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What is the Bochner Technique and Where is it Applied

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Abstract—In the present article we consider the generalized Bochner technique that is a natural development of the classical Bochner technique. As an illustration, we prove some Liouville-type theorems for Killing and Killing–Yano tensors, as well as for projective and conformal mappings of complete Riemannian manifolds, using the L^q-Liouville theorems for subharmonic and superharmonic functions.

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INTRODUCTION

In mathematics, subharmonic and superharmonic functions are important classes of functions widely used in partial differential equations, complex analysis, and potential theory. A special place is occupied by the geometric theory of such functions on complete Riemannian manifolds. Since Huber's 1957 paper [19], mathematicians have been interested in the relationship between the geometry of a complete Riemannian manifold—primarily the curvature—and the global behavior of its subharmonic and superharmonic functions. In particular, an important question is what subspaces of subharmonic and superharmonic functions consist of constants. All such results are called *Liouville-type theorems* or, in other words, *L^q-Liouville theorems* and belong to the *generalized Bochner technique* (see, for example, [34]).

Here, we discuss this analytical method of differential geometry. The article is intended to geometers studying Riemannian geometry *in the large* (following [19], we omit the quotes) of manifolds and submanifolds: it will acquaint them with the generalized Bochner technique, which, in contrast to the classical Bochner technique ([33, pp. 333–364]; [42]; [43]), has not yet become widespread. In the first section, we give a brief survey of advances in the geometry of subharmonic and superharmonic functions on complete Riemannian manifolds. In the other three sections we prove some Liouville-type theorems for the Killing and Killing–Yano tensors on complete Riemannian manifolds, as well as Liouville-type theorems for projective and conformal mappings of complete Riemannian manifolds, using L^q-Liouville theorems for subharmonic and superharmonic functions. In conclusion, note that Liouville-type theorems on harmonic mappings of complete manifolds have been known for a long time (see [34, pp. 127–146]).

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1. LIOUVILLE-TYPE THEOREMS FOR SUBHARMONIC AND SUPERHARMONIC FUNCTIONS

The oldest result on applications subharmonic functions to differential geometry in the large is due to A. Huber [19], who proved that a complete two-dimensional Riemannian manifold with nonnegative curvature does not admit a non-constant negative subharmonic function. This research was continued by S. T. Yau [48], who was looking for conditions under which the *Poisson equation* $\Delta f = u$ for smooth functions f and $u \geq 0$ has only trivial solutions on a complete Riemannian manifold, and proved Liouville-type theorems on harmonic and subharmonic functions, which later became widely known. According to [48], the main reason to study the Poisson equation is an attempt to generalize the familiar Bochner's method to non-compact manifolds. As a result of the study of several mathematicians, a new direction of research in differential geometry appeared, which is now called *generalized Bochner technique* [34] for complete Riemannian manifolds, in contrast to the classical *Bochner technique* (see, for example, [18, 28, 29, 35, 42, 43] and [33, pp. 333–364]) for closed manifolds.

In this section, we briefly survey advances in geometry in the large of subharmonic and superharmonic functions on a complete Riemannian manifold. Let (M, g) be a closed (i.e., complete without boundary) n-dimensional Riemannian manifold with the Levi-Civita connection ∇ of metric g. Recall that *Laplace–Beltrami operator* Δ on any open subset $U \subset M$ is given in its local coordinates $x^1, \ldots, x^n$ by the formula $\Delta = 1/\sqrt{\det g}\, \partial_i(\sqrt{\det g}\; g^{ij}\partial_j)$, where $\partial_j = \partial/\partial x^j$, and the components of Riemannian metric tensor are denoted by $g_{ij} = g(\partial_i, \partial_j)$ and $(g^{ij}) = (g^{ij})^{-1}$, and summation here is assumed over pairs of repeated indices. On the other hand, for any smooth real-valued function f on an open subset $U \subset M$ with local coordinates $x^1, \ldots, x^n$, the gradient ∇f is a vector field of the form $(\nabla f)^i = g^{ij}\partial_j f$. Therefore, the action of the Laplace–Beltrami operator on a C^2-function f is defined by equalities

$$\Delta f = \operatorname{div}(\nabla f) = \operatorname{trace}_g \nabla df,$$

where ∇df is the Hessian of f.

The theory of *subharmonic functions* on Riemannian manifolds can be approached at various levels of generality. The equation of a subharmonic function $f \in C^2(M)$ has the well-known form $\Delta f \geq 0$ on an open subset $U \subset M$. Moreover, the equation of a harmonic function $f \in C^2(M)$ has the form $\Delta f = 0$. The classical *Liouville theorem* asserts that a subharmonic (or a harmonic) function defined over $\mathbb{R}^n$ with Euclidean metric and bounded above is constant. An important question: what subspaces of subharmonic and superharmonic functions on complete Riemannian manifolds contain only constant functions and, in particular, identically equal to zero.

Throughout the article, integrals are always computed with respect to the Riemannian measure (the volume form)

$$d\,vol_g = \sqrt{\det(g_{ij})}\; dx^1 \wedge \cdots \wedge dx^n.$$

Moreover, the L^q-norm of a function or a tensor, say, φ is denoted by $\int_M \|\varphi\|^q d\,vol_g$ being understood that it is referred to the whole manifold M, where to simplify the notation, we write $\|\varphi\| = \sqrt{g(\varphi, \varphi)}$.

The next most famous Liouville-type theorem for subharmonic functions on complete Riemannian manifolds belongs to S.T. Yau.

Theorem 1 [48]. *Let f be a nonnegative smooth subharmonic function on a complete Riemannian manifold (M, g). Then $\int_M f^q d\,vol_g = \infty$ for $q > 1$, unless f is a constant.*

The next wonderful theorem is a generalization of Theorem 1.

Theorem 2 [50]. *Let f be a smooth function on a complete Riemannian manifold (M, g) such that $f \geq 0$ and $(q-1) f\, \Delta f \geq 0$, where q is a positive number. Then for $q \neq 1$, either $\int_M f^q d\,vol_g = \infty$ or $f = \mathrm{const}$.*

We end our Yau's citation with the following statement.

Theorem 3 [48]. *If a subharmonic function f on a complete Riemannian manifold (M, g) satisfies $\int_M \|\nabla f\|\, d\,vol_g < \infty$, then f is harmonic.*

Remark 1. In contrast to Theorems 1 and 2, the Hopf maximum principle shows us the following (see [43, p. 30]): any subharmonic function on a compact Riemannian manifold is constant.

For a long time, mathematicians were interested in the relationship between the curvature of a complete Riemannian manifold and the existence of subharmonic functions on it (see, for example, [34] and [42]). Recall that a complete manifold with nonnegative Ricci curvature does not admit a non-constant positive harmonic function, [47]. As a generalization of these assertions, we formulate the following Liouville-type theorem, which we will quote several times in our paper.

Theorem 4 [25]. *a) A complete simply connected Riemannian manifold of nonpositive sectional curvature does not carry a non-constant nonnegative subharmonic L^q-function for $q > 0$. b) A complete Riemannian manifold of nonnegative Ricci curvature does not carry a non-constant nonnegative subharmonic L^q-function for $q > 0$.*

Corollary 1. *A complete non-compact Riemannian manifold of nonnegative Ricci curvature does not carry a non-zero nonnegative subharmonic L^q-function for $q > 0$.*

Proof. By Theorem 1, if f is a nonnegative subharmonic L^q-function for $q > 0$ on a complete Riemannian manifold, then f is constant. In this case, we have $Vol(M, g) < \infty$ for the volume of (M, g), because $f \in L^q(M)$ for $q > 0$. On the other hand, any non-compact complete Riemannian manifold of nonnegative Ricci curvature has an infinite volume, see [48, p. 667]. In this case, $f \equiv 0$. □

Recall that the Riemannian symmetric space is a Riemannian manifold (M, g), such that for each its point x there is an involutive geodesic symmetric s_x such that x is an isolated fixed point of s_x. Further, (M, g) is said to be *Riemannian locally symmetric* if its geodesic symmetries are in fact isometries. A Riemannian locally symmetric space is called a *Riemannian (globally) symmetric space* if, in addition, its geodesic symmetries are defined on all (M, g). A Riemannian symmetric space is complete (see [23, p. 244]). Moreover, a complete and simply connected Riemannian locally symmetric space is a Riemannian symmetric space (see [23, p. 244]). Riemannian symmetric spaces can be classified using their isometry groups. The classification distinguishes three basic types of such spaces: spaces of so-called *compact type*, spaces of so-called *non-compact type* and spaces of *Euclidean type* (see, e.g., [23, p. 252]). In turn, a Riemannian symmetric space of non-compact type has nonpositive sectional curvature. Therefore, we formulate another corollary of Theorem 1.

Corollary 2. *A Riemannian symmetric space of non-compact type (M, g) does not carry a non-constant nonnegative subharmonic L^q-function for any $q > 0$.*

Remark 2. Recall that a Riemannian symmetric space of compact type is a compact manifold. The Hopf maximum principle shows the following: any subharmonic function is constant on a Riemannian symmetric space of compact type.

Consider one application of the concept of a subharmonic function. Recall that a scalar function $f \in C^2(M)$ is *convex* if its Hessian ∇df is positive semidefinite at every point of M. In differential geometry the existence of convex functions on a (M, g) is a long standing problem. The first solution of this problem can be found in the article [5] of R.I. Bishop and B. O'Neill. They proved that if (M, g) is complete and has finite volume, then it does not possess a non-constant smooth convex function. Let us modify their result and state the following.

Corollary 3. *A complete Riemannian manifold does not possess any non-constant nonnegative smooth convex L^q-function for $q > 1$.*

Proof. If a scalar function $f \in C^2(M)$ is convex, then $\Delta f \geq 0$, i.e., f is a subharmonic function. In turn, we known from Theorem 1 that on any complete non-compact Riemannian manifold (M, g) there are no non-constant nonnegative L^q-subharmonic functions for any $q > 1$. In our case, this means that $f = \text{const}$. □

Remark 3. Many other results on subharmonic functions and their applications on complete Riemannian manifolds have been obtained by many authors (for example, R. Greene and H. H. Wu [15, 16], A. Huber [19], L. Karp [21, 22], S. T. Yau [45–49], L. O. Chung [11], S. Pigola at al. [34]; P. Li [26], etc).

In the second part of this section, we will look at superharmonic functions. In contrast to the Yau's problem [48] mentioned above, we can consider the *Poisson equation* $\Delta f = u$ for $u \leq 0$ on complete Riemannian manifolds. In accordance with this problem, recall that a function $f \in C^2M$ is called *superharmonic* if $\Delta f \leq 0$ on an open subset $U \subset M$ or, in particular, on (M, g). It is easy to see that a function f is superharmonic if $-f$ is subharmonic. Thus, the theory of superharmonic functions is dual to the theory of superharmonic functions in many of its aspects.

Nonetheless, the theory of superharmonic functions has its own peculiarities, which we consider below. The Hopf maximum principle shows us the following (see [43, p. 30] and also Remark 1): any superharmonic function on a compact Riemannian manifold is constant. The following statement belongs to S. R. Adams.

Theorem 5 [2]. *A complete Riemannian manifold of finite volume does not carry non-constant positive superharmonic functions.*

Let us formulate and prove several assertions about superharmonic functions that follow from the corresponding propositions of S. T. Yau [48, 50].

Corollary 4. *Let f be a nonnegative superharmonic function on a complete Riemannian manifold (M, g), then either $\int_M f^q d\,vol_g = \infty$ for $0 < q < 1$ or $f = \text{const}$.*

Proof. Due to Theorem 2, we can assume that q may even be less than one in this statement (see also [48, p. 664]). In this case, the inequalities $0 < q < 1$, $f \geq 0$ and $\Delta f \leq 0$ satisfy the conditions of the theorem. In particular, the inequality $(q-1)\, f \Delta f \geq 0$ is valid. By Theorem 2, we conclude that either $\int_M f^q d\,vol_g = \infty$ or $f = \text{const}$. □

Secondary, we get the following corollary, which is a modification of Theorem 3.

Corollary 5. *Let f be a superharmonic function on a complete non-compact, oriented Riemannian manifold such that $\|\nabla f\| \in L^1(M)$, then f is a harmonic function.*

Proof. If X is a smooth vector field on a complete and oriented Riemannian manifold (M, g) such that $\|X\| \in L^1(M)$ and $\operatorname{div} X \leq 0$ (or $\operatorname{div} X \geq 0$), then $\operatorname{div} X = 0$ (see [6]; [7]). On the other hand, if $X = \nabla f$ for $f \in C^2(M)$, then $\operatorname{div} X = \Delta f$; hence, the condition $\operatorname{div} X \leq 0$ can be rewritten the form $\Delta f \leq 0$. In this case, f is a superharmonic function. Then from the condition $\|\nabla f\| \in L^1(M)$ we conclude that $\Delta f = 0$, i.e., f is a harmonic function. □

It is known that any harmonic function $f \in L^q(M)$ for some $0 < q < \infty$ on a complete Riemannian manifold of nonnegative Ricci curvature is constant (see [25]). At the same time, any complete non-compact Riemannian manifold of nonnegative Ricci curvature has an infinite volume (see [48]). Thus, any harmonic function $f \in L^q(M)$ for some $0 < q < \infty$ on a complete non-compact Riemannian manifold of nonnegative Ricci curvature is zero. Using Corollary 5, we can state the following L^q-Liouville statement.

Corollary 6. *Let f be a superharmonic L^q-function for some $0 < q < \infty$ defined on a complete non-compact, oriented Riemannian manifold (M, g) of nonnegative Ricci curvature. Moreover, if $\|\nabla f\| \in L^1(M)$, then $f \equiv 0$.*

2. PROJECTIVE MAPPINGS OF COMPLETE RIEMANNIAN MANIFOLDS

The theory of projective (or geodesic) diffeomorphisms and pointwise projectively equivalent metrics has a very long and fascinating history, presented in more detail in a voluminous monograph [30].

Let (M, g) and $(\bar{M}, \bar{g})$ be smooth Riemannian manifolds of the same dimension $n \geq 2$ with the Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. A diffeomorphism $F: (M, g) \to (\bar{M}, \bar{g})$ is called *projective* if it maps the unparameterized geodesic of the metric g to a geodesic of the metric $\bar{g}$. A diffeomorphism $F\colon M \to \bar{M}$ is called *affine* if it maps ∇ to $\bar{\nabla}$. In this case, the metric tensor $\bar{g}$ is parallel (with respect to ∇,), i.e., $\nabla\bar{g} = 0$. The converse is also true.

If $U \subset M$ and $\bar{U} = f(U) \subset \bar{M}$ are connected open subsets with local coordinates $(x^1, \ldots, x^n)$ and $(\bar{x}^1, \ldots, \bar{x}^n)$, then the diffeomorphism $F\colon (M, g) \to (\bar{M}, \bar{g})$ can be defined by the equations $\bar{x}^1 = x^1, \ldots, \bar{x}^n = x^n$. In this case, the tangent map $F_{*x}\colon T_x M \to T_{f(x)} \bar{M}$ is represented by a unit matrix in local coordinates at any point $x \in U$. Moreover, in a common coordinate system $x^1, \ldots, x^n$ with respect to the projective diffeomorphism $F: (M, g) \to (\bar{M}, \bar{g})$ the Levi-Civita connections ∇ and $\bar{\nabla}$ are related by the equation (see [13, p. 132], [30, p. 292])

$$\bar{\nabla} = \nabla + \mathrm{Id}_{TM} \otimes d\psi + d\psi \otimes \mathrm{Id}_{TM} \tag{1}$$

for the scalar function (see [13, p. 132], [30, p. 292])

$$\psi = \frac{1}{n+1} \ln(d\,vol_g / d\,vol_{\bar{g}}),$$

where

$$d\,vol_g = \sqrt{\det(g_{ij})}\; dx^1 \wedge \cdots \wedge dx^n \quad \text{and} \quad d\,vol_{\bar{g}} = \sqrt{\det(\bar{g}_{ij})}\; dx^1 \wedge \cdots \wedge dx^n$$

are Riemannian measures of (M, g) and $(\bar{M}, \bar{g})$, respectively. Further, a projective diffeomorphism $F: (M, g) \to (\bar{M}, \bar{g})$ is called trivial if $d\psi \equiv 0$, i.e., F is affine.

Theorem 6. *Let $F: (M, g) \to (\bar{M}, \bar{g})$ be a projective diffeomorphism between Riemannian manifolds, then*

$$f = (d\,vol_{\bar{g}} / d\,vol_g)^{\frac{1}{n+1}} \tag{2}$$

is an eigenfunction of the Laplace–Beltrami operator $\Delta f = \lambda_f\, f$ corresponding to the eigenvalue

$$\lambda_f = (n-1)^{-1}(s - \mathrm{trace}_g\, \overline{Ric}), \tag{3}$$

which is generally non constant.

Proof. Using equation (1), we obtain (see [13, p. 135], [30, p. 299])

$$\overline{Ric} = Ric + (n-1)\,(\nabla, d\psi - d\psi \otimes d\psi) \tag{4}$$

for the Ricci tensors Ric and $\overline{Ric}$ of (M, g) and $(\bar{M}, \bar{g})$, respectively. Next, we define the scalar function f by the equality $\psi = -\ln f$, then (2) is valid. In this case, (4) can be written in the following form

$$\nabla_i \nabla_j\, f = \frac{1}{n-1}\,(Ric - \overline{Ric})\, f. \tag{5}$$

From (5) we obtain the following equation

$$\Delta f = \frac{1}{n-1}\,(s - \mathrm{trace}_g\, \overline{Ric})\, f, \tag{6}$$

where f is an eigenfunction of the Laplace–Beltrami operator corresponding to the eigenvalue (3), which is generally not constant. □

Remark 4. On the application of the classical Bochner technique to the theory of geodesic mappings, V. S. Matveev wrote the following (see [27, p. 405]): In the theory of geodesically equivalent metrics, the standard way to use the Bochner technique is to apply tensor calculus to canonically obtained nonconstant function f such as $\Delta f = \mathrm{const} f$, where const$\geq 0$, which cannot exist on a closed Riemannian manifold. The first application of this technique in the theory of geodesically equivalent metrics is associated with the geometric school of Yano, Tanno, and Obata.

From (6) we conclude that if $F: (M,g) \to (\bar{M},\bar{g})$is a non-trivial projective diffeomorphism $Ric \geq \overline{Ric}$ at each point $x \in M$, then f is a non-constant positive convex function. Recall the following result in [46]: If (M,g) is a complete Riemannian manifold, on which there exists a non-constant convex function, then the volume of (M,g) is infinite. Thus, the following theorem holds.

Theorem 7. *Let $F: (M,g) \to (\bar{M},\bar{g})$ be a non-trivial projective diffeomorphism between Riemannian manifolds such that $Ric \geq \overline{Ric}$ for the Ricci tensors of (M,g) and $(\bar{M},\bar{g})$, respectively. Then (M,g) has an infinite volume.*

Next, we consider the conditions under which the Poisson-type equation (6) has only trivial solutions, that is the function f from (2) is a positive constant. From (5) we conclude that if

$$s \leq \mathrm{trace}_g\, \overline{Ric}, \tag{7}$$

then the function (2) is superharmonic. In this case, using Theorem 5 and Corollary 4, we obtain the following Liouville-type theorem.

Theorem 8. *Let $F: (M,g) \to (\bar{M},\bar{g})$ be a projective diffeomorphism between Riemannian manifolds such that the scalar curvature s of g and the Ricci tensor $\overline{Ric}$ of $\bar{g}$ satisfy the inequality (7). Then f given in (2) is a positive superharmonic function. Therefore, if (M,g) is a complete manifold and one of the following statements holds: (M,g) has finite volume or $f \in L^q(M)$ for some $0 < q < 1$, then F is an affine mapping. Moreover, if (M,g) is irreducible, then F is a homothetic mapping.*

Remark 5. Recall that a Riemannian manifold (M,g) is *irreducible* if it cannot be represented as a nontrivial Riemannian product. It is irreducible if and only if its holonomy group $\mathrm{Hol}(g_x)$ acts irreducibly on T_x for any $x \in M$. In this case, if $\varphi \in C^\infty(S^2M)$ is parallel (with respect to the Levi-Civita connection of g), then $\varphi = C\,g$ for some $C \in \mathbb{R}$.

On the other hand, from (5) we conclude that if

$$s \geq \mathrm{trace}_g\, \overline{Ric}, \tag{8}$$

then the function from (2) is subharmonic. Therefore, by Theorem 1, if (M,g) is complete and $f \in L^q(M)$ for some $q > 1$, then $f = \mathrm{const}$ and, hence, $\psi = \mathrm{const}$. In this case, the projective diffeomorphism $F: (M,g) \to (\bar{M},\bar{g})$ is an affine mapping, because in this case $\bar{\nabla} = \nabla$, see (1). In turn, if the holonomy representation of $\mathrm{Hol}(g)$ is irreducible, then the metric tensor $\bar{g}$ is proportional to g; hence, F is a homothetic mapping. We proved the following L^q-Liouville theorem.

Theorem 9. *Let $F: (M,g) \to (\bar{M},\bar{g})$ be a projective diffeomorphism between Riemannian manifolds such that the scalar curvature s of g and the Ricci tensor $\overline{Ric}$ of $\bar{g}$ satisfy the inequality (8). Then f given in (2) is a positive subharmonic function. Therefore, if (M,g) is a complete manifold and $f \in L^q(M)$ for some $q > 1$, then F is an affine mapping. Moreover, if (M,g) is irreducible, then F is a homothetic mapping.*

Theorems 8 and 9 generalize a similar result [9, Theorem 1.3], which was proved for compact Riemannian manifolds by the classical Bochner method.

3. CONFORMAL MAPPINGS OF COMPLETE RIEMANNIAN MANIFOLDS

Conformal mapping is one of the important concepts of Riemannian geometry. Conformal mappings of Riemannian (or, pseudo-Riemannian) manifolds have been investigated by many authors (see the reviews in [30], [24] and [37]). In general relativity, conformal mappings are important, because they preserve causal structure up to time orientation and light-like geodesics up to parametrization. Liouville's theorem made it clear back in the 19th century that conformal mappings are more rigid in dimensions $n \geq 3$ than in dimension 2. Moreover, more than forty five years ago, Yau [45] showed the effectiveness of the Liouville-type theorems in the theory of conformal mappings of an n-dimensional ($n \geq 3$) complete

Riemannian manifold. In turn, conformal mappings of complete Riemannian manifolds were studied using the generalized Bochner technique in [34].

A diffeomorphism $F: (M, g) \to (\bar{M}, \bar{g})$ between n-dimensional ($n \geq 3$) Riemannian manifolds is called *conformal* if their metric tensors in a common coordinate system $x^1, \ldots, x^n$ with respect to the diffeomorphism F are related by the equation $\bar{g} = e^{2\sigma} g$ for some smooth scalar function σ (see [13, p. 89], [4, p. 58]).

Theorem 10. *Let $F: (M, g) \to (\bar{M}, \bar{g})$ be a conformal diffeomorphism between Riemannian manifolds such that $\bar{g} = e^{2\sigma} g$. Then*

$$f = e^{(n-2)\sigma/2} \tag{9}$$

is an eigenfunction of the Laplace–Beltrami operator $\Delta f = \lambda_f f$ corresponding to the eigenvalue

$$\lambda_f = \frac{n-2}{4(n-1)} \left(s - e^{2\sigma} \bar{s}\right), \tag{10}$$

which is generally not constant.

Proof. From the definition of conformal diffeomorphisms, we obtain (see, for example, [4, p. 59]; [30, p. 271]) the following equation

$$\Delta\sigma = \frac{1}{2(n-1)} \left(s - e^{2\sigma} \bar{s}\right) - \frac{n-2}{2} \|\nabla\sigma\|^2 \tag{11}$$

for the scalar curvature $\bar{s} = \mathrm{trace}_{\bar{g}}\, \overline{Ric}$ of $\bar{g}$. Next, we define the scalar function f by the equality $\sigma = 2/(n-2) \ln f$, then (9) is valid. In this case, (11) can be written in the form

$$\Delta f = \frac{n-2}{4(n-1)} \left(s - e^{2\sigma} \bar{s}\right) f, \tag{12}$$

where f is an eigenfunction of Laplace–Beltrami operator Δ corresponding to the eigenvalue λ_f, see (10), which generally is not constant. □

We are looking for conditions under which the *Poisson-type equation* (12) has only trivial solutions. This means that the function f, given in (9), is a positive constant on (M, g). In this case, the conformal diffeomorphism $F: (M, g) \to (\bar{M}, \bar{g})$ is a homothetic mapping. From (11) we conclude that if $s \leq e^{2\sigma} \bar{s}$, then f is a positive superharmonic function. This fact allows us to state the following.

Theorem 11. *Let $F: (M, g) \to (\bar{M}, \bar{g})$ be a conformal diffeomorphism between Riemannian manifolds such that $\bar{g} = e^{2\sigma} g$ and the scalar curvatures of g and $\bar{g}$ satisfy the inequality $s \leq e^{2\sigma} \bar{s}$, then (9) defines a superharmonic function. If (M, g) is a complete non-compact manifold with one of the following conditions: (M, g) has finite volume or $f \in L^q(M)$ for some $0 < q < 1$, then F is a homothetic mapping.*

On the other hand, from (12) we find that if $e^{2\sigma} \bar{s} \leq s$, then the function from (12) is positive subharmonic. In this case, we state another L^q-Liouville theorem.

Theorem 12. *Let F be a conformal diffeomorphism of a complete Riemannian manifold (M, g) onto a Riemannian manifold $(\bar{M}, \bar{g})$ such that $\bar{g} = e^{2\sigma} g$ and scalar curvatures of (M, g) and $(\bar{M}, \bar{g})$ satisfy the inequality $e^{2\sigma} \bar{s} \leq s$. If $f \in L^q(M)$ for some $q > 1$ and f given in (9), then F is a homothetic mapping.*

4. KILLING TENSORS ON COMPLETE RIEMANNIAN MANIFOLDS

Symmetric and skew-symmetric Killing p-tensors (or, in other words, Killing p-forms, see [1]), introduced into physics by Penrose et al. in the early 1970s (see [20, 32, 41]), play an important role in the development of mathematical physics and differential geometry. For example, such tensors find a wide range of applications in classical and quantum physics [8], in studying various integrability properties of black holes (see [14, 44]), naturally arise in relativistic electrodynamics [31] and in the string theory [10]. We present here some applications of the results on subharmonic functions to the geometry of symmetric and skew-symmetric Killing p-tensors on complete Riemannian manifolds.

A symmetric Killing p-tensor is defined by the equation $\delta^*\varphi = 0$, where $\delta^* : C^\infty(S^pM) \to C^\infty(S^{p+1}M)$ is the differential operator of degree 1, defined on the bundle of symmetric p-tensors S^p by the formula $\delta^*\varphi = (n+1)^{-1}\mathrm{Sym}(\nabla\varphi)$ for the standard symmetry operator Sym: $T^*M \otimes S^pM \to S^{p+1}M$ and any function $\varphi \in C^\infty(S^pM)$. Next, we shall consider a traceless symmetric Killing p-tensor φ on (M, g), determined by conditions $\delta^*\varphi = 0$ and $\delta\varphi = 0$ (see [38]). In this case, set $\|\varphi\| = \sqrt{g(\varphi,\varphi)}$. Then we get the classical *Bochner–Weitzenböck formula* for any tensor $\varphi \in C^\infty(S^pM)$ (see also [38])

$$\frac{1}{2}\Delta\|\varphi\|^2 = \|\nabla\varphi\|^2 - g(\Re_p(\varphi), \varphi), \tag{13}$$

where $\Re_p$: $S^pM \to S^pM$ is an algebraic symmetric operator that depends linearly in a known way on the curvature tensor R and the Ricci tensor Ric of the metric g. On the other hand, a direct calculation yields the second *inequality of Kato* [3, p. 380] $\|d\,\|T\|\,\| \le \|\nabla T\|$, which can be rewritten in the form

$$\frac{1}{2}\,\Delta\|T\|^2 \le \|T\|\,\Delta|T\| + \|\nabla T\|^2 \tag{14}$$

for any tensor field $T \in C^\infty(\otimes^p T^*M)$ and, in particular, for any $\varphi \in C^\infty(S^pM)$. From (13) and (14) we obtain the inequality $-g(\Re_p(\varphi), \varphi) \le \|\varphi\|\,\Delta|\varphi\|$. We rewrite it in the following form

$$f\,\Delta\,f \ge -g(\Re_p(\varphi), \varphi) \tag{15}$$

for $f = \|\varphi\|$, which allows us to use Theorem 2.

It is possible to prove that non-positive sectional curvature implies $g(\Re_p(\varphi), \varphi) \le 0$ for any $\varphi \in C^\infty(S^pM)$ and $p > 1$. The proof of this fact follows from [17]. Using this condition on sectional curvature of (M, g), we conclude from (15) that $f\,\Delta\,f \ge 0$. Then, based on Theorem 2, from (13) and (15) we obtain $\nabla\varphi = 0$ if $f = \|\varphi\|$ is a L^q-function for some $q > 1$ on a complete Riemannian manifold (M, g). In particular, if $p = 2$ and (M, g) is irreducible, then $\varphi \equiv 0$. As a result, we have the following L^q-Liouville theorem.

Theorem 13. *Let φ be a symmetric, traceless Killing p-tensor φ on a complete Riemannian manifold of nonpositive sectional curvature (M, g). If $\|\varphi\| \in L^q(M)$ for some $q > 1$, then φ is parallel. In particular, if $p = 2$ and (M, g) is irreducible, then $\varphi \equiv 0$.*

If φ has no zeros on M then we can use Theorem 5, because in this case the inequality $\|\varphi\|\,\Delta\|\varphi\| \ge 0$ is equivalent to the inequality $\Delta\|\varphi\| \ge 0$. Thus, we state the following.

Theorem 14. *Let φ be a symmetric, traceless Killing p-tensor on a complete Riemannian manifold (M, g) of nonpositive sectional curvature. If φ has no zeros on M and $\|\varphi\| \in L^q(M, g)$ for some $q > 0$, then φ is parallel. Moreover, if $p = 2$ then (M, g) is not irreducible.*

A *Cartan–Hadamard manifold* is an n-dimensional Riemannian manifold (M, g) that is complete simply connected and has non-positive sectional curvature (see, for example, [26]). Recall that by *Cartan–Hadamard theorem* all Cartan–Hadamard manifolds are diffeomorphic to the Euclidean space $\mathbb{R}^n$. Globally symmetric spaces of non-compact type are examples of Cartan–Hadamard manifolds. In the case of Cartan–Hadamard manifolds, the following holds.

Corollary 7. *Let φ be a symmetric, traceless Killing p-tensor φ on a Cartan–Hadamard manifold (M, g) and, in particular, on a Riemannian symmetric spaces of non-compact type, then the scalar function $\|\varphi\|$ is subharmonic. If additionally $\|\varphi\| \in L^q(M)$ for some $q > 1$, then φ is parallel. In particular, if $p = 2$ and (M, g) is irreducible, then $\varphi \equiv 0$.*

Remark 6. If the rank r of a symmetric space is greater than 2, then it contains a totally geodesic r-dimensional flat submanifold. So its sectional curvature attains zero at some tangent plane. Therefore, one can formulate a consequence from Corollary 7 for any symmetric, traceless Killing p-tensor φ on a Riemannian symmetric space of non-compact type of rank r.

In the second part of this section, we consider Killing p-tensors (or, Killing–Yano p-tensors) on n-dimensional Riemannian manifolds for $1 < p < n$. By definition (see [39, p. 559]), a *Killing–Yano p-tensor* ω is a smooth section of the bundle of skew-symmetric p-tensors $\Lambda^p M$ on (M, g) such that $\nabla\omega \in C^\infty(\Lambda^{p+1}M)$.

Denote by Δ: $C^\infty(\Lambda^p M) \to C^\infty(\Lambda^p M)$ the well-known Hodge–de Rham Laplacian. Then for any $\omega \in C^\infty(\Lambda^p M)$ we have (see, for example, [40])

$$\Delta\omega = \bar{\Delta}\omega + \Re_p(\omega). \tag{16}$$

In this case, we consider the scalar function $\|\omega\| = \sqrt{g(\omega,\omega)}$. Then we can write the well-known *Bochner–Weitzenböck formula* for the tensor $\omega \in C^\infty(\Lambda^p M)$ (see also [40])

$$\frac{1}{2}\Delta\|\omega\|^2 = -g(\bar{\Delta}\omega,\omega) + \|\nabla\omega\|^2 + g(\Re_p(\omega),\omega).$$

By virtue of the Kato inequality (14), from (16) we obtain

$$\|\omega\|\,\Delta\|\omega\| \geq g(\bar{\Delta}\omega,\omega) - g(\Re_p(\omega),\omega). \tag{17}$$

In turn, for any Killing–Yano tensor $\omega \in C^\infty(\Lambda^p M)$ we have (see also [40])

$$\Delta\omega = \frac{p+1}{p}\,\Re_p(\omega).$$

In this case, from (16) and (17) we obtain the inequality

$$\|\omega\|\,\Delta, \|\omega\| \geq -\frac{p-1}{p}\,g(\Re_p(\omega),\omega),$$

which we rewrite in the following form

$$f\,\Delta\,f \;\geq -\frac{p-1}{p}\,g(\Re_p(\omega),\omega), \tag{18}$$

where $f = \sqrt{g(\omega,\omega)}$. The inequality (18) allows us to use Theorem 2.

Recall that the Riemannian curvature tensor R of (M, g) defines the *curvature operator* $\bar{R}$, that is a symmetric algebraic operator $\bar{R}$: $\Lambda^2(T_x\,M) \to \Lambda^2(T_x\,M)$ on the vector space $\Lambda^2(T_x M)$ of 2-forms over tangent space $T_x\,M$ at every point $x \in M$ (see [33, pp. 82–83]). We say that a Riemannian manifold (M, g) has a nonpositive curvature operator $\bar{R}$ if $g(\bar{R}(\theta),\theta) \leq 0$ for all nonzero two-forms $\theta \in \Lambda^2(TM)$.

Thus, if ω is a nonzero Killing–Yano tensor on a Riemannian manifold with nonnegative curvature operator, then from (16)) we obtain $f\,\Delta\,f \geq 0$ for $f = \|\omega\| \geq 0$. In this case, based on Theorem 2, we conclude that $\nabla\omega = 0$ if $f = \|\omega\|$ is a L^q-function on a complete Riemannian manifold (M, g) for some $q > 1$. Thus, the following L^q-Liouville theorem holds.

Theorem 15. *Let ω be a Killing–Yano p-tensor on a complete Riemannian manifold (M, g) with nonpositive curvature operator. If $\|\omega\| \in L^q(M, g)$ for some $q > 1$, then ω is parallel. Moreover, if the volume of (M, g) is infinite, then $\omega = 0$ on M.*

If ω has no zeros on M, then we can use Theorem 5, because in this case the inequality $\|\omega\|\Delta\,\|\omega\| \geq 0$ is equivalent to the following inequality $\Delta\,\|\omega\| \geq 0$. Therefore, the following theorem holds.

Theorem 16. *Let ω be a Killing–Yano p-tensor on a complete Riemannian manifold (M, g) with nonpositive curvature operator. If ω has no zeros on M and $\|\omega\| \in L^q(M, g)$ for some $q > 0$, then ω is parallel. Moreover,* $\mathrm{Vol}(M, g) < \infty$.

We will remind once again that a Riemannian symmetric space of non-compact type has nonpositive sectional curvature. Moreover, a Riemannian symmetric space of non-compact type is a complete Riemannian manifold. Then, based on Corollary 1, we state another corollary.

Corollary 8. *Let (M, g) be a Riemannian symmetric space of non-compact type. Then the function $\|\omega\|$ is subharmonic for any Killing–Yano p-tensor ω on (M, g). If $\|\omega\|$ is a L^q-function for some $q > 1$, then ω is parallel. Moreover, if the volume of (M, g) is infinite, then $\omega = 0$ on M.*

Remark 7. One of the authors of this article classified connected complete, irreducible Riemannian manifolds with nonpositive curvature operator, which admit nonzero Killing–Yano L^2-tensors [36].

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