, and it is invariant with respect to the choice of F_n. □

Lemma 4. *If* $\det Y_n(A, c) = 0$*, there exists an eigenvalue of the matrix Φ_n that is invariant with respect to the choice of F_n, and this eigenvalue is an eigenvalue of the matrix A^n.*

The proof is similar to the proof of Lemma 3.

Hence, if $\det X_n(A, b) = 0$ or $\det Y_n(A, c) = 0$, the eigenvalues of Φ_n cannot be given arbitrarily.

Let $a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of the matrix A.

Lemma 5. *Let* $\det X_n(A,b) \neq 0$. *Then the matrix* Φ_n *is similar to the matrix*

$$\Psi_n = (\bar{A} + \bar{b}\bar{c}f_1)(\bar{A} + \bar{b}\bar{c}f_2)\cdots(\bar{A} + \bar{b}\bar{c}f_n),$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}.$$

Proof. Consider the nonsingular matrix

$$P = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and the matrix $Q = X_n(A,b)P$. Note that the matrix Q is also nonsingular. The following relations are valid [3]: $\bar{A} = Q^{-1}AQ$, $\bar{b} = Q^{-1}b$. The vector $\bar{c} = cQ$ will have a general form. Thus,

$$\begin{aligned} \Phi_n &= (A + bcf_1)(A + bcf_2)\cdots(A + bcf_n) \\ &= Q(\bar{A} + \bar{b}\bar{c}f_1)Q^{-1}Q(\bar{A} + \bar{b}\bar{c}f_2)Q^{-1}\cdots Q(\bar{A} + \bar{b}\bar{c}f_n)Q^{-1} = Q\Psi_n Q^{-1}. \end{aligned}$$

Hence, the matrices Φ_n and Ψ_n are similar. □

Consider two particular cases of the problem for second- and third-order matrices.

3. THE PROBLEM FOR A SECOND-ORDER MATRIX

Given: two real 2×2, 2×1, and 1×2 matrices A, b, and c, respectively. Required: numbers f_1 and f_2 at which the eigenvalues of the matrix $\Phi_2 = (A + bcf_1)(A + bcf_2)$ coincide with given values λ_1 and λ_2, or, equivalently, the characteristic polynomial of the matrix Φ_2 coincides with the given polynomial

$$q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + q_1\lambda + q_2.$$

Theorem 1. *All eigenvalues of the matrix* Φ_2 *can be arbitrarily specified by choosing values of* f_1 *and* f_2, *which are complex in the general case, if and only if*

$$\det X_2(A,b) \neq 0, \qquad \det Y_2(A,c) \neq 0, \qquad cAb + a_1cb \neq 0. \tag{1}$$

Proof. The necessity of the first two conditions follows from Lemmas 3 and 4. Let $\det X_2(A,b) \neq 0$. The coefficients of the characteristic polynomial of Φ_2 linearly depend on the coefficients of the polynomial

$$p(\lambda) = (\lambda - f_1)(\lambda - f_2) = \lambda^2 + p_1\lambda + p_2.$$

Clearly,

$$\det(\lambda E - \Phi_2) = \det(\lambda E - \Psi_2),$$

since the matrices Φ_2 and Ψ_2 are similar (Lemma 5). Taking into account that

$$\Psi_2 = (\bar{A} + \bar{b}\bar{c}f_1)(\bar{A} + \bar{b}\bar{c}f_2),$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix},$$

we obtain

$$\det(\lambda E - \Phi_2) = \lambda^2 + ((c_1 - a_1c_2)p_1 - c_2^2p_2 + 2a_2 - a_1^2)\lambda + c_1a_2p_1 + c_1^2p_2 + a_2^2,$$

where

$$p_1 = -(f_1 + f_2), \qquad p_2 = f_1f_2.$$

Equating the coefficients of the polynomials $\det(\lambda E - \Phi_2)$ and $q(\lambda)$, we obtain a system of linear algebraic equations:

$$R\bar{p} = r, \tag{2}$$

where

$$R = \begin{bmatrix} c_1 - a_1c_2 & -c_2^2 \\ c_1a_2 & c_1^2 \end{bmatrix}, \qquad r = \begin{bmatrix} q_1 + a_1^2 - 2a_2 \\ q_2 - a_2^2 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

Thus, the algorithm of assigning the eigenvalues of Φ_2 is as follows: Specify the polynomial $q(\lambda)$. Solve the system of equations (2). Find the roots of the polynomial $p(\lambda) = \lambda^2 + p_1\lambda + p_2$, which are the sought-for values of f_1 and f_2.

The system of equations (2) has a unique solution at any q_1, q_2 if and only if the matrix R is nonsingular. It is easy to verify that $\det R = c_1(c_1^2 - a_1c_1c_2 + a_2c_2^2)$. Here

$$c_1 = cAb + a_1cb, \quad c_1^2 - a_1c_1c_2 + a_2c_2^2 = \det Y_2(\bar{A}, \bar{c}) = \det Y_2(A, c)\det Q.$$

Hence, the conditions (1) are necessary and sufficient for arbitrarily assigning the eigenvalues of the matrix Φ_2. □

Remark 2. The roots of the polynomial $p(\lambda)$ may be complex. In this case no solution to the problem exists in the set of real numbers with the given polynomial $q(\lambda)$. There is a condition imposed on the coefficients of the polynomial $q(\lambda)$ at which the roots of the polynomial $p(\lambda)$ are real. The condition is obtained from the inequality $p_1^2 \geq 4p_2$, and has the following form:

$$\begin{aligned}(c_1^2(q_1 + a_1^2 - 2a_2) + c_2^2(q_2 - a_2^2))^2 \\ \geq\ 4((c_1 - a_1c_2)(q_2 - a_2^2) - c_1a_2(q_1 + a_1^2 - 2a_2))c_1(c_1^2 - a_1c_1c_2 + a_2c_2^2).\end{aligned} \tag{3}$$

Example 1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 5 & 2 \end{bmatrix}.$$

The desired eigenvalues of the matrix Φ_2 are specified equal to

$$\Lambda = \{0.1 + 3i; 0.1 - 3i\}.$$

The conditions of Theorem 1 are satisfied. Therefore, the system of equations (2) has a unique solution: $p_1 = -5.2314$; $p_2 = -1.8922$. The roots of the polynomial $p(\lambda) = \lambda^2 + p_1\lambda + p_2$ are $f_1 = 5.5710$ and $f_2 = -0.33964$.

Let $\Lambda = \{6; 12\}$. In this case $p(\lambda) = \lambda^2 - 2.2615\lambda + 1.8154$. The roots of the polynomial $p(\lambda)$ are complex: $f_1 = 1.1308 + 0.73263i$ and $f_2 = 1.1308 - 0.73263i$. The condition (3) is not satisfied.

4. THE PROBLEM FOR A THIRD-ORDER MATRIX

Consider the matrix $\Phi_3 = (A + bcf_1)(A + bcf_2)(A + bcf_3)$. Here $n = 3$. Required: numbers f_1, f_2, and f_3 at which the eigenvalues of the matrix Φ_3 coincide with given values λ_1, λ_2, and λ_3.

Theorem 2. *All eigenvalues of the matrix Φ_3 can be arbitrarily specified by choosing values of f_1, f_2, and f_3, which are complex in the general case if and only if*

$$\det X_3(A, b) \neq 0, \qquad \det Y_3(A, c) \neq 0, \tag{4}$$

$$c_1 \neq 0, \qquad c_1^2 - a_1c_1c_2 + a_3c_2c_3 \neq 0,$$

where

$$c_1 = cA^2b + a_1cAb + a_2cb, \qquad c_2 = cAb + a_1cb, \qquad c_3 = cb.$$

Proof. The proof is similar to the proof of Theorem 1. The necessity of the inequalities (4) follows from Lemmas 3 and 4. Let $\det X_3(A, b) \neq 0$. According to Lemma 5, the matrix Φ_3 is similar to the matrix

$$\Psi_3 = (\bar{A} + \bar{b}\bar{c}f_1)(\bar{A} + \bar{b}\bar{c}f_2)(\bar{A} + \bar{b}\bar{c}f_3),$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{c} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}.$$

The characteristic polynomial of the matrix Ψ_3 is

$$\begin{aligned} \det(\lambda E - \Psi_3) \;=\; & \lambda^3 + ((a_1^2c_3 - a_1c_2 - a_2c_3 + c_1)p_1 \\ & + (a_1c_3^2 - c_2c_3)p_2 + c_3^3p_3 + a_1^3 - 3a_1a_2 + 3a_3)\lambda^2 \\ & + ((a_2^2c_2 + 2a_3c_1 - a_1a_2c_1 - a_1a_3c_2 - a_2a_3c_3)p_1 \\ & + (a_2c_2^2 - a_1c_1c_2 - a_2c_1c_3 - a_3c_2c_3 + c_1^2)p_2 \\ & + (c_2^3 - 3c_1c_2c_3)p_3 + a_2^3 - 3a_1a_2a_3 + 3a_3^2)\lambda \\ & + a_3^2c_1p_1 + a_3c_1^2p_2 + c_1^3p_3 + a_3^3, \end{aligned}$$

where p_1, p_2, and p_3 are coefficients of the polynomial

$$p(\lambda) = (\lambda - f_1)(\lambda - f_2)(\lambda - f_3) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3.$$

Equating coefficients of the polynomials $\det(\lambda E - \Psi_3)$ and

$$q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3,$$

we obtain a system of linear algebraic equations:

$$R\bar{p} = r, \tag{5}$$

where

$$r = \begin{bmatrix} q_1 - a_1^3 + 3a_1a_2 - 3a_3 \\ q_2 - a_2^3 + 3a_1a_2a_3 - 3a_3^2 \\ q_3 - a_3^3 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

The elements of the matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

are

$$\begin{aligned}
&r_{11} = a_1^2 c_3 - a_1 c_2 - a_2 c_3 + c_1, \\
&r_{12} = a_1 c_3^2 - c_2 c_3, \quad r_{13} = c_3^3, \\
&r_{21} = a_2^2 c_2 + 2a_3 c_1 - a_1 a_2 c_1 - a_1 a_3 c_2 - a_2 a_3 c_3, \\
&r_{22} = a_2 c_2^2 - a_1 c_1 c_2 - a_2 c_1 c_3 - a_3 c_2 c_3 + c_1^2, \\
&r_{23} = c_2^3 - 3c_1 c_2 c_3, \\
&r_{31} = a_3^2 c_1, \quad r_{32} = a_3 c_1^2, \quad r_{33} = c_1^3.
\end{aligned}$$

The following equality is valid:

$$\det R = c_1(c_1^2 - a_1 c_1 c_2 + a_3 c_2 c_3) \det Y_3(\bar{A}, \bar{c}).$$

Hence, Eq. (5) has a unique solution at any q_1, q_2, q_3 if and only if $\det Y_3(\bar{A}, \bar{c}) \neq 0$, $c_1 \neq 0$, $c_1^2 - a_1 c_1 c_2 + a_3 c_2 c_3 \neq 0$. Note that

$$\det Y_3(\bar{A}, \bar{c}) = \det Y_3(A, c) \det Q,$$

$$c_1 = cA^2 b + a_1 cAb + a_2 cb, \quad c_2 = cAb + a_1 cb, \quad c_3 = cb.$$

□

Remark 3. As in the case of a second-order matrix, the roots of the polynomial $p(\lambda)$ may be complex. A condition under which the polynomial $p(\lambda)$ does not have complex roots is rather complicated and, therefore, omitted here.

Example 2. Consider the following matrices:

$$A = \begin{bmatrix} 3.8506 & -8.7682 & 2.1573 \\ 1.1334 & 5.6035 & 4.8251 \\ -2.0696 & -3.2483 & -7.9037 \end{bmatrix}, \qquad b = \begin{bmatrix} -7.4422 \\ 0.99080 \\ -0.29541 \end{bmatrix},$$

$$c = \begin{bmatrix} 7.8095 & 5.9792 & 4.6868 \end{bmatrix}.$$

The characteristic polynomial of the matrix A is

$$a(\lambda) = \lambda^3 - 1.5504\lambda^2 - 23.070\lambda + 84.096.$$

The matrices $\bar{A}$, $\bar{b}$, and $\bar{c}$ have the following form:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -84.096 & 23.070 & 1.5504 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\bar{c} = \begin{bmatrix} 941.16 & -171.26 & -53.580 \end{bmatrix}.$$

Specify the required eigenvalues of the matrix Φ_3 as $\Lambda = \{0.1; 0.2; 0.3\}$. Form and solve the system of equations (5). We obtain

$$\bar{p} = \begin{bmatrix} -0.48129 \\ 0.066259 \\ -0.0027912 \end{bmatrix}.$$

The sought-for numbers f_1, f_2, and f_3 are found as the roots of the polynomial

$$p(\lambda) = \lambda^3 - 0.48129\lambda^2 + 0.066259\lambda - 0.0027912.$$

These numbers are: $f_1 = 0.28061$; $f_2 = 0.11132$; $f_3 = 0.089353$.

Remark 4. In the general case with $n > 3$, the statement that the coefficients of the characteristic polynomial of the matrix Φ_n linearly depend on the coefficients of the polynomial $p(\lambda)$ is not true. As an example, let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \det(\lambda E - \Phi_4) &= \lambda^4 + (-5 - f_1 - f_2 - f_3 - f_4)\lambda^3 \\ &\quad + (2(f_1 + f_2 + f_3 + f_4) + f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4)\lambda^2 \\ &\quad + (-f_1 f_2 - f_1 f_4 - f_2 f_3 - f_3 f_4 - f_1 f_2 f_3 - f_1 f_3 f_4 - f_2 f_3 f_4)\lambda + f_1 f_2 f_3 f_4 \\ &= \lambda^4 + (-5 + p_1)\lambda^3 + (-2p_1 + p_2)\lambda^2 + (-p_2 + f_1 f_3 + f_2 f_4 + p_3)\lambda + p_4. \end{aligned}$$

In this case, the coefficients of the characteristic polynomial of the matrix Φ_4 do not linearly depend on the coefficients of the polynomial

$$p(\lambda) = (\lambda - f_1)(\lambda - f_2)(\lambda - f_3)(\lambda - f_4) = \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4,$$

since the coefficient at λ has the terms $f_1 f_3$ and $f_2 f_4$.

5. CONCLUSIONS

In this paper, an approach to solving an inverse eigenvalue problem for the matrices Φ_2 and Φ_3 has been proposed. It is in successively handling the following two problems: solving a system of linear algebraic equations and finding the roots of a polynomial. The system of linear algebraic equations has been obtained, and necessary and sufficient conditions for the existence of a unique solution to the system have been defined.

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