ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 3, pp. 642–652. © Pleiades Publishing, Ltd., 2022.

On the Solvability of Boundary Value Problems for the Inhomogeneous Schrödinger Equation on Model Riemannian Manifolds

A. Losev[1*] and E. Mazepa[1]**

(Submitted by F. G. Avkhadiev)

[1] *Institute of Mathematics and Information Technologies, Volgograd State University, Volgograd, 400062 Russia*

Received December 29, 2021; revised January 19, 2022; accepted January 26, 2022

Abstract—The paper investigates the asymptotic behavior of solutions of the inhomogeneous stationary Schrödinger equation on non-compact model Riemannian manifolds. Exact conditions for the unique solvability of the Dirichlet problem for the equation with continuous boundary data at «infinity» and some other boundary value problems are found.

DOI: 10.1134/S1995080222060208

Keywords and phrases: *Dirichlet problem, nonhomogeneous elliptic equations, model Riemannian manifold, asymptotic behavior of the solutions.*

1. INTRODUCTION

In recent years, the field of mathematics, which is involved in the study of the properties of solutions to various partial differential equations on spaces in connection with the geometry of the underlying space, become increasingly popular. In particular, solutions of the equations on non-compact Riemannian manifolds, on graphs, on fractals, etc. are studied. In this case most of the research is devoted to finding conditions for the fulfillment of Liouville-type theorem which states the triviality of solutions of the equations from some given classes of functions. A much smaller number of papers are devoted to the study of the solvability of boundary value problems with boundary conditions at infinity. The latter is due to the fact that even the formulation of a boundary value problem on an arbitrary non-compact Riemannian manifold may cause difficulties.

One of the origin of this topic is the classification theory of two-dimensional non-compact Riemannian surfaces. A distinctive property of two-dimensional surfaces of parabolic type is the fulfillment of the Liouville theorem for them, which states that every positive superharmonic function on the surface is an identical constant (see, for example, [1], [2]).

Exactly this property served as a basis for the extension of the concept of parabolicity for arbitrary Riemannian manifolds. Namely, manifolds on which any lower bounded superharmonic function is constant are called *parabolic manifolds* (see [3]).

One of the first results in determining the type of a Riemannian manifold using geometric characteristics is the theorem of S. Y. Cheng and S. T. Yau [4], which states that a complete manifold is parabolic if the volume of a geodesic ball of radius R grows no faster than R^2 at $R \to \infty$. However, there are manifolds of parabolic type with an arbitrary increase in the volume of the geodesic ball.

In [5], A. Grigor'yan proved that the parabolic type of a complete Riemannian manifold M is equivalent to the fact that the variational capacity of any compact in M is zero. The search for signs

*E-mail: alexander.losev@volsu.ru
**E-mail: elena.mazepa@volsu.ru

of parabolic type of a Riemannian manifold M has a long history. A general idea of modern research on this topic can be obtained, for example, from the paper of A. Grigor'yan [3].

Problems of the existence of nontrivial harmonic and superharmonic functions naturally lead to Liouville-type theorems. The classical formulation of Liouville's theorem states that every bounded harmonic function in Euclidean space is constant identically. At present, the following approach to the problems is applied.

Let A be a class of functions and L be an elliptic operator on the Riemannian manifold M. We say that the (A, L)-Liouville property is satisfied on M if any solution $u \in A$ of the equation $Lu = 0$ is constant identically (or trivial). If A is a linear space and L is a linear operator there are studies devoted to the estimation of the dimensions of spaces of the solutions $u \in A$ for the equation $Lu = 0$.

However, the class of Riemannian manifolds containing fairly ample sets of bounded harmonic functions is quite extensive. In this regard, the solvability of the Dirichlet problem is of serious interest. Note that even the formulation of boundary-value problems for elliptic differential equations (in particular, the Dirichlet problem) on non-compact Riemannian manifolds can be problematic, since it is unclear how we interpret the boundary data. One possible way to solve this problem is to use the "Martin boundary" (see, for example, [6]), or to state boundary value problems using classes of equivalent functions (see, for example, [7]).

In some cases, the geometric compactification of a manifold allows one to do this in a classical formulation. One of the classes of Riemannian manifolds, which has natural geometric compactification, is manifolds with negative sectional curvature. For example, in [8] and [9] it is shown that for a simply connected Riemannian manifold with negative sectional curvature sect M satisfying conditions

$$-b^2 \leq \mathrm{sect} M \leq -a^2 < 0,$$

there exists a geometric compactification that adds the sphere at infinity $S(\infty)$, and it is proved that the Dirichlet problem with continuous boundary data on $S(\infty)$ is solvable on $\overline{M} = M \cup S(\infty)$. Another class of manifolds admitting traditional statement of the Dirichlet problem is the class of model Riemannian manifolds. A number of papers have been published in recent years, devoted to the study of the behaviour of solutions of various elliptic equations on such manifolds and some of their generalizations (see, for example, [10–14]). These manifolds will be described in more detail below.

Now, we note that most of the papers on this topic are devoted to solvability of various boundary-value problems for harmonic functions, for solutions of stationary Schrödinger equation

$$Lu \equiv \Delta u - c(x)u = 0, \qquad \text{where} \qquad c(x) \geq 0 \tag{1}$$

and some other homogeneous linear and quasilinear elliptic equations. But studies of inhomogeneous elliptic equations are of a single nature (see, for example, [15–18] and are mainly devoted to the study of the asymptotic behavior of the solutions of these equations and not to the solvability of boundary-value problems.

In this paper, we consider non-compact Riemannian manifolds, that are representable in the form $M_g = B \cup D$, where B is some precompact with the non-empty interior, and D is isometric to the Cartesian product $[r_0; +\infty) \times S$ ($r_0 > 0$ and S is a closed Riemannian manifold (for example, a sphere)) with the metric

$$ds^2 = dr^2 + g^2(r)\, d\theta^2. \tag{2}$$

Here $g(r)$ is an arbitrary positive, smooth function on $[r_0; +\infty)$ and $d\theta^2$ is Riemannian metric on S. The manifolds M_g are usually called model manifolds or manifolds with metric horns. Examples of such manifolds are Euclidean space R^n, hyperbolic space H^n, surfaces of revolution, and others.

Theorem 1. *Manifold M_g is parabolic if and only if*

$$K = \int_{r_0}^{\infty} g^{1-n}(t)dt = \infty. \tag{3}$$

This assertion has been proved in various versions by a number of authors (see, for example, [1], [2], [5], [11]).

Everywhere else, we assume that the condition $0 \leq c(r,\theta) \equiv c(r)$ is fulfilled on D, and introduce the following notation

$$I = \int\limits_{r_0}^{\infty} g^{1-n}(t) \left(\int\limits_{r_0}^{t} g^{n-3}(\beta) d\beta \right) dt + J,$$

where

$$J = \int\limits_{r_0}^{\infty} g^{1-n}(t) \left(\int\limits_{r_0}^{t} c(\beta) g^{n-1}(\beta)\, d\beta \right) dt.$$

and $r_0 = \mathrm{const} > 0$, $n = \dim M_g$.

It is easy to verify (see also [12]) that exactly one of the conditions is satisfied on D

$$\alpha)\ I < \infty; \quad \beta)\ I = \infty,\ J < \infty; \quad \gamma)\ K = \infty; \quad \delta)\ J = \infty,\ K < \infty.$$

Theorem 2. *The following statements are true for M_g.*

1) Let the condition α) be fulfilled on the Riemannian manifold M_g, i.e. $I < \infty$. Then for any function $\Phi(\theta) \in C(S)$ on M there exists a unique bounded solution of the Schrödinger equation (1) *such that $\lim\limits_{r\to\infty} u(r,\theta) = \Phi(\theta)$.*

2) Let the condition α) be fulfilled on M_g, i.e. $I < \infty$. Then for any functions $\Phi_1(\theta) \in C(S)$ and $\Phi_2(\theta) \in C(S)$ on D there exists a unique bounded solution of the equation (1) *such that*

$$u(r_0,\theta) = \Phi_1(\theta) \qquad \text{and} \qquad \lim_{r\to\infty} u(r,\theta) = \Phi_2(\theta).$$

3) Let the condition β) be fulfilled on M_g, i.e. $I = \infty$, $J < \infty$. Then on M_g there exists nontrivial bounded solution $u(x)$ of the equation (1) *such that there is a finite limit $\lim_{r\to\infty} u(r,\theta)$ that does not depend on θ.*

4) Let the condition β) be fulfilled on M_g, i.e. $I = \infty$, $J < \infty$. Then for any function $\Phi(\theta) \in C(S)$ and constant C on D there exists a unique bounded solution of the equation (1) *such that*

$$u(r_0,\theta) = \Phi(\theta) \quad \text{and} \quad \lim_{r\to\infty} u(r,\theta) = C.$$

5) Let at least one of the conditions γ) or δ) be fulfilled on M_g, i.e. $K = \infty$ or $J = \infty$. Then every bounded solution of the equation (1) *on M is identically equal to zero.*

6) Let the condition γ) be fulfilled on M_g, i.e. $K = \infty$, and additionally

$$\int\limits_{r_0}^{\infty} c(t) g^{n-1}(t) dt = \infty,$$

or the condition δ) be fulfilled on M_g, i.e. $K < \infty$, $J = \infty$. Then there exists nontrivial bounded solution $u(x)$ on D of the equation (1) *such that $u(r_0,\theta) = \Psi(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = 0$.*

Remark. All statements of Theorem 2 were proved earlier in [11, 12]. If $c(x) \equiv 0$, then the assertions 3), 5) and 6) of the Theorem 2 are replaced by the following statement proved earlier in [11]: *If M_g is such that $I = \infty$, then every bounded harmonic function on M_g is constant identically.*

When proving the statements, we used a specific form of the Laplace–Beltrami operator on D in the coordinates (r,θ) (see, for example, [11, 12]):

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\frac{g'(r)}{g(r)}\frac{\partial}{\partial r} + \frac{1}{g^2(r)}\Delta_\theta,$$

where Δ_θ is the interior Laplacian on S.

In this paper we study the asymptotic behavior of bounded solutions $u \in C^2(M_g)$ of the inhomogeneous Schrödinger equation

$$Lu \equiv \Delta u - c(x)u = f(x), \tag{4}$$

where $c(x) \geq 0$ on M_g, $c(x),\ f(x) \in C^\alpha(G)$ for any precompact subset $G \subset M_g$, $0 < \alpha < 1$, $f(x) \not\equiv 0$.

Earlier in [19], the behavior of solutions of the Poisson equation $\Delta u = f(x)$, where f is a sufficiently smooth function, on model manifolds was investigated. Let us describe the result of this paper in more detail. Let

$$G^p(M_g) = \{\phi \in C(M_g) : t \in [r_0, \infty), \phi(t,\theta) \in C^p(S)\}$$

be the subset of the space of continuous functions on M_g formed by the functions that are p times continuously differentiable with respect to the second argument on D. In [19] it is assumed that $f \in G^{[\frac{3n}{2}]}(M_g)$ and denoted

$$\varphi_0(r) = ||f(r,\theta)||_{L^1(S)}, \qquad \varphi_m(r) = ||\Delta_\theta^m f(r,\theta)||_{L^2(S)},$$

where $m = [\frac{3n}{4}]$. Further, let

$$I_\varphi = \int\limits_{r_0}^{\infty} g^{1-n}(t) \left(\int\limits_{r_0}^{t} \left(\frac{1}{g^2(\xi)} + \varphi_0(\xi) + \varphi_m(\xi) \right) g^{n-1}(\beta)\, d\beta \right) dt.$$

In [19], sharp conditions for the unique solvability of the Dirichlet problem for the Poisson equation with continuous boundary data at "infinity" are found. Namely, the following theorem is proved: *if the Riemannian manifold M_g and the right-hand side f of the Poisson equation are such that $I_\varphi < \infty$ then for each function $\Phi(\theta) \in C(S)$ on M_g there exists a unique bounded solution of the Poisson equation such that on D $\lim\limits_{r\to\infty} u(r,\theta) = \Phi(\theta)$.*

Accordingly, in present paper everywhere else we assume $c(x) \not\equiv 0$ on M_g, and moreover $c(x) > 0$ in some neighborhood of the precompact B. Also, let the conditions $f_1(r) \leq f(x) \leq f_2(r)$ be fulfilled on D, where $f_i(r) \in C^\alpha(G)$ for $i = 1, 2$. Let us introduce additional notation

$$I_f = I + \int\limits_{r_0}^{\infty} g^{1-n}(t) \left(\int\limits_{r_0}^{t} \max\{|f_1(\xi)|, |f_2(\xi)|\} g^{n-1}(\xi) d\xi \right) dt, \tag{5}$$

$$J_f = J + \int\limits_{r_0}^{\infty} g^{1-n}(t) \left(\int\limits_{r_0}^{t} \max\{|f_1(\xi)|, |f_2(\xi)|\} g^{n-1}(\xi) d\xi \right) dt. \tag{6}$$

We formulate the main result of present paper, its proof is given in Section 3.

Theorem 3. *1) Let the manifold M_g and the right-hand side f of the Schrödinger equation* (2) *such that $I_f < \infty$. Then for any function $\Phi(\theta) \in C(S)$ on M_g there exists a unique bounded solution of the equation* (2) *such that on D $\lim\limits_{r\to\infty} u(r,\theta) = \Phi(\theta)$.*

2) Let the manifold M_g and the right-hand side f of the Schrödinger equation (2) *such that $I_f = \infty$, $J_f < \infty$. Then for any constant C on M_g there exists a unique bounded solution of the equation* (2) *such that on D $\lim\limits_{r\to\infty} u(r,\theta) = C$.*

2. SOLVABILITY OF THE DIRICHLET PROBLEM ON MODEL ENDS

At first, we study the solvability of the Dirichlet problem for the equation (2) on the so-called model ends $D = M_g \setminus B$. Consider the inhomogeneous equation

$$Lu = f_1(r), \tag{7}$$

where the Schrödinger operator L is defined above in (2). The following statements are true on D.

Lemma 1. *1) Let the manifold M_g and the right-hand side f_1 of the Schrödinger equation (7) such that $I_f < \infty$. Then for any functions $\Phi_1(\theta) \in C(S)$ and $\Phi_2(\theta) \in C(S)$ on D there exists a unique bounded solution of the equation (7) such that $u(r_0, \theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r, \theta) = \Phi_2(\theta)$.*

2) Let the manifold M_g and the right-hand side f_1 of the Schrödinger equation (7) such that $I_f = \infty$, $J_f < \infty$. Then for any function $\Phi_1(\theta) \in C(S)$ and constant C on D there exists a unique bounded solution of the equation (7) such that $u(r_0, \theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r, \theta) = C$.

Proof. 1) Let us prove the first assertion of the Lemma 1. We seek the solution of the boundary value problem in the form $u(r, \theta) = u_1(r, \theta) + u_2(r)$, where the function $u_1(r, \theta)$ is a solution of the problem

$$\begin{cases} Lu_1 = 0, \\ u_1(r_0, \theta) = \Phi_1(\theta), \\ \lim_{r\to\infty} u_1(r, \theta) = \Phi_2(\theta), \end{cases} \tag{8}$$

and the function $u_2(r)$ is a radially symmetric solution of the problem

$$\begin{cases} Lu_2 = f_1(r), \\ u_2(r_0) = 0, \\ \lim_{r\to\infty} u_2(r) = 0. \end{cases} \tag{9}$$

Let the parameters I_f and J_f be determined by the formulas (5) and (6), respectively. The condition $I < \infty$ immediately follows from the condition $I_f < \infty$. Then we obtain the unique solvability of the boundary value problem (8) by Theorem 2.

Set $f_1^+(r) = \max\{f_1(r), 0\}$, $f_1^-(r) = \max\{-f_1(r), 0\}$. It is clear that $f_1^+ \geq 0$ and $f_1^- \geq 0$.

We seek the solution of the problem (9) in the form $u_2(r) = u^+(r) - u^-(r)$, where the functions $u^+(r)$ and $u^-(r)$ are solutions of the boundary value problems

$$\begin{cases} Lu^+ = f_1^+(r), \\ u^+(r_0) = 0, \\ \lim_{r\to\infty} u^+(r) = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} Lu^- = f_1^-(r), \\ u^-(r_0) = 0, \\ \lim_{r\to\infty} u^-(r) = 0. \end{cases} \tag{11}$$

From the representation of the Laplace-Beltrami operator on D and the condition $c(r, \theta) \equiv c(r)$ it is easy to verify that the function $u^+(r)$ must be a solution of the following boundary value problem

$$\begin{cases} (u^+(r))'' + (n-1)\frac{g'(r)}{g(r)}(u^+(r))' - c(r)u^+(r) = f_1^+(r), \\ u^+(r_0) = 0, \\ \lim_{r\to\infty} u^+(r) = 0. \end{cases} \tag{12}$$

We write the ordinary differential equation in (12) as

$$\left((u^+(r))' g^{n-1}(r)\right)' = c(r)g^{n-1}(r)u^+(r) + f_1^+(r)g^{n-1}(r).$$

It follows that

$$(u^+(r))' = \frac{1}{g^{n-1}(r)} \int_{r_0}^{r} \left(c(t)u^+(t) + f_1^+(t) \right) g^{n-1}(t)dt + \frac{(u^+(r_0))' g^{n-1}(r_0)}{g^{n-1}(r)}. \tag{13}$$

Let us prove the existence of a solution of the boundary value problem (12). We denote by $l(r)$ the solution of the equation (13) with initial conditions $l(r_0) = 0$, $l'(r_0) = 1$. It immediately follows from (13) that the function $l(r)$ is monotone increasing. Let us prove that it is bounded above.

Assume the opposite, i.e. $l(r)$ monotonically tends to infinity. Then, without loss of generality, we can assume that $l(r) \geq 1$ for all $r \geq r_1 \geq r_0$. Since the function $l(r)$ is monotone increasing, we obtain the inequality

$$l'(r) \leq \frac{l(r)}{g^{n-1}(r)} \int_{r_0}^{r} c(t)g^{n-1}(t)dt + \frac{l(r)}{g^{n-1}(r)} \int_{r_0}^{r} f_1^+(t)g^{n-1}(t)dt + \frac{l'(r_0)g^{n-1}(r_0)}{g^{n-1}(r)}.$$

Taking into account the initial condition $l'(r_0) = 1 \leq l(r)$, we get

$$l'(r) \leq \frac{l(r)}{g^{n-1}(r)} \int_{r_0}^{r} c(t)g^{n-1}(t)dt + \frac{l(r)}{g^{n-1}(r)} \int_{r_0}^{r} f_1^+(t)g^{n-1}(t)dt + \frac{l(r)g^{n-1}(r_0)}{g^{n-1}(r)}$$

for all $r \geq r_1 \geq r_0$.

Let us rewrite the last inequality in the form

$$\frac{l'(r)}{l(r)} \leq \frac{1}{g^{n-1}(r)} \left(\int_{r_0}^{r} (c(t)g^{n-1}(t) + f_1^+(t)g^{n-1}(t))dt + g^{n-1}(r_0) \right)$$

and integrate this expression on the interval $r \geq r_1$. We get the following chain of inequalities

$$l(r) \leq l(r_1) \exp \left(\int_{r_1}^{r} \frac{dt}{g^{n-1}(t)} \int_{r_0}^{t} (c(z)g^{n-1}(z) + f_1^+(z)g^{n-1}(z))dz \right)$$

$$\times \exp \left(g^{n-1}(r_0) \int_{r_1}^{r} \frac{dt}{g^{n-1}(t)} \right)$$

$$\leq l(r_1) \exp \left(\int_{r_1}^{r} \frac{dt}{g^{n-1}(t)} \int_{r_0}^{t} (c(z)g^{n-1}(z) + |f_1(z)|g^{n-1}(z))dz \right) \exp \left(g^{n-1}(r_0) \int_{r_1}^{r} \frac{dt}{g^{n-1}(t)} \right)$$

$$\leq l(r_1) \exp \left(\int_{r_1}^{\infty} \frac{dt}{g^{n-1}(t)} \int_{r_0}^{t} (c(z) + |f_1(z)|)g^{n-1}(z))dz \right) \exp \left(g^{n-1}(r_0) \int_{r_1}^{\infty} \frac{dt}{g^{n-1}(t)} \right).$$

The convergence of integrals K and I_f (see (3) and (5)) implies the boundedness of the function $l(r)$, and hence, the existence of a limit $\lim_{r\to\infty} l(r) = b$. In addition, the convergence of the integral I_f implies the convergence of the integral $I < \infty$, which implies the existence of a solution $m(r)$ of the equation (see also [11, 12])

$$m''(r) + (n-1)\frac{g'(r)}{g(r)}m'(r) - c(r)m(r) = 0,$$

such that $m(r_0) = 0$ and $\lim_{r\to\infty} m(r) = -b$. Then the function $u^+(r) = l(r) + m(r)$ is the solution of the boundary value problem (12), and therefore $u^+(r)$ is the solution of the boundary value problem (10).

Similarly, it is possible to prove the existence of a solution $u^-(r)$ to the boundary value problem (11). And so the function $u_2(r) = u^+(r) - u^-(r)$ is the solution of the boundary value problem (9), that proves the first part of Lemma 1.

2) Let us prove the second assertion of the lemma. As above, we consider the solution of the boundary value problem in the form $u(r,\theta) = u_1(r,\theta) + u_2(r)$, where $u_1(r,\theta)$ is a solution of the next problem

$$\begin{cases} Lu_1 = 0, \\ u_1(r_0,\theta) = \Phi_1(\theta), \\ \lim_{r\to\infty} u_1(r,\theta) = C, \end{cases} \tag{14}$$

and $u_2(r)$ is the radially symmetric solution of the problem (9). Its existence is shown in assertion 1) of the Lemma 1, i.e.

$$\begin{cases} Lu_2 = f_1(r), \\ u_2(r_0) = 0, \\ \lim_{r\to\infty} u_2(r) = 0. \end{cases}$$

We prove that under the conditions of assertion 2) of the Lemma 1 there is a solution of the problem (14). First, from the definition of the integrals I_f, J_f (see (5), and (6)), and the conditions $I_f = \infty$ and $J_f < \infty$, it follows the fulfillment of conditions $I = \infty$ and $J < \infty$. Then, taking into account assertion 4) of Theorem 2, we obtain the unique solvability of the boundary value problem (14), and hence the solvability of the boundary value problem in assertion 2). □

Next, we consider an inhomogeneous equation on D

$$Lu = f_2(r). \tag{15}$$

As above, the following statement holds.

Lemma 2. *1) Let the manifold M_g and the right-hand side f_2 of the Schrödinger equation* (15) *such that $I_f < \infty$. Then for any functions $\Phi_1(\theta) \in C(S)$ and $\Phi_2(\theta) \in C(S)$ on D there exists a unique bounded solution of the equation* (15) *such that $u(r_0,\theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = \Phi_2(\theta)$.*

2) Let the manifold M_g and the right-hand side f_2 of the Schrödinger equation (15) *such that $I_f = \infty$, $J_f < \infty$. Then for any function $\Phi_1(\theta) \in C(S)$ and constant C on D there exists a unique bounded solution of the equation* (15) *such that $u(r_0,\theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = C$.*

Now, we prove the solvability of the Dirichlet problem for the equation (2) on the model end D.

Lemma 3. *1) Let the manifold M_g and the right-hand side f of the Schrödinger equation* (2) *such that $I_f < \infty$. Then for any functions $\Phi_1(\theta) \in C(S)$ and $\Phi_2(\theta) \in C(S)$ on D there exists a unique bounded solution of the equation* (2) *such that $u(r_0,\theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = \Phi_2(\theta)$.*

2) Let the manifold M_g and the right-hand side f of the Schrödinger equation (2) *such that $I_f = \infty$, $J_f < \infty$. Then for any function $\Phi_1(\theta) \in C(S)$ and constant T on D there exists a unique bounded solution of the equation* (2) *such that $u(r_0,\theta) = \Phi_1(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = C$.*

Proof. We seek the solution of the boundary value problem of assertion 1) in the form $u(r,\theta) = u_1(r,\theta) + u_2(r,\theta)$, where $u_1(r,\theta)$ is the solution of the problem (8)

$$\begin{cases} Lu_1 = 0, \\ u_1(r_0,\theta) = \Phi_1(\theta), \\ \lim_{r\to\infty} u_1(r,\theta) = \Phi_2(\theta), \end{cases}$$

and $u_2(r,\theta)$ is the solution of the next problem

$$\begin{cases} Lu_2 = f(r,\theta), \\ u_2(r_0,\theta) = 0, \\ \lim_{r\to\infty} u_2(r,\theta) = 0. \end{cases} \tag{16}$$

As in Lemma 1, the existence of a solution $u_1(r,\theta)$ follows from the convergence of the integral $I < \infty$ by the Theorem 1. We show the unique solvability of the boundary value problem (16).

Due to the convergence of integral $I_f < \infty$ by Lemmas 1 and 2 there are bounded solutions $v_0(r,\theta)$ and $u_0(r,\theta)$ for the corresponding problems

$$\begin{cases} Lv_0 = f_1(r), \\ v_0(r_0,\theta) = 0, \\ \lim_{r\to\infty} v_0(r,\theta) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Lu_0 = f_2(r), \\ u_0(r_0,\theta) = 0, \\ \lim_{r\to\infty} u_0(r,\theta) = 0. \end{cases}$$

Since $f_1(r) \le f_2(r)$, then $Lv_0 \le Lu_0$. Therefore, according to the comparison theorem for linear elliptic equations (see, for example, [20, p. 41]), the inequality $v_0 \ge u_0$ is fulfilled on D.

Let $\{B_k\}_{k=1}^{\infty}$ be an exhaustion of M_g, i.e. a sequence of nonempty precompact open subsets of M_g such that $\overline{B_k} \subset B_{k+1}$ and $M_g = \cup_{k=1}^{\infty} B_k$. Throughout the sequel, we assume that boundaries ∂B_k are C^1-smooth submanifolds and $B \subset B_k$ for all k.

We consider the sequences of functions u_k and v_k that are solutions of the problems

$$\begin{cases} Lu_k = f \quad \text{in} \quad B_k \setminus B, \\ u_k|_{\partial B_k} = u_0|_{\partial B_k}, \\ u_k|_{\partial B} = 0, \end{cases} \qquad \begin{cases} Lv_k = f \quad \text{in} \quad B_k \setminus B, \\ v_k|_{\partial B_k} = v_0|_{\partial B_k}, \\ v_k|_{\partial B} = 0. \end{cases} \tag{17}$$

Since on D the condition $f_1(r) \le f(r,\theta) \le f_2(r)$ is fulfilled, hence we have in $B_k \setminus B$

$$Lv_0 \le Lv_k = Lu_k \le Lu_0,$$

$$v_k|_{\partial B_k} = v_0|_{\partial B_k} \ge u_0|_{\partial B_k} = u_k|_{\partial B_k},$$

$$v_k|_{\partial B} = v_0|_{\partial B} = u_0|_{\partial B} = u_k|_{\partial B}.$$

Then, according to the comparison theorem for linear elliptic equations in $B_k \setminus B$ (see, for example, [20, p. 41]) for all k, we get

$$v_0 \ge v_k \ge u_k \ge u_0. \tag{18}$$

The boundedness of solutions v_0 and u_0 implies the uniform boundedness of the families $\{u_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ for an arbitrary compact subset $G \subset D$. Uniform boundedness of the sequences and Schauder's internal estimates (see, for example, [20, p. 94–95]) imply the compactness of these families of functions in the class $C^2(G)$ for an arbitrary compact subset $G \subset D$. In turn, compactness of families $\{u_k\}_{k=1}^{\infty}$, $\{v_k\}_{k=1}^{\infty}$ entails the existence of limit functions $v = \lim\limits_{k\to\infty} v_k$ and $u = \lim\limits_{k\to\infty} u_k$, which are solutions of the equation (2) on D. Taking the limit in (18) for $k \to \infty$, we have $v_0 \ge v \ge u \ge u_0$ on D.

Since $\lim_{r\to\infty} v_0(r,\theta) = \lim_{r\to\infty} u_0(r,\theta) = 0$, then $\lim_{r\to\infty} v(r,\theta) = \lim_{r\to\infty} u(r,\theta) = 0$. In addition, given the boundary conditions of the problem (17), we get $v|_{\partial B} = u|_{\partial B} = 0$ or $v(r_0,\theta) = u(r_0,\theta)$. So, by the comparison theorem for linear elliptic equations we have $u \equiv v$ on D. We denote the common limit function by $u_2 = \lim\limits_{k\to\infty} v_k = \lim\limits_{k\to\infty} u_k$, which is the desired solution to the problem (16).

The proof of the second assertion of the Lemma 3 is done analogy. Lemma 3 is proved. □

3. SOLVABILITY OF THE DIRICHLET PROBLEM ON MODEL MANIFOLDS

This section is devoted to the proof of Theorem 3. First, we formulate an auxiliary statement from [13], that plays an essential role in the proof of the Theorem 3.

Lemma 4. ([13]) *Let G be a precompact subset in M_g, and a function $u \in C(\overline{G}) \cap C^2(G)$ satisfies the equation (2) on G, where $f \in C_0(\overline{G})$, $\Omega := \operatorname{supp} f$ and $\Omega \subset\subset G$, $c \geq 0$ on $\overline{G}$ and $c \not\equiv 0$ on Ω. Then*

$$\sup_G |u| \leq \sup_{\partial G} |u| + \sup_\Omega \frac{|f|}{c}.$$

Now, we go directly to the proof of the Theorem 3.

Proof. Also, as in the above proof of Lemma 3, by the assertion 1) and 3) of Theorem 2, it is sufficient to show the solvability of the boundary value problem

$$\begin{cases} Lu = f, \\ \lim_{r\to\infty} u(r,\theta) = 0. \end{cases} \tag{19}$$

Lemma 3 implies the existence of a function $u_0(r,\theta)$, such that

$$\begin{cases} Lu_0 = f(r,\theta), \\ u_0(r_0,\theta) = 0, \\ \lim_{r\to\infty} u_0(r,\theta) = 0. \end{cases}$$

Denote by B'' the neighborhood of compact set B, where $c(x) > 0$. We consider a function $U_0 \in C^{2,\alpha}(M)$, such that $U_0 = u_0$ outside B'', $U_0 = 0$ on some subset $B' \subset\subset B$. It is clear that $LU_0 = f_0(x)$ on M_g, where the function $f_0(x) \in C^\alpha(M_g)$ satisfies the following conditions: $f_0(x) = 0$ in B', $f_0(x) \equiv f(x)$ outside B'', $f_0(x) \not\equiv f(x)$ in $B'' \setminus B'$.

Consider now the sequence of functions φ_k that are solutions of the problems

$$\begin{cases} L\varphi_k = f(x) \quad \text{in} \quad B_k, \\ \varphi_k \,|_{\partial B_k} = u_0|_{\partial B_k} \end{cases}$$

and the sequence of functions $\psi_k = \varphi_k - U_0$. It is clear, that ψ_k are solutions of the problems

$$\begin{cases} L\psi_k = f(x) - f_0(x) \quad \text{in} \quad B_k, \\ \psi_k \,|_{\partial B_k} = 0, \end{cases}$$

where the function $f(x) - f_0(x) \in C^\alpha(M_g)$ and satisfies the following conditions: $f(x) - f_0(x) = f(x)$ on the compact set B', $f(x) - f_0(x) = 0$ outside of B''. Thus, $\Omega := \operatorname{supp}\{f(x) - f_0(x)\}$ is a compact and $\Omega \subset B''$.

By Lemma 4, for all k for $x \in B_k$ we have

$$|\psi_k| \leq \sup_{B_k} |\psi_k| \leq \sup_{\partial B_k} |\psi_k| + \sup_\Omega \frac{|f(x) - f_0(x)|}{c(x)} = \sup_\Omega \frac{|f(x) - f_0(x)|}{c(x)},$$

which implies the uniform boundedness of the family of functions $\{\psi_k\}_{k=1}^\infty$ on M_g. Hence, as above, we obtain compactness of this family in the class $C^2(G)$ for an arbitrary compact subset $G \subset M_g$. One implies the existence of the limit function $\psi = \lim_{k\to\infty} \psi_k$ on M_g such that $L\psi = f(x) - f_0(x)$ on M_g.

The existence of the function $\psi = \lim_{k\to\infty} \psi_k$ implies the existence of the limit function $u = \lim_{k\to\infty} \varphi_k$ such that $Lu = f(x)$ on M_g.

To finish the proof of the theorem, we need to show the fulfillment of the boundary condition $\lim_{r\to\infty} u(r,\theta) = 0$. We denote $A = \max_S |u(r_0,\theta)|$. The existence of the limit of sequence $\{\varphi_k\}$ implies that we have $-(A+1) < \varphi_k|_{\partial B} < A + 1$ for sufficiently large values k.

Obviously, the following inequality holds

$$-(A+1) < u_0|_{\partial B} = u_0(r_0,\theta) < A+1.$$

By Lemma 3, we obtain the existence of solutions to the following boundary value problems on D

$$\begin{cases} Lu_1 = f(r,\theta), \\ u_1(r_0,\theta) = -(A+1), \\ \lim_{r\to\infty} u_1(r,\theta) = 0. \end{cases} \quad \text{and} \quad \begin{cases} Lu_2 = f(r,\theta), \\ u_2(r_0,\theta) = A+1, \\ \lim_{r\to\infty} u_2(r,\theta) = 0. \end{cases}$$

Then applying the comparison theorem for linear elliptic equations as above, we get on D

$$u_1(r,\theta) \le u_0(r,\theta) \le u_2(r,\theta),$$

and for sufficiently large values k on $B_k \setminus B$ $u_1 \le \varphi_k \le u_2$. Passing to the limit as $k \to \infty$ in last inequality on D, we get $u_1 \le u \le u_2$. According to the asymptotic behavior of functions u_1 and u_2, we have $\lim_{r\to\infty} u(r,\theta) = 0$.

So, the function u is the desired solution of the boundary value problem (19). To complete the proof, we note that under the conditions of this theorem, the conditions of assertions 1) and 3) of Theorem 2 are fulfilled, i.e. there are bounded solutions on M_g with a given asymptotic behavior for the stationary Schrödinger equation (1). These solutions, combined with the solution u of the problem (19), give the desired solutions to the boundary value problems in assertions 1) and 2) of Theorem 3. □

REMARK

When studying the solvability of boundary value problems for solutions of inhomogeneous elliptic equations on non-compact Riemannian manifolds, in addition to Hölder continuity, some additional conditions are imposed on the right-hand side. For example, the condition of infinite differentiability was imposed on the right-hand side of the Poisson equation and it was supplemented with a condition on the rate of decay to zero at infinity in [17]. In [15], the nonnegativity condition was imposed on the right-hand side of the equation, and the boundedness condition, in addition to Hölder continuity, was imposed on the right-hand side of the equation in [18]. In [19], a smoothness condition for the right-hand side of $f \in C^{[\frac{3n}{2}]}(M_g)$, where $n = \dim(M_g)$, is imposed. It is sufficient for the solvability of the Dirichlet problem on a model manifold for the Poisson equation.

In this paper, similar results are obtained for a more general inhomogeneous equation, while the conditions for the smoothness of the right-hand side $f(x)$ of the inhomogeneous equation (2) is weakened due to the requirement of its boundedness from above and below on D. Moreover, these majorants and minorants depend only on the radial coordinate, i.e. $f_1(r) \le f(x) \le f_2(r)$.

FUNDING

This work was partially supported by the Ministry of Science and Higher Education of the Russian Federation (government task no. 0633-2020-0003).

REFERENCES

1. L. V. Ahlfors, "Sur le type d'une surface de Riemann," C. R. Acad. Sci. Paris **201**, 30–32 (1935).
2. R. Nevanlina, "Ein Satz über offenen Riemannsche Flächen," Ann. Acad. Sci. Fenn., Part A **54**, 1–18 (1940).
3. P. Grigor'yan, "Analitic and geometric background of recurence and non-explosion of the Brownian motion on Riemannian manifolds," Bull. Am. Math. Soc. **36**, 135–249 (1999).
4. S. Y. Cheng and S. T. Yau, "Differential equations on Riemannian manifolds and their geometric applications," Comm. Pure Appl. Math. **28**, 333–354 (1975).
5. A. Grigor'yan, "On the existance of positive fundamental solution of the Laplace equation on Riemannian manifolds," Mat. Sb. **128**, 354–363 (1985).
6. A. Ancona, "Negative curved manifolds, elliptic operators, and the Martin boundary," Ann. Math. **125**, 495–536 (1987).
7. E. A. Mazepa, "Boundary value problems for stationary Schrödinger equation on Riemannian manifolds," Sib. Math. J. **43**, 591–599 (2002).

8. M. T. Anderson, “The Dirichlet problem at infinity for manifolds with negative curvature,” J. Differ. Geom. **18**, 701–722 (1983).
9. D. Sullivan, “The Dirichlet problem at infinity for a negatively curved manifolds,” J. Differ. Geom. **18**, 723–732 (1983).
10. S. Y. Cheng, “The Dirichlet problem at infinity for nonpositively curved manifolds,” Commun. Anal. Geom. **1**, 101–112 (1983).
11. A. G. Losev, “Some Liouville theorems on Riemannian manifolds of a special type,” Sov. Math. (Iz. VUZ) **35** (12), 15–23 (1991).
12. A. G. Losev and E. A. Mazepa, “On the behavior of bounded solutions of the Schrödinger equation on non-compact Riemannian manifolds,” Vestn. Volgogr. Univ., Ser. 1: Mat. Fiz. **3**, 32–43 (1998).
13. A. G. Losev, E. A. Mazepa, and V. Y. Chebanenko, “Unbounded solutions of the stationary Schrödinger equation on Riemannian manifolds,” Comput. Methods Funct. Theory **3**, 443–451 (2003).
14. A. G. Losev and V. V. Filatov, “Bounded solutions of the stationary Schrödinger equation with finite energy integral on model manifolds,” Mat. Fiz. Komp’yut. Simul. **24** (3), 5–17 (2021).
15. L. Ni, Y. Shi, and L.-F. Tam, “Poisson equation, Poincare–Lelong equation and curvature decay on complete Kahler manifolds,” J. Diff. Geom. **57**, 339–388 (2001).
16. A. Grigor’yan and I. Verbitsky, “Pointwise estimates of solutions to semilinear elliptic equations and inequalities,” J. Anal. Math. **137**, 559–601 (2019).
17. O. Munteanu and N. Sesum, “The Poisson equation on complete manifolds with positive spectrum and applications,” Adv. Math. **223**, 198–219 (2010).
18. P. Mastrolia, D. D. Monticelly, and F. Punzo, “Elliptic and parabolic equations with Dirichlet conditions at infinityu on Riemannian manifolds,” arXiv: 1511.09023v1 (2015), pp. 1–15.
19. A. G. Losev, “On the solvability of the Dirichlet problem for the Poisson equation on certain noncompact Riemannian manifolds,” Differ. Equat. **53**, 1643–1652 (2017).
20. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer, Berlin, 1983).