ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 2, pp. 293–302. © Pleiades Publishing, Ltd., 2022.

Nonlocal Boundary Value Problem for a Fourth Order Differential Equation

D. Amanov[1*] **and O. Sh. Kilichov**[2**]

(Submitted by T. K. Yuldashev)

[1] *Fergana State University, Fergana, 150100 Uzbekistan*

[2] *V. I. Romanovskii Institute of Mathematics of Academy of Sciences of Uzbekistan, Tashkent, 100174 Uzbekistan*

Received October 20, 2021; revised November 28, 2021; accepted December 15, 2021

Abstract—In this article, a nonlocal boundary value problem for a fourth order partial differential equation is solved. The method of separation of variables is used. The solution is constructing in the form of Fourier series. Theorems on the existence and uniqueness of the solution to the problem are proved.

DOI: 10.1134/S1995080222050043

Keywords and phrases: *boundary value problem, Fourier series, eigenvalues, eigenfunctions, uniqueness of a solution, existence of a solution.*

1. INTRODUCTION

A nonlocal problem for a fourth order differential equation is solved for the case, when initial and final conditions are on the higher-order derivatives over the second argument y, i.e. the order of conditions are exceeding the order of the given differential equation. A. N. Tikhonov was the first who studied the problem of high order derivatives on a part of the domain boundary. In [1], for the homogeneous heat equation, he was investigated the problem with the following conditions

$$\sum_{k=0}^{\infty} a_k \frac{\partial^k u}{\partial x^k}(0,t) = f(t), \quad u(x,0) = 0$$

in domain $(0 < x < \infty,\, t > 0)$. In [2], A. V. Bitsadze was investigated in n-dimensional bounded domain D the problem

$$\Delta u(x) = 0, \quad \frac{d^m u}{dv^m} = f(x), \quad x \in D$$

and proved its Fredholm property. Boundary value problems with boundary conditions containing higher order derivatives for the partial differential equations were studied in the works of I. I. Bavrin [3], V. V. Karachik and B. Kh. Turmetov [4], V. B. Sokolovsky [5] and others. Boundary value problems for many kind of higher order partial differential equations were studied in the works of many authors (see, for example [6–18]).

The mixed problem for the heat conduction equation with initial conditions on the higher order derivatives was studied in [19], and the mixed problem for the vibrating string equation with initial conditions on the higher order derivatives was studied in [20]. Mixed problems for the fourth order differential equations were studied, for example, in [21–29] and in other publications. The most complete bibliography on these issues could be found in [21].

*E-mail: damanov@yandex.ru

**E-mail: oybek2402@mail.ru

2. FORMULATION OF THE PROBLEM

In this article, in the domain $\Omega = \{(x, y) : 0 < x < p,\ 0 < y < q\}$ we consider the following differential equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial x^4} = f(x, y), \tag{1}$$

where $f(x, y)$ is a given continuous function in $\overline{\Omega} = \{(x, y) : 0 \le x \le p,\ 0 \le y \le q\}$.

Problem *A.* Find a solution $u(x, y) \in C^{4,k+1}_{x,y}(\overline{\Omega})$ to the differential equation (1) with conditions

$$u(0, y) = 0, \quad u(p, y) = 0, \quad 0 \le y \le q, \tag{2}$$

$$u_{xx}(0, y) = 0, \quad u_{xx}(p, y) = 0, \quad 0 \le y \le q, \tag{3}$$

$$\left.\frac{\partial^k u}{\partial y^k}\right|_{y=0} = \left.\frac{\partial^k u}{\partial y^k}\right|_{y=q}, \quad 0 \le x \le p, \tag{4}$$

$$\left.\frac{\partial^{k+1} u}{\partial y^{k+1}}\right|_{y=0} = \left.\frac{\partial^{k+1} u}{\partial y^{k+1}}\right|_{y=q}, \quad 0 \le x \le p, \tag{5}$$

where $1 \le k$ is a fixed natural number.

3. UNIQUENESS OF THE SOLUTION TO PROBLEM A

We will prove that the following theorem is true.

Theorem 1. *A solution to Problem A is unique, if it exists.*

Proof. We suppose that $f(x, y) = 0$ in $\overline{\Omega}$. Let us show that $u(x, y) = 0$ in $\overline{\Omega}$. Following [30], we consider the following integral

$$\alpha_n(y) = \int_0^p u(x, y)\, X_n(x) dx, \quad 0 \le y \le q, \tag{6}$$

with eigenfunctions

$$X_n(x) = \sqrt{\frac{2}{p}} \sin \lambda_n x, \quad n = 1, 2, ... \tag{7}$$

and eigenvalues $\lambda_n = \frac{n\pi}{p}$, $n = 1, 2, ...$ We note that the eigenfunctions (7) form a complete orthonormal system in $L_2(0, p)$. Differentiating the function (6) twice in y, from the following homogeneous equation $u_{yy} - u_{xxxx} = 0$, we find

$$\alpha_n''(y) = \int_0^p u_{yy}(x, y) X_n(x) dx \quad \text{or} \quad \alpha_n''(y) = \int_0^p u_{xxxx}(x, y) X_n(x) dx. \tag{8}$$

Integrating the right-hand side of (8) by parts fourth times, we obtain

$$\alpha_n''(y) - \lambda_n^4\, \alpha_n(y) = 0. \tag{9}$$

The general solution of the countable system of second order ordinary differential equation (9) is written as

$$\alpha_n(y) = a_n\, e^{\lambda_n^2 y} + b_n\, e^{-\lambda_n^2 y}, \tag{10}$$

where a_n and b_n are arbitrary constants. We differentiate the presentation (10) k and $k + 1$ times over y:

$$\alpha_n^{(k)}(y) = (\lambda_n^2)^k\, a_n\, e^{\lambda_n^2 y} + (-\lambda_n^2)^k\, b_n\, e^{-\lambda_n^2 y},$$
$$\alpha_n^{(k+1)}(y) = (\lambda_n^2)^{k+1}\, a_n\, e^{\lambda_n^2 y} + (-\lambda_n^2)^{k+1}\, b_n\, e^{-\lambda_n^2 y}.$$

Using the formulas (6) and (7), the conditions (4) and (5) we rewrite as

$$\alpha_n^{(k)}(y)\Big|_{y=0} = \alpha_n^{(k)}(y)\Big|_{y=q}, \quad 0 \le x \le p,$$

$$\alpha_n^{(k+1)}(y)\Big|_{y=0} = \alpha_n^{(k+1)}(y)\Big|_{y=q}, \quad 0 \le x \le p.$$

By virtue of last two conditions, for the unknown coefficients a_n and b_n we obtain $a_n = 0$, $b_n = 0$. Then from the presentation (10) it follows that $\alpha_n(y) = 0$. Consequently, from the presentation (6) we obtain

$$\int_0^p u(x,y)\, X_n(x) dx = 0.$$

Since $X_n(x)$, $n = 1, 2, \ldots$ is the complete orthonormal system in $L_2(0,p)$, then $u(x,y) = 0$ almost everywhere in Ω. From the inclusion $u(x,y) \in C_{x,y}^{4,k+1}(\overline{\Omega})$ it follows that $u(x,y) \equiv 0$ in $\overline{\Omega}$. Theorem 1 is proved. □

4. THE EXISTENCE OF A SOLUTION TO PROBLEM A

Taking the boundary value conditions (2) and (3) into account, we will seek the solution to equation (1) in the form of Fourier series

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y)\, X_n(x), \tag{11}$$

where eigenfunctions $X_n(x)$ defines from the (7). We also expand the function $f(x,y)$ in a Fourier series by functions $X_n(x), n = 1, 2, \ldots$:

$$f(x,y) = \sum_{n=1}^{\infty} f_n(y)\, X_n(x), \tag{12}$$

where

$$f_n(y) = \int_0^p f(x,y)\, X_n(x) dx. \tag{13}$$

Substituting the Fourier series (11) and (12) into the equation (1), we obtain

$$\sum_{n=1}^{\infty} \left[u_n''(y)\, X_n(x) - u_n(y)\, \lambda_n^4 X_n(x) \right] = \sum_{n=1}^{\infty} f_n(y)\, X_n(x).$$

Hence, we have the second order countable system of ordinary differential equations

$$u_n''(y) - \lambda_n^4 u_n(y) = f_n(y).$$

The general solution of this system has the following form

$$u_n(y) = a_n(0)\, e^{\lambda_n^2 y} + b_n(0)\, e^{-\lambda_n^2 y} + \frac{1}{\lambda_n^2} \int_0^y \sinh \lambda_n^2 (y - \tau)\, f_n(\tau) d\tau, \tag{14}$$

where $a_n(0)$ and $b_n(0)$ are the unknown constants.

It is easy to verify that from the presentation (14) we obtain

$$u_n^{(k)}(y) = (\lambda_n^2)^k\, a_n(0)\, e^{\lambda_n^2 y} + (-\lambda_n^2)^k\, b_n(0)\, e^{-\lambda_n^2 y} + \sum_{s=0}^{[\frac{k-2}{2}]} \lambda_n^{4s} f_n^{(k-2-2s)}(y)$$

$$+\lambda_n^{2(k-1)}\int\limits_0^y \frac{e^{\lambda_n^2(y-\tau)}+(-1)^{k+1}\,e^{-\lambda_n^2(y-\tau)}}{2}\,f_n(\tau)d\tau, \tag{15}$$

$$u_n^{(k+1)}(y)=(\lambda_n^2)^{k+1}\,a_n(0)\,e^{\lambda_n^2 y}+(-\lambda_n^2)^{k+1}\,b_n(0)\,e^{-\lambda_n^2 y}+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}f_n^{(k-1-2s)}(y)$$
$$+\lambda_n^{2(k-1)}\,\frac{1+(-1)^{k+1}}{2}f_n(y)+\lambda_n^{2k}\int\limits_0^y\frac{e^{\lambda_n^2(y-\tau)}+(-1)^{k+2}\,e^{-\lambda_n^2(y-\tau)}}{2}\,f_n(\tau)d\tau. \tag{16}$$

The conditions (4) and (5) we rewrite in the form of Fourier coefficients:

$$u_n^{(k)}(y)\Big|_{y=0}=u_n^{(k)}(y)\Big|_{y=q},\quad 0\le x\le p,$$

$$u_n^{(k+1)}(y)\Big|_{y=0}=u_n^{(k+1)}(y)\Big|_{y=q},\quad 0\le x\le p.$$

Using these conditions, from the presentations (15) and (16) we have

$$a_n(0)=\frac{1}{2\,(\lambda_n^2)^{k+1}\,(1-e^{\lambda_n^2 q})}\left[\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\Big(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\Big)+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\Big(f_n^{(k-1-2s)}(q)\right.$$
$$\left.-f_n^{(k-1-2s)}(0)\Big)+(\lambda_n^2)^{k-1}\,\frac{1+(-1)^{k+1}}{2}\,\Big(f_n(q)-f_n(0)\Big)+\lambda_n^{2k}\int\limits_0^q e^{\lambda_n^2(q-\tau)}\,f_n(\tau)d\tau\right], \tag{17}$$

$$b_n(0)=\frac{(-1)^k}{2\,(\lambda_n^2)^{k+1}(1-e^{-\lambda_n^2 q})}\left[\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\Big(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\Big)-\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\Big(f_n^{(k-1-2s)}(q)\right.$$
$$\left.-f_n^{(k-1-2s)}(0)\Big)-(\lambda_n^2)^{k-1}\,\frac{1+(-1)^{k+1}}{2}\Big(f_n(q)-f_n(0)\Big)-\lambda_n^{2k}\int\limits_0^q(-1)^k e^{-\lambda_n^2(q-\tau)}\,f_n(\tau)d\tau\right]. \tag{18}$$

Substituting the values (17) and (18) into the presentation (14), we determine the Fourier coefficients of unknown function $u\,(x,y)$:

$$u_n(y)=K_{0n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k}\Big[f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\Big]$$
$$+M_{0n}(y)\left[\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k-2}\Big(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\Big)+\frac{1+(-1)^{k+1}}{2\lambda_n^4}\Big(f_n(q)-f_n(0)\Big)\right]$$
$$+\frac{1}{\lambda_n^2}\int\limits_0^q K_{1n}\,(y,\tau)\;f_n(\tau)d\tau, \tag{19}$$

where, if k is an even number, then

$$K_{0n}(y)=\frac{\cosh\lambda_n^2 y-\cosh\lambda_n^2(y-q)}{2(1-\cosh\lambda_n^2 q)},\quad M_{0n}(y)=\frac{\sinh\lambda_n^2 y-\sinh\lambda_n^2(y-q)}{2(1-\cosh\lambda_n^2 q)}$$

and, if k is an odd number, then

$$K_{0n}(y)=\frac{\sinh\lambda_n^2 y-\sinh\lambda_n^2(y-q)}{2(1-\cosh\lambda_n^2 q)},\quad M_0(y)=\frac{\cosh\lambda_n^2 y-\cosh\lambda_n^2(y-q)}{2(1-\cosh\lambda_n^2 q)},$$

$$K_{1n}(y,\tau)=\begin{cases}\frac{\sinh\lambda_n^2(y-\tau)+\sinh\lambda_n^2(q-y+\tau)}{2\left(1-\cosh\lambda_n^2 q\right)}, & \tau\in[0,y],\\ \frac{\sinh\lambda_n^2(q+y-\tau)-\sinh\lambda_n^2(y-\tau)}{2\left(1-\cosh\lambda_n^2 q\right)}, & \tau\in[y,q].\end{cases}$$

Substituting the presentation (19) into the series (11), we obtain the formal solution of the Problem A:

$$u(x,y)=\sum_{n=1}^{\infty}X_n(x)\Bigg[K_{0n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k}\left(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\right)$$
$$+M_{0n}(y)\left(\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k-2}\left(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\right)+\frac{1+(-1)^{k+1}}{2\lambda_n^4}\left(f_n(q)-f_n(0)\right)\right)$$
$$+\frac{1}{\lambda_n^2}\int_0^q K_{1n}(y,\tau)\ f_n(\tau)d\tau\Bigg],\tag{20}$$

where and hereinafter $\sum\limits_{s=0}^{m}(\ldots)=0$ for $m<0$.

We need to prove the absolute and uniform convergence of the series (20) and following series

$$\frac{\partial^2 u}{\partial y^2}=\sum_{n=1}^{\infty}X_n(x)\Bigg[K_0(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k+4}\left(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\right)$$
$$+M_{0n}(y)\left(\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k+2}\left(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\right)+\frac{1+(-1)^{k+1}}{2}\left(f_n(q)-f_n(0)\right)\right)+f_n(y)$$
$$+\lambda_n^2\int_0^q K_{1n}(y,\tau)\ f_n(\tau)d\tau\Bigg],\tag{21}$$

$$\frac{\partial^4 u}{\partial x^4}=\sum_{n=1}^{\infty}X_n(x)\Bigg[K_{0n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k+4}\left(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\right)$$
$$+M_{0n}(y)\left(\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k+2}\left(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\right)+\frac{1+(-1)^{k+1}}{2}\left(f_n(q)-f_n(0)\right)\right)$$
$$+\lambda_n^2\int_0^q K_{1n}(y,\tau)\,f_n(\tau)d\tau\Bigg],\tag{22}$$

$$\frac{\partial^k u}{\partial y^k}=\sum_{n=1}^{\infty}X_n(x)\Bigg[K_{3n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\left(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\right)$$
$$+K_{4n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2}\left(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\right)$$
$$+K_{4n}(y)\ \frac{1+(-1)^{k+1}}{2}\lambda_n^{2k-4}\left(f_n(q)-f_n(0)\right)+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}f_n^{(k-2-2s)}(y)$$

$$+\lambda_n^{2(k-1)}\int\limits_0^q K_1(y,\tau)\,f_n(\tau)d\tau\Bigg],\tag{23}$$

$$\frac{\partial^{k+1}u}{\partial y^{k+1}}=\sum_{n=1}^{\infty}X_n(x)\Bigg[K_{4n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\Big(f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\Big)$$
$$+K_{3n}(y)\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\Big(f_n^{(k-1-2s)}(q)-f_n^{(k-1-2s)}(0)\Big)$$
$$+K_{3n}(y)\,\frac{1+(-1)^{k+1}}{2}\,\lambda_n^{2k-2}\Big(f_n(q)-f_n(0)\Big)+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\,f_n^{(k-1-2s)}(y)$$
$$+\frac{1+(-1)^{k+1}}{2}\,\lambda_n^{2k-2}\,f_n(y)+\lambda_n^{2k}\int\limits_0^q K_2(y,\tau)\,f_n(\tau)d\tau\Bigg],\tag{24}$$

where, if k is an even number, then

$$K_{1n}(y,\tau)=\begin{cases}\frac{\sinh\lambda_n^2(y-\tau)+\sinh\lambda_n^2(q-y+\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[0,y],\\ \frac{\sinh\lambda_n^2(q+y-\tau)-\sinh\lambda_n^2(y-\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[y,q],\end{cases}$$

$$K_{2n}(y,\tau)=\begin{cases}\frac{\cosh\lambda_n^2(y-\tau)-\cosh\lambda_n^2(q-y+\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[0,y],\\ \frac{\cosh\lambda_n^2(q+y-\tau)-\cosh\lambda_n^2(y-\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[y,q],\end{cases}$$

and, if k is an odd number, then

$$K_{1n}(y,\tau)=\begin{cases}\frac{\cosh\lambda_n^2(y-\tau)-\cosh\lambda_n^2(q-y+\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[0,y],\\ \frac{\cosh\lambda_n^2(q+y-\tau)-\cosh\lambda_n^2(y-\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[y,q],\end{cases}$$

$$K_{2n}(y,\tau)=\begin{cases}\frac{\sinh\lambda_n^2(y-\tau)+\sinh\lambda_n^2(q-y+\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[0,y],\\ \frac{\sinh\lambda_n^2(q+y-\tau)-\sinh\lambda_n^2(y-\tau)}{2(1-\cosh\lambda_n^2 q)}, & \tau\in[y,q],\end{cases}$$

$$K_{3n}(y)=\frac{\cosh\lambda_n^2 y-\cosh\lambda_n^2(q-y)}{2\,(1-\cosh\lambda_n^2 q)},\quad K_{4n}(y)=\frac{\sinh\lambda_n^2 y+\sinh\lambda_n^2(q-y)}{2\,(1-\cosh\lambda_n^2 q)}.$$

We first estimate the functional series (20) for absolute and uniform convergence:

$$|u(x,y)|\le\sum_{n=1}^{\infty}\Bigg[\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k}\Big|f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\Big|+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s-2k-2}\Big|f_n^{(k-1-2s)}(q)$$
$$-f_n^{(k-1-2s)}(0)\Big|+\frac{1+(-1)^{k+1}}{\lambda_n^4}\Big|f_n(q)-f_n(0)\Big|+\frac{1}{\lambda_n^2}\int\limits_0^q\Big|f_n(\tau)\Big|d\tau\Bigg].\tag{25}$$

If $f(x,y)\in C_{x,y}^{0,k-1}(\overline{\Omega})$, then the number series on the right-hand side of (25) converges. By the Weierstrass criterion, the convergence of a numerical series implies absolute and uniform convergence of the series (20).

Now, we will investigate the absolute and uniform convergence of the series (21)–(24). According to the comparison criterion, if the series (24) converges, then series (21)–(23) with terms less than the corresponding terms of the series (24) converge absolutely and uniformly. Let us show the absolute and uniform convergence of the series (24):

$$\left|\frac{\partial^{k+1}u}{\partial y^{k+1}}\right| \leq \sum_{n=1}^{\infty}\Bigg[\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\left|f_n^{(k-2-2s)}(q)-f_n^{(k-2-2s)}(0)\right|+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\Big|f_n^{(k-1-2s)}(q)$$
$$-f_n^{(k-1-2s)}(0)\Big|+\frac{1+(-1)^{k+1}}{2}\lambda_n^{2k-2}\left|f_n(q)-f_n(0)+f_n(y)\right|+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s}\left|f_n^{(k-1-2s)}(y)\right|$$
$$+\lambda_n^{2k}\int_0^q\left|K_2(y,\tau)\,f_n(\tau)\right|d\tau\Bigg]. \qquad (26)$$

We estimate the terms in the right-hand side of the series (26).

Lemma 1. *Let*

1) k be an even number and $f(x,y)\in W_2^{(2k-1,k-2)}(\Omega)$, $\frac{\partial^{2l}f(0,y)}{\partial x^{2l}}=\frac{\partial^{2l}f(p,y)}{\partial x^{2l}}=0,\ l=\overline{0,k-1}$,

2) k be an odd number and $f(x,y)\in W_2^{(2k-3,k-2)}(\Omega)$, $\frac{\partial^{2l+1}f(0,y)}{\partial y\partial x^{2l}}=\frac{\partial^{2l+1}f(p,y)}{\partial y\partial x^{2l}}=0, l=\overline{0,k-1}$.

Then the series

$$\sum_{n=1}^{\infty}\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\left|f_n^{(k-2-2s)}(q)\right| \qquad (27)$$

and

$$\sum_{n=1}^{\infty}\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\left|f_n^{(k-2-2s)}(0)\right| \qquad (28)$$

converge absolutely and uniformly in $\overline{\Omega}$.

Proof. To prove the convergence of the series (27) and (28), we use the following series:

$$\sum_{n=1}^{\infty}\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{4s+2}\left|f_n^{(k-2-2s)}(y)\right|=\sum_{n=1}^{\infty}\left[\lambda_n^2\left|f_n^{(k-2)}(y)\right|+\lambda_n^6\left|f_n^{(k-4)}(y)\right|\right]+\ldots$$
$$+\sum_{n=1}^{\infty}\begin{cases}\lambda_n^{2k-2}\left|f_n(y)\right|, & k=2m,\quad m\in\mathbb{N},\\ \lambda_n^{2k-4}\left|f_n'(y)\right|, & k=2m+1,\quad m\in\mathbb{N}.\end{cases} \qquad (29)$$

Let k be an even number. If the series

$$\sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(y)\right| \qquad (30)$$

converges, then the series (29) also converges. Integrating the Fourier coefficients (13) by parts $(2k-1)$ times, we have

$$\left|f_n(y)\right|=\frac{1}{\lambda_n^{2k-1}}\left|f_n^{(2k-1,0)}(y)\right|, \qquad (31)$$

where

$$f_n^{(2k-1,0)}(y)=\int_0^p\frac{\partial^{2k-1}f(x,y)}{\partial x^{2k-1}}\sqrt{\frac{2}{p}}\sin\Big((2k-1)\frac{\pi}{2}+\lambda_n x\Big)dx.$$

By virtue of (31), the series (30) takes the form

$$\sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(y)\right|=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(2k-1,0)}(y)\right|.$$

Applying Cauchy–Schwartz inequality for the sum on the right-hand side of the last equality, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(2k-1,0)}(y)\right|\le\frac{p}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|f_n^{(2k-1,0)}(y)\right|^2\right)^{1/2}=\frac{p}{\sqrt{6}}\left(\sum_{n=1}^{\infty}\left|f_n^{(2k-1,0)}(y)\right|^2\right)^{1/2}.$$

Applying Bessel's inequality, we find

$$\left(\sum_{n=1}^{\infty}\left|f_n^{(2k-1,0)}(y)\right|^2\right)^{1/2}\le\left\|\frac{\partial^{2k-1}f}{\partial x^{2k-1}}\right\|_{L_2(\Omega)}<\infty.$$

So, we obtain

$$\sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(y)\right|\le\frac{p}{\sqrt{6}}\left\|\frac{\partial^{2k-1}f}{\partial x^{2k-1}}\right\|_{L_2(\Omega)}<\infty.$$

The convergence of series (30) for even k is proved.

Let k be an odd number. If the series $\sum\limits_{n=1}^{\infty}\lambda_n^{2k-4}\left|f_n'(y)\right|$ converges, then the series (29) also converges for odd k. The proof of this assertion is similar to the proof of the convergence of series (30). Lemma 1 is proved. □

Lemma 2. *Let*

1) k be an even number and $f(x,y)\in W_2^{(2k-3,k-1)}(\Omega)$, $\frac{\partial^{2l+1}f(0,y)}{\partial y\partial x^{2l}}=\frac{\partial^{2l+1}f(p,y)}{\partial y\partial x^{2l}}=0,\quad l=\overline{0,k-2}$,

2) k be an odd number and $f(x,y)\in W_2^{(2k-5,k-1)}(\Omega)$, $\frac{\partial^{2l+2}f(0,y)}{\partial y^2\partial x^{2l}}=\frac{\partial^{2l+2}f(p,y)}{\partial y^2\partial x^{2l}}=0,\quad l=\overline{0,k-3}$.

Then the following series

$$\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{4s}\left|f_n^{(k-1-2s)}(q)\right|,\tag{32}$$

$$\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{4s}\left|f_n^{(k-1-2s)}(0)\right|\tag{33}$$

and

$$\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{4s}\left|f_n^{(k-1-2s)}(y)\right|\tag{34}$$

converge absolutely and uniformly in $\overline{\Omega}$.

Proof. The series (34) we write as following series

$$\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{4s}\left|f_n^{(k-1-2s)}(y)\right|$$

$$=\sum_{n=1}^{\infty}\left[\left|f_n^{(k-1)}(y)\right|+\lambda_n^4\left|f_n^{(k-2)}(y)\right|\right]+\ldots+\sum_{n=1}^{\infty}\begin{cases}\lambda_n^{2k-4}\left|f_n'(y)\right|, & k=2m,\quad m\in\mathbb{N},\\ \lambda_n^{2k-6}\cdot\left|f_n''(y)\right|, & k=2m+1,\quad m\in\mathbb{N}.\end{cases}\tag{35}$$

Let k be an even number. If the series

$$\sum_{n=1}^{\infty}\lambda_n^{2k-4}\left|f_n'(y)\right|\tag{36}$$

converges, then the series (35) also converges. Integrating the Fourier coefficients (13) by parts $(2k-3)$ times, we have the equality

$$\left|f_n'(y)\right| = \frac{1}{\lambda_n^{2k-3}}\left|f_n^{(2k-3,1)}(y)\right|, \tag{37}$$

where

$$f_n^{(2k-3,1)}(y) = \int\limits_0^p \frac{\partial^{2k-2} f(x,y)}{\partial y \partial x^{2k-3}}\sqrt{\frac{2}{p}}\sin\Big((2k-3)\frac{\pi}{2}+\lambda_n x\Big)dx.$$

By virtue of the formula (37), the series (36) takes the form

$$\sum_{n=1}^{\infty}\lambda_n^{2k-4}\left|f_n'(y)\right| = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(2k-3,1)}(y)\right|.$$

Applying Cauchy–Schwartz inequality for the series on the right-hand side of the last equality, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(2k-3,1)}(y)\right| \le \frac{p}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|f_n^{(2k-3,1)}(y)\right|^2\right)^{1/2} = \frac{p}{\sqrt{6}}\left(\sum_{n=1}^{\infty}\left|f_n^{(2k-3,1)}(y)\right|^2\right)^{1/2}.$$

Applying Bessel's inequality, we find

$$\sum_{n=1}^{\infty}\lambda_n^{2k-4}\left|f_n'(y)\right| \le \frac{p}{\sqrt{6}}\left|\left|\frac{\partial^{2k-2} f}{\partial y \partial x^{2k-3}}\right|\right|_{L_2(\Omega)} < \infty.$$

The convergence of series (32)–(34) is proved for the even k.

Let k be an odd number. If the series $\sum\limits_{n=1}^{\infty}\lambda_n^{2k-6}\left|f_n''(y)\right|$ converges, then the series (35) also converges for odd k. The proof of this assertion is similar to the proof of the convergence of the series (36). Lemma 2 is proved. □

We also need in following two lemmas.

Lemma 3. *We suppose that* $f(x,y) \in W_2^{(2k-1,0)}(\Omega)$, $\frac{\partial^{2l} f(0,y)}{\partial x^{2l}} = \frac{\partial^{2l} f(p,y)}{\partial x^{2l}} = 0$, $l = \overline{0,k-1}$. *Then the following series*

$$\sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(q)\right|,\quad \sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(0)\right|,\quad \sum_{n=1}^{\infty}\lambda_n^{2k-2}\left|f_n(y)\right|$$

converge absolutely and uniformly in $\overline{\Omega}$.

Lemma . *Assume that* $f(x,y) \in W_2^{(2k+1,0)}(\Omega)$, $\frac{\partial^{2l} f(0,y)}{\partial x^{2l}} = \frac{\partial^{2l} f(p,y)}{\partial x^{2l}} = 0$, $l = \overline{0,k}$. *Then the series*

$$\sum_{n=1}^{\infty}\lambda_n^{2k}\left|\int\limits_0^q K_2(y,\tau)\, f_n(\tau)\, d\tau\right|$$

converges absolutely and uniformly in $\overline{\Omega}$.

The proofs of these last two lemmas are similar to those of the previous two lemmas. Therefore, we will not present their proofs here.

Theorem 2. *We suppose that* $f(x,y) \in W_2^{(2k+1,k-1)}(\Omega)$, $\frac{\partial^{2l} f(0,y)}{\partial x^{2l}} = \frac{\partial^{2l} f(p,y)}{\partial x^{2l}} = 0$, $l = \overline{0,k}$, $\frac{\partial^{2l+1} f(0,y)}{\partial y \partial x^{2l}} = \frac{\partial^{2l+1} f(p,y)}{\partial y \partial x^{2l}} = 0$, $l = \overline{0,k-1}$, $\frac{\partial^{2l+2} f(0,y)}{\partial y^2 \partial x^{2l}} = \frac{\partial^{2l+2} f(p,y)}{\partial y^2 \partial x^{2l}} = 0$, $l = \overline{0,k-3}$. *Then the series (12), (20)–(24) converge absolutely and uniformly in* $\overline{\Omega}$. *The solution (20) satisfies the given equation (1) and conditions (2)–(5).*

Proof. By virtue of the lemmas proved that the series (20)–(24) converge absolutely and uniformly. Subtracting the series (21) from (22), we make sure that the solution (20) satisfies differential equation (1). The conditions (2) and (3) are satisfied due to the properties of function $X_n(x)$. From the (23) and (24) we make sure that the conditions (4) and (5) are satisfied. Theorem 2 is proved. □

REFERENCES

1. A. N. Tikhonov, “On boundary conditions containing derivatives of order exceeding the order of equation,” Mat. Sb. **26** (1), 35–56 (1950).
2. A. V. Bitsadze, “On the Neumann problem for harmonic functions,” Dokl. Akad. Nauk SSSR **311**, 11–13 (1990).
3. I. I. Bavrin, “Operators for harmonic functions and their applications,” Differ. Uravn. **21** (1), 9–15 (1985).
4. V. V. Karachik and B. Kh. Turmetov, “On a problem for a harmonic equation,” Izv. Akad. Nauk UzSSR, Ser. Fyz.-Mat. Nauki, №4, 17–21 (1990).
5. V. B. Sokolovsky, “A generalization of the Neumann problem,” Differ. Uravn. **34**, 714–716 (1998).
6. R. R. Ashurov and A. T. Mukhiddinova, “Initial-boundary value problems for hyperbolic equations with an elliptic operator of arbitrary order,” Vestn. KRAUNC, Fiz.-Mat. Nauki **30** (1), 8–19 (2020).
7. O. Costin and S. Tanveer, “Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane,” Commun. Pure Appl. Math. **53**, 1067–1091 (2000).
8. V. A. Galaktionov, E. Mitidieri and S. I. Pohozaev, “Global sign-changing solutions of a higher order semilinear heat equation in the subcritical Fujita range,” Adv. Nonlin. Studies **12**, 569–596 (2012).
9. S. Hwang, “Kinetic decomposition for singularly perturbed higher order partial differential equations,” J. Differ. Equat. **200**, 191–205 (2004).
10. Sh. T. Karimov, “The Cauchy problem for the degenerated partial differential equation of the high even order,” Sib. Elektron. Mat. Izv. **15**, 853–862 (2018).
11. W. Littman, “Decay at infinity of solutions to higher order partial differential equations: Removal of the curvature assumption,” Israel J. Math. **8**, 403–407 (1970).
12. K. B. Sabitov, “The Dirichlet problem for higher-order partial differential equations,” Math. Notes **97**, 255–267 (2015).
13. J. Yan and C.-W. Shu, “Local discontinuous Galerkin methods for partial differential equations with higher order derivatives,” J. Sci. Comput. **17**, 27–47 (2002).
14. T. K. Yuldashev and B. J. Kadirkulov, “Inverse problem for a partial differential equation with Gerasimov–Caputo-type operator and degeneration,” Fractal Fract. **5** (2), 58-1–13 (2021).
15. T. K. Yuldashev and K. Kh. Shabadikov, “Mixed problem for a higher-order nonlinear pseudoparabolic equation,” J. Math. Sci. **254**, 776–787 (2021).
16. T. K. Yuldashev and K. Kh. Shabadikov, “Initial-value problem for a higher-order quasilinear partial differential equation,” J. Math. Sci. **254**, 811–822 (2021).
17. T. K. Yuldashev, “Integro-differential equation with a higher-order two-dimensional Whitham operator,” J. Math. Sci. **254**, 823–832 (2021).
18. H. Zhao, C. Zhu, and Z. Yu, “Existence and convergence of solutions to a singularly perturbed higher order partial differential equation,” Nonlin. Anal. **24**, 1435–1455 (1995).
19. D. Amanov, “On a generalization of the first initial boundary value problem for the heat conduction equation,” Contemp. Anal. Appl. Math. **2**, 88–97 (2014).
20. D. Amanov, G. Ibragimov, and A. Kilicman, “On a generalization of the initial-boundary problem for the vibrating string equation,” Symmetry **11** (73), 88–97 (2019).
21. T. D. Dzhuraev and A. Sopuev, *To the Theory of Partial Differential Equations of the Fourth Order* (Fan, Tashkent, 2000) [in Russian].
22. M. S. Salakhitdinov and D. Amanov, “Solvability and spectral properties of a self-adjoint problem for a fourth-order equation,” Uzbek. Mat. Zh., №3, 72–77 (2005).
23. D. Amanov and A. V. Yuldasheva, “Solvability and spectral properties of a self-adjoint problem for a fourth-order equation,” Uzbek. Mat. Zh., №4, 3–8 (2007).
24. D. Amanov and M. B. Murzambetova, “A boundary value problem for a fourth order equation with a lower term,” Vestn. Udmurt. Univ., Mat. Mekh. Komp’yut. Nauki, №1, 3–10 (2013).
25. D. Amanov, “On a nonlocal problem for the heat equation,” Uzbek. Math. J., No. 2, 21–25 (2016).
26. O. Sh. Kilichov, “A boundary value problem for a fourth order equation,” Bull. Inst. Math. **4** (2), 61–69 (2021).
27. D. Amanov, “Solvability and spectral properties of the boundary value problem for degenerating higher order parabolic equation,” Appl. Math. Comput. **268**, 1282–1291 (2015).
28. T. K. Yuldashev, “Nonlocal mixed-value problem for a Boussinesq-type integro-differential equation with degenerate kernel,” Ukr. Math. J. **68**, 1278–1296 (2016).
29. T. K. Yuldashev, “Mixed problem for pseudoparabolic integro-differential equation with degenerate kernel,” Differ. Equat. **53**, 99–108 (2017).
30. E. I. Moiseev, “On the solution of a nonlocal boundary value problem by the spectral method,” Differ. Uravn. **35**, 1094–1100 (1999).