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Stability of Equilibrium Points for a Hamiltonian Systems with Two Degrees of Freedom in the Problem of Parametric Resonance

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Abstract—The paper considers the problem of the stability of an equilibrium position of nonlinear periodic Hamiltonian systems with two degrees of freedom depending on a small parameter. The main attention is paid to the study of critical cases when the unperturbed linearized equation has second-order resonances. The main cases leading to these resonances are considered. Sufficient conditions under which the equilibrium point of a Hamiltonian system will be formally stable or unstable in the sense of Lyapunov are indicated as well. The results obtained are formulated in terms of the original equations and brought to effective formulas and algorithms.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

A nonlinear periodic Hamiltonian system with two degrees of freedom is considered depending on a small parameter ε in the form of

$$\frac{dx}{dt} = J\nabla H(x,t,\varepsilon), \quad x \in R^4; \tag{1}$$

in which Hamiltonian $H(x,t,\varepsilon)$ can be represented as $H(x,t,\varepsilon) = H_2(x,t,\varepsilon) + H_3(x,t,\varepsilon) + \ldots$; here $H_j(x,t,\varepsilon)$ — are homogeneous order of j with respect to x and T-periodic in t functions;

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \nabla H(x,t,\varepsilon) = \begin{bmatrix} H'_{x_1} \\ H'_{x_2} \\ H'_{x_3} \\ H'_{x_4} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Let $x = 0$ be the equilibrium position of the analytic Hamiltonian system (1).

It is assumed that $H_2 = H_2(x,t,\varepsilon)$ can be represented as $H_2 = H_{20}(x) + \varepsilon H_{21}(x,t)$, where $H_{20}(x), H_{21}(x,t)$ — are quadratic forms in x. In this case, the system (1) can be represented as

$$\frac{dx}{dt} = J\left[A_0 + \varepsilon A_1(t)\right]x + a(x,\varepsilon,t), \quad x \in \mathbb{R}^4, \tag{2}$$

where A_0, $A_1(t)$ are real symmetric matrices, $A_1(t)$ is a T-periodic matrix (that is, $A_1(t+T) \equiv A_1(t)$), the function $a(x,\varepsilon,t)$ satisfies condition: $||a(x,\varepsilon,t)|| = O(||x^2||)$ for $x \to 0$, uniformly in ε and t. All

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functions in the system (1) are assumed to be continuous in t and smooth (continuously differentiable) in ε.

Along with the equation (2) we consider a linear periodic Hamiltonian system (LPHS) of the form

$$\frac{dx}{dt} = J\left[A_0 + \varepsilon A_1(t)\right]x, \quad x \in \mathbb{R}^4. \tag{3}$$

By virtue of the general properties of Hamiltonian systems (cf., for instance, [1, 2], the following is true: if the matrix JA_0 has at least one eigenvalue with a nonzero real part, then the LPHS (3) will be unstable for all small $|\varepsilon|$. The equilibrium point $x = 0$ the nonlinear system (2) will be also unstable.

Further, we will assume that the eigenvalues of the matrix JA_0 are purely imaginary, namely, they are numbers

$$\pm i\omega_1, \quad \pm i\omega_2, \quad \text{where} \quad \omega_j > 0. \tag{4}$$

One of the most interesting problems in studying systems (2) and (3) is the problem of parametric resonance (cf., for example, [3]). As applied to the systems under consideration, the problem of parametric resonance consists in studying the stability of the equilibrium point $x = 0$ of the system (2) (stability of the system (3)) in the situation when one of the conditions:

S1) among the eigenvalues (4) of the matrix JA_0 there is at least one $i\omega_0$ such that the simple resonance condition is satisfied

$$\omega_0 = \frac{\pi k_0}{T} \quad \text{for some integer} \quad k_0;$$

S2) the eigenvalues (4) the combinational resonance condition is satisfied

$$\omega_1 - \omega_2 = \frac{2\pi k_0}{T} \quad \text{for some integer} \quad k_0. \tag{5}$$

In this case, it is assumed that when the risk conditions S1 (S2) are met, the detection of S2 (S1) is not performed.

The problem of studying the stability of Hamiltonian systems (2) and (3) with a periodic perturbation and, in particular, the problem of parametric resonance is the subject of many works. Most of the research is based on the methods of normalization of Hamiltonian systems. A number of important results have been obtained in this direction (cf., for instance, [1, 3–9]).

In the work [10] the problem of parametric resonance in the main resonances for LPHS of the form (3) was studied. First approximation formulas were proposed for perturbations of multiple definite and indefinite multipliers and their applications to the stability analysis of LPHS. This paper develops the results of this work as applied to the problem of analyzing the stability of the equilibrium point $x = 0$ of a nonlinear Hamiltonian system with two degrees of freedom (2).

2. AUXILIARY FACTS

Let us give some auxiliary information about the stability of the equilibrium points of nonlinear periodic Hamiltonian systems of the form

$$\frac{dx}{dt} = J\nabla H(x,t), \quad x \in R^4. \tag{6}$$

We assume that the Hamiltonian $H(x,t)$ can be represented as $H(x,t) = H_2(x) + H_3(x,t) + \ldots$, where (as above) H_j are homogeneous forms of order j with respect to x, while the quadratic form H_2 does not depend on t, and the forms H_3, H_4, $\ldots$ are T-periodic in t. Let $x = 0$ be the equilibrium position of the system (6).

In this paper, along with the notion of Lyapunov stability of the equilibrium point $x = 0$ of the system (6), we also use the more general notion of formal stability (cf., for example, [11, 12]). The equilibrium point $x = 0$ of the system (6) is called formally stable if there exists a formal sign-definite integral $G(x,t) = G_2(x) + G_3(x,t) + G_4(x,t) + \ldots$ of the system (6), and the homogeneous form $G_2(x)$ vanishes only for $x = 0$. The formality is understood in the sense that the series $G(x,t)$ can diverge.

In most physical problems, formal stability is quite sufficient (for a discussion cf., for example, [1, 11]). In the presence of formal stability, if there are trajectories that go far from the unperturbed motion, then the motion along them is extremely slow.

Let us present some assertions regarding stability criteria for the equilibrium point $x = 0$ of the system (6). Let the matrix JA_0 of the linearized system

$$\frac{dx}{dt} = JA_0 x, \quad x \in \mathbb{R}^4,$$

has its own values (4). According to Birkhoff (cf., for example, [1]) in this case the system (6) can be reduced to normal form, in which the quadratic form $H_2(x)$ will look like:

1^0 if $\omega_1 \neq \omega_2$ or $\omega_1 = \omega_2 = \omega_0$, where $i\omega_0$ is a semi-simple eigenvalue of the matrix JA_0, then

$$H_2(x) = \frac{\omega_1}{2}(x_1^2 + x_3^2) + \sigma\frac{\omega_2}{2}(x_2^2 + x_4^2); \quad \text{where} \quad \sigma = 1 \quad \text{or} \quad \sigma = -1; \tag{7}$$

2^0 if $\omega_1 = \omega_2 = \omega_0$, where $i\omega_0$ is a non-semi-simple eigenvalue of the matrix JA_0, then

$$H_2(x) = \omega_0(x_1x_4 - x_2x_3) + \sigma\frac{1}{2}(x_3^2 + x_4^2); \quad \text{where} \quad \sigma = 1 \quad \text{or} \quad \sigma = -1. \tag{8}$$

A resonance of order p (here p is a natural number) is said to take place if there are integers m_1 and m_2 such that $|m_1| + |m_2| = p$, wherein

$$m_1\omega_1 + m_2\omega_2 = \frac{2\pi k_0}{T} \tag{9}$$

for some integer k_0.

Let the quantities ω_1 and ω_2 be not related by any resonance relation (9). Then the equilibrium $x = 0$ of the system (6) is formally stable according to J. Moser's theorem (cf., for example, [13]).

3. THE STABILITY OF THE EQUILIBRIUM POSITION OF A NONLINEAR HAMILTONIAN SYSTEM

Let us return to the main question about the stability of the equilibrium point $x = 0$ of the nonlinear Hamiltonian system (2) in the above cases S_1 and S_2. Note that in each of these cases for the system (2) for $\varepsilon = 0$ there is a resonance of order 2.

3.1. Case S_1

Let's start studying the problem with the case S_1, i.e. let one of the eigenvalues (4) of the matrix JA_0 in the system (2) be represented as $\omega_0 = \pi k_0/T$ for some natural k_0. It is also assumed that another purely imaginary eigenvalue of the matrix JA_0 does not satisfy a similar relation.

There is a nonzero vector $e + ig \in \mathbb{C}^4$ (where $e, g \in \mathbb{R}^4$) such that

$$JA_0(e + ig) = i\omega_0(e + ig);$$

for which the relation $(e, Jg) \neq 0$ holds. Let us assume that $\nu = 1/(e, Jg)$.

Define a constant matrix

$$S_0 = \int_0^T A_1(t)\,dt, \tag{10}$$

where $A_1(t)$ is the matrix involved in the system (3). Let

$$\Delta_1 = a^2 + b_1 b_2, \tag{11}$$

where

$$a=\int_0^T \{\cos(2\omega_0 t)\,(A_1(t)e,g)-\frac{1}{2}\sin(2\omega_0 t)\,[(A_1(t)g,g)-(A_1(t)e,e)]\}dt,$$

$$b_1=\int_0^T [\cos^2(\omega_0 t)\,(A_1(t)g,g)+\sin^2(\omega_0 t)\,(A_1(t)e,e)+\sin(2\omega_0 t)\,(A_1(t)e,g)]dt,$$

$$b_2=b_1-[(S_0e,e)+(S_0g,g)].$$

Theorem 1. *Let $\Delta_1>0$. Then for all small $\varepsilon\neq 0$ the equilibrium point $x=0$ the system (2) is Lyapunov unstable.*

Theorem 2. *Let $\Delta_1<0$. Then for almost all small $\varepsilon\neq 0$ the equilibrium point of the system (2) is formally stable.*

The proofs of these and other main statements of the article are given in Section 4.

3.2. Case S_2

Consider now the case S_2. This case is divided into subcases when in (5) $k_0=0$ or $k_0\neq 0$.

3.2.1. Subcase $k_0\neq 0$. In this subcase, there is a pair of non-zero linearly independent vectors $e,g\in\mathbb{C}^4$ such that the following equalities hold

$$JA_0e=i\omega_1 e,\quad JA_0g=i\omega_2 g. \tag{12}$$

In this case, the vectors e, g can be chosen satisfying the equality $(e,Jg)=0$.

It can be assumed that the quadratic form $H_{20}(x)$ of the Hamiltonian $H(x,t,\varepsilon)$ of the system (1) is chosen in accordance with the formula (7). There are two mutually exclusive cases when $\sigma=1$ or $\sigma=-1$.

Lemma 1. *If $\sigma=1$, then the vectors e,g from (12) can be normalized according to one and only one pair of equalities $(iJe,e)=(iJg,g)=1$ or $(iJe,e)=(iJg,g)=-1$.*

If $\sigma=-1$, then the vectors e,g can be normalize according to the equalities $(iJe,e)=-1$, $(iJg,g)=1$.

Further let

$$a=(S_0e,e),\quad b=(S_0g,g),\quad c=\int_0^T e^{-2\pi ik_0t/T}(A_1(t)g,e)dt;$$

where S_0 is matrix (10). Note that the numbers a and b are real, while the number c, is generally complex.

Consider first the case when $\sigma=1$. In this case, let $\Delta_2=(a-b)^2+4c\bar{c}$.

Theorem 3. *Let $\sigma=1$. If $\Delta_2\neq 0$, then for almost all small $\varepsilon\neq 0$ the equilibrium point $x=0$ is formally stable.*

Let now $\sigma=-1$. In this case let $\Delta_3=(a+b)^2-4c\bar{c}$.

Theorem 4. *Let $\sigma=-1$. Then if $\Delta_3<0$, then for all small $\varepsilon\neq 0$ the equilibrium point $x=0$ of the system (2) is Lyapunov unstable. If $\Delta_3>0$, then for almost all small $\varepsilon\neq 0$ the equilibrium point $x=0$ of the system (2) is formally stable.*

3.2.2. Subcase $k_0=0$. In the situation under consideration, the matrix JA_0 has a multiple eigenvalue $i\omega_0=i\omega_1=i\omega_2$, where $\omega_0\neq\pi k/T$ for natural k.

If $i\omega_0$ is a semi-simple eigenvalue, then the study can be carried out according to the same scheme that was given for the subcase $k_0\neq 0$.

Let now the eigenvalue $i\omega_0$ be non-semi-simple. In this subcase, there is a pair of non-zero linearly independent vectors $e, g \in \mathbb{C}^4$ such that the equalities

$$JA_0e = i\omega_0 e, \quad JA_0 g = i\omega_0 g + e.$$

Then $(e, Je) = 0$ and $(e, Jg) \neq 0$, and the number (e, Jg) is real. The vector g can be chosen from the condition that the equality $(g, Jg) = 0$.

We can assume that the quadratic form $H_{20}(x)$ of the Hamiltonian $H(x, t, \varepsilon)$ system (1) is chosen according to the formula (7).

Theorem 5. *Let $\varepsilon\sigma(S_0e, e) > 0$. Then for almost all small $\varepsilon \neq 0$ the equilibrium $x = 0$ of the system (2) is formally stable.*

Theorem 6. *Let $\varepsilon\sigma(S_0e, e) < 0$. Then for almost all small nonzero $|\varepsilon|$ the equilibrium $x = 0$ of the system (2) is unstable.*

4. PROOF OF MAIN STATEMENTS

Let's limit to to giving the proof of the Theorems 1 and 2; Theorems 3–6 are proved similarly.

Denote by $V(\varepsilon)$ the monodromy matrix of the system (3). Then $V_0 = e^{JA_0T}$ is the monodromy matrix of the system (3) for $\varepsilon = 0$. In the case of S_1 the matrix V_0 has a semi-simple eigenvalue μ_0 of multiplicity 2, where $\mu_0 = 1$ (if k_0 is even) or $\mu_0 = -1$ (if k_0 is odd). In the research [10, 14] it was determined that for a small $|\varepsilon|$ matrix $V(\varepsilon)$ has a pair of eigenvalues $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$, they can be represented as

$$\mu_1(\varepsilon) = \mu_0 + \mu_1^{(1)}\varepsilon + O(\varepsilon^{3/2}), \quad \mu_2(\varepsilon) = \mu_0 + \mu_1^{(2)}\varepsilon + O(\varepsilon^{3/2}). \tag{13}$$

The coefficients $\mu_1^{(1)}$ and $\mu_1^{(2)}$ are numbers

$$\mu_1^{(1)} = \nu\mu_0\sqrt{\Delta_1}, \quad \mu_1^{(2)} = -\mu_1^{(1)}; \tag{14}$$

here the number Δ_1 is defined according to (11).

If $\Delta_1 > 0$, then for small nonzero $|\varepsilon|$ we have $|\mu_1(\varepsilon)| \neq 1$ and $|\mu_2(\varepsilon)| \neq 1$. Therefore, for small nonzero $|\varepsilon|$, the equilibrium point $x = 0$ of the system (2) is Lyapunov unstable. Theorem 1 is proved.

Now let $\Delta_1 < 0$. Then for small $|\varepsilon|$ we have $|\mu_1(\varepsilon)| = |\mu_2(\varepsilon)| = 1$. Moreover, for almost all small $|\varepsilon| \neq 0$ in the system (2), there are no resonances of all orders. This fact follows from the formulas (13) and (14). Therefore, by Moser's theorem, the equilibrium point $x = 0$ will be formally stable for almost all small $|\varepsilon| \neq 0$. Theorem 2 is proved.

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