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On the Solvability of the Dirichlet Problem for the Heat Equation in a Degenerating Domain

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Abstract—A domain, degenerating at the initial moment of time, is considered. A boundary value problem of heat conduction in this domain is studied. By virtue of the isotropy property, the solvability theorems for given boundary value problem are established in weight spaces of essentially bounded functions. The proof of the theorems is based on the solvability conditions of a nonhomogeneous integral equation of the third kind. Using the Fourier series method, the problem splits into families of boundary value problems. The method of representation of the solution to the boundary value problem in the form of sum of constructed thermal potentials is used. The given problem is reduced to the problems of solvability of integral equations. In addition, the solvability theorems for the boundary value problems are proved also for the case, when the axial symmetry property is absent.

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1. INTRODUCTION

In modern electrical devices, super-strong and ultra-weak currents are used very often, so there is a need to study new phenomena that were not previously observed when using currents of the normal, medium range. For example, it was experimentally established that when the contacts of the current circuit breakers open, a liquid-metal bridge appears for a short time, which significantly affects the erosion of the contact material [1]. In [2], a mathematical model is presented that describes the transitional phenomena accompanying a vacuum short arc at the initial stage of contact opening. This allows the authors to describe the evolution of the transitional short anode dominant arc, which appears immediately after the fracture of the molten bridge.

Earlier, the boundary value problems (BVP) of heat conduction in one-dimensional degenerating domains is studied (see [3–6]). The application of the method of thermal potentials [7] allows us to reduce the boundary-value problem with a moving boundary to an Volterra type integral equation of the second kind. The integral equation was singular, since the corresponding homogeneous equation (and hence the original homogeneous boundary-value problem) had nonzero solutions [5, 6, 8, 9]. Moreover, the method of successive approximations is not applicable to solve the integral equation in our case. So, we have applied the Carleman–Vekua regularization method. The existence of a solution of the considered problem is reduced to the investigation of a singular integral equation. The unique solvability of BVP is also considered in [10, 11]. In [12], the BVP is equivalently reduced (in the meaning uniqueness and existence of the solution) to Volterra integral equation of the second kind. Note that integral equations with similar singularities arise in the study of BVP with loaded equations or problems with boundary value conditions containing the derivatives [13, 14].

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In this paper, we study a two-dimensional BVP with respect to spatial variables in an inverted cone $G = \{(x;y,t) : x^2 + y^2 < t^2, 0 < t < T\}$ for the equation

$$\frac{\partial u(x,y,t)}{\partial t} = a^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) \tag{1}$$

on the lateral surface of the cone

$$u(x,y,t) = u_c(x,y,t), \quad \sqrt{x^2+y^2} = t, \quad 0 < t < T, \tag{2}$$

where $u_c(x,y,t)$ is a given function.

The BVP (1)–(2) simulates the temperature field in a plasma body of an electrical discharge between high-voltage disconnecting contacts [15]. These contacts were initially in the closed state. Taking into account the short duration of the process, there are no instruments that can measure the specified temperature field. It is necessary, at least qualitatively, to evaluate the nature of the carrying out of these thermal processes using methods of mathematical modeling.

In [16], the solvability of BVP (1), (2) in the case, when the isotropy property on an angular coordinate holds, is studied. In other word, a BVP of heat conduction in polar coordinates is studied in [16]. The method of representation of the solution to the BVP in the form of a sum of constructed thermal potentials is used. The problem was reduced to the study of a degenerating Abel integral equation. The fundamental solution of the auxiliary initial-boundary value problem for the thermal potentials is studied in [17]. In this case, the BVP for the ordinary differential equation is obtained. In [18], using the method of the degenerate kernel, the BVP is also integrated as an ordinary differential equation.

This paper consists of an introduction, two sections, and a conclusion. In the first section, we present results on the solvability issues for boundary value problem (1), (2) in the case of the isotropy property. These results are stated in Theorem 2 (classes of solutions to the BVP of heat conduction in polar coordinates) and Theorem 3 (classes of solutions to BVP (1), (2). The proofs of these theorems are based on Theorem 1 regarding the solvability of integral equation to which the posed boundary value problem is reduced. In [16] these results are presented without proofs. In this paper, we give a complete proof of Theorem 3.

The second section provides a process for solving the original BVP that is a process for solving the BVP in the absence of axial symmetry. To solve the problem, it is split into two families of BVP, for which the solvability issues were studied in the previous section. The results of this section are formulated as two theorems (Theorem 4 and Theorem 5).

2. SOME BACKGROUND

In this section we consider the case of the isotropy property on an angular coordinate for BVP (1), (2). Then the BVP is reduced to an integral equation ([16]) and solvability theorems are formulated for the obtained integral equation and the posed boundary value problem.

2.1. Reducing a Boundary Value Problem to an Integral Equation

Converting to polar coordinates in problem (1), (2) and assuming that the isotropy property is fulfilled along the angular coordinate (case of axial symmetry), we encounter the following problem:

In the domain $\Omega = \{(r,t) : 0 < r < t, 0 < t < T\}$, to find a solution to BVP:

$$\frac{\partial u(r,t)}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u(r,t)}{\partial r} \right), \tag{3}$$

$$\lim_{r\to 0} \frac{u(r,t)}{\ln(1/r)} = u_0(t), \quad 0 < t < T, \tag{4}$$

$$\lim_{r\to t} u(r,t) = u_1(t) \equiv u_c(x,y,t)\big|_{\sqrt{x^2+y^2}=t}, \quad 0 < t < T. \tag{5}$$

Usually, instead of condition (4), a limit relation on the boundedness of the solution is required, that is, $|u(r,t)| \neq \infty$ as $r \to 0$. We assume that the solution $u(r,t)$ has a singularity as $r \to 0$, that is, we assume that $u(r,t)$ may have some growth order as $r \to 0$. We associate this assumption with the property of a fundamental solution to the Laplace operator in the center of the circle. Thus, we admit the presence of some growth property of the required solution $u(r,t)$ to equation (3) (this will be specified below in Theorems 2 and 3.

As is known (see [19, p. 76, Problem 1.2.2-7]), the function

$$G(r,\xi,t) = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2}{4a^2t}\right\} I_0\left(\frac{r\xi}{2a^2t}\right)$$

is a fundamental solution to equation (3), where ξ is a parameter. $I_\mu(\eta)$ is the modified Bessel function. Using Green's formula as in [7, p. 476–480], we write the integral representation of the solution to equation (3). Satisfying this solution representation to conditions (4), (5), we get a degenerating Abel integral equation (an integral equation of the third kind) [16]:

$$t\varphi(t) - \frac{\lambda}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\varphi(\tau)d\tau = f(t), \quad 0 < t < T, \quad \lambda = \frac{a}{2}, \tag{6}$$

where the right-hand side $f(t)$ of equation (6) contains a unknown function $\varphi(t)$.

Equations of the form (6) have been the subject of study for many authors. Here, we indicate only the following papers [20–23] and note the numerous studies that are cited in them. In [22], the unique solvability of the integral equation is established, under the assumption that a order of degeneracy is strictly less than unity when degeneracy is determined by the degree of the independent variable t. In '[20], an operator acts in the space of quadratically summable functions. Necessary and sufficient conditions are established for representation of the operator as the sum of an operator of multiplication by a bounded function and an integral operator. This sum is called the integral operator of the third kind.

In [21, 23–25] some Volterra-type integral equations are considered. Kernels of the integral operators do not have singularities. Kernels of the integral operators with singularities are considered in [5].

The order of degeneracy in equation (6) is equal to unity, and the kernel of the integral operator has a weak singularity and determines the Abel integral operator. We had studied [16] the solvability of equation (6) in the weight class of essentially bounded functions. In general, when studying BVP, approach based on reduction the problem to integral equations is used often. In [26] the existence of a regular solution of the Gellerstedt spectral problem is also proved by the method of integral equations.

2.2. Solvability Theorems

The theorem on the solvability of integral equation (6) is valid:

Theorem 1. *Let $t^{-1/2}f(t) \in L_\infty(0,T)$. Then integral equation (6) has a general solution*

$$\varphi(t) = C\varphi_{hom}(t) + \varphi_{part}(t) \in L_\infty((0,T); t^{-1/2}),$$

i.e. $t^{-1/2}\varphi(t) \in L_\infty(0,T)$, where $C = const$, $\varphi_{1hom}(t)$ and $\varphi_{1part}(t)$ are solutions to homogeneous (when $f(t) \equiv 0$) and nonhomogeneous integral equations (6), respectively.

In [16], using the assertion of Theorem 1, we find the class of solutions to boundary value problem (3)–(5).

Theorem 2. *Let $t^{-1}u_0(t)$, $t^{-1/2}u_1(t) \in L_\infty(0,T)$. Then, BVP (3)–(5) has a general solution*

$$u(r,t) = Cu_{hom}(r,t) + u_{part}(r,t) \in L_\infty(\Omega; r^{1/2}),$$

i.e. $r^{1/2}u(r,t) \in L_\infty(\Omega)$, where $C = const$, $u_{hom}(r,t)$ and $u_{part}(r,t)$ are solutions to homogeneous (when $u_0(t) \equiv 0$, $u_1(t) \equiv 0$) and nonhomogeneous boundary value problems (3)–(5), respectively.

For the axisymmetric case, the following result follows from Theorem 2.

Theorem 3. *Let $t^{-1/2}u_1(t) \equiv t^{-1/2}u_c(x,y,t)\big|_{\sqrt{x^2+y^2}=t} \in L_\infty(0,T)$. Then, BVP (1), (2) has a general solution*

$$u(x,y,t) = Cu_{hom}(x,y,t) + u_{part}(x,y,t) \in L_\infty(G; (x^2+y^2)^{1/4}),$$

i.e. $(x^2+y^2)^{1/4}u(x,y,t) \in L_\infty(G)$, *where* $C = const$, $u_{hom}(x,y,t)$ *and* $u_{part}(x,y,t)$ *are solutions to (homogeneous, when* $u_c(x,y,t) \equiv 0$*) nonhomogeneous boundary value problems (1), (2).*

The proofs of these two theorems are based on Theorem 1 regarding the solvability of integral equation (6).

2.3. Proof of Solvability Theorem for Integral Equation (6)

We study solvability issues in the class of essentially bounded functions $\psi(t) \in L_\infty(0;+\infty)$ for the following degenerating Abel's equation (integral equation of the third kind):

$$t\psi(t) - \frac{\lambda}{\sqrt{\pi}} \int\limits_0^t \frac{\psi(\tau)d\tau}{\sqrt{t-\tau}} = F(t), \quad t > 0, \tag{7}$$

where λ is a given positive constant and $F(t)$ is a given function such that $F(t)/t \in L_\infty(0;+\infty)$.

2.3.1. Solution to a homogeneous integral equation for (7). The homogeneous integral equation

$$t\psi(t) - \frac{\lambda}{\sqrt{\pi}} \int\limits_0^t \frac{1}{\sqrt{t-\tau}}\psi(\tau)d\tau = 0, \quad \psi(t) \in L_\infty(0;+\infty), \tag{8}$$

along with a trivial solution has a nontrivial solution

$$\psi_{hom}(t) = \frac{\lambda}{\sqrt{\pi}} t^{-3/2} \exp\left\{-\frac{\lambda^2}{t}\right\}, \quad t > 0, \tag{9}$$

up to a constant factor. Indeed, applying the Laplace transform to equation (8), we obtain

$$-\frac{d\hat{\psi}_{hom}(p)}{dp} - \lambda\frac{1}{\sqrt{p}}\hat{\psi}_{hom}(p) = 0, \tag{10}$$

where $\hat{\psi}_{hom}(p)$ is the Laplace image of the function $\psi_{hom}(t)$.

A solution to equation (10) is determined by the formula

$$\hat{\psi}_{hom}(p) = \exp\left\{-2\lambda\sqrt{p}\right\}. \tag{11}$$

A simple substitution shows that function (9) really satisfies the homogeneous integral equation (8).

2.3.2. Particular solution to the nonhomogeneous integral equation (7): Construction of the resolvent. Applying the Laplace transform to equation (7), we obtain

$$-\frac{d\hat{\psi}_{part}(p)}{dp} - \lambda\frac{1}{\sqrt{p}}\hat{\psi}_{part}(p) = \hat{F}(p), \tag{12}$$

where $\hat{\psi}_{part}(p)$ is the Laplace image of the function $\psi_{part}(t)$. Using the solution (11) of homogeneous differential equation (10), by the method of variation of a constant we find a particular solution $\hat{\psi}_{part}(p)$ of the nonhomogeneous equation (12):

$$\hat{\psi}_{part}(p) = C(p)\exp\left\{-2\lambda\sqrt{p}\right\}, \tag{13}$$

where from (12), it follows that

$$C(p) = \int\limits_p^\infty \hat{F}(q)\exp\left\{2\lambda\sqrt{q}\right\}dq. \tag{14}$$

Hence, from (13), (14), for the Laplace image of the particular solution, we obtain

$$\hat{\psi}_{part}(p) = \int\limits_0^\infty \hat{R}^*(p,\tau)F(\tau)d\tau, \tag{15}$$

where

$$\hat{F}(q) = \int\limits_0^{\infty} F(\tau) \exp\{-q\tau\}\, d\tau,$$

$$\hat{R}^*(p,\tau) = \exp\{-2\lambda\sqrt{p}\} \int\limits_p^{\infty} \exp\{-q\tau + 2\lambda\sqrt{q}\}\, dq = \exp\{-2\lambda\sqrt{p}\}\, \hat{I}(p,\tau). \tag{16}$$

Integrating in parts and using [27, p. 336, formula 3.322.1] for $\hat{I}(p,\tau)$, we obtain

$$\hat{I}(p,\tau) = \int\limits_p^{\infty} \exp\{-q\tau + 2\lambda\sqrt{q}\}\, dq$$

$$= \frac{\exp\left\{-\tau p + 2\lambda\sqrt{p}\right\}}{\tau} + \lambda\sqrt{\pi}\frac{\exp\left\{\lambda^2/\tau\right\}}{\tau^{3/2}} \operatorname{erfc}\left(\sqrt{\tau p} - \lambda/\sqrt{\tau}\right). \tag{17}$$

From (16), (17), we obtain

$$\hat{R}^*(p,\tau) = \frac{\exp\{-\tau p\}}{\tau} + \hat{R}(p,\tau), \tag{18}$$

where

$$\hat{R}(p,\tau) = \exp\{-2\lambda\sqrt{p}\}\left[\frac{\lambda\sqrt{\pi}\exp\left\{\lambda^2/\tau\right\}}{\tau^{3/2}} \operatorname{erfc}\left(\sqrt{\tau p} - \lambda/\sqrt{\tau}\right)\right]. \tag{19}$$

We now find an original for the image $\hat{R}(p,\tau)$ (19). Obviously, the required original can be found as a convolution of originals of the following two images $\exp\left\{-2\lambda\sqrt{p}\right\}$ and $\hat{E}(p,\tau) = \operatorname{erfc}\left(\sqrt{\tau p} - \lambda/\sqrt{\tau}\right)$.

Applying Jordan's Lemma ([28, p. 410–412], we have

$$\hat{E}(p,\tau) \div E(t,\tau) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \exp\{pt\} \operatorname{erfc}\left(\sqrt{\tau p} - \lambda/\sqrt{\tau}\right) dp$$

$$= \left\| \begin{matrix} I:\ p = xe^{-i\pi} = -x, & II:\ p = xe^{i\pi} = -x, \\ \sqrt{p} = -i\sqrt{x}, & \sqrt{p} = i\sqrt{x} \\ dp = -dx & dp = -dx \end{matrix} \right\|$$

$$= \lim_{r\to 0}\lim_{R\to\infty}\left[\frac{1}{2\pi i}\int\limits_R^r \exp\{-xt\} \operatorname{erfc}\left(-i\sqrt{\tau x} - \lambda/\sqrt{\tau}\right)(-dx)\right.$$

$$\left. + \frac{1}{2\pi i}\int\limits_r^R \exp\{-xt\} \operatorname{erfc}\left(i\sqrt{\tau x} - \lambda/\sqrt{\tau}\right)(-dx)\right]$$

$$= \frac{i}{2\pi}\int\limits_0^{\infty} \exp\{-xt\}\left[\operatorname{erfc}\left(i\sqrt{\tau x} - \lambda/\sqrt{\tau}\right) - \operatorname{erfc}\left(-i\sqrt{\tau x} - \lambda/\sqrt{\tau}\right)\right] dx$$

$$= \left\| \begin{matrix} \text{the replacement} z = \sqrt{x}, \\ \text{we use}([29, Vol.\ 2, formula 1.5.3.10]) \end{matrix} \right\|$$

$$= \frac{i}{\pi}\int\limits_0^\infty z\exp\left\{-tz^2\right\}\left[\operatorname{erfc}\left(i\sqrt{\tau}z-\lambda/\sqrt{\tau}\right)-\operatorname{erfc}\left(-i\sqrt{\tau}z-\lambda/\sqrt{\tau}\right)\right]dz$$

$$= \frac{i}{\pi}\Bigg[-\frac{i\sqrt{\tau}}{2t\sqrt{t-\tau}}\exp\left\{-\frac{\lambda^2 t}{\tau(t-\tau)}\right\}\operatorname{erf}\left(z\sqrt{t-\tau}-\frac{i\lambda}{\sqrt{t-\tau}}\right)$$

$$-\frac{1}{2t}\exp\left\{-tz^2\right\}\operatorname{erfc}\left(zi\sqrt{\tau}-\frac{\lambda}{\sqrt{\tau}}\right)-\frac{i\sqrt{\tau}}{2t\sqrt{t-\tau}}\exp\left\{-\frac{\lambda^2 t}{\tau(t-\tau)}\right\}$$

$$\times\operatorname{erf}\left(z\sqrt{t-\tau}+\frac{i\lambda}{\sqrt{t-\tau}}\right)+\frac{1}{2t}\exp\left\{-tz^2\right\}\operatorname{erfc}\left(-zi\sqrt{\tau}-\frac{\lambda}{\sqrt{\tau}}\right)\Bigg]\Bigg|_{z=0}^{z=\infty}$$

$$= \left\|\begin{array}{c}\text{in the last relation, the terms with} \operatorname{erfc}(\cdot) \quad \text{are excluded,}\\ \text{since first, as}\quad z\to\infty,\quad \text{they become equal to zero}\\ = \text{and second, as}\quad z\to 0,\quad \text{they are mutually annihilating terms}\end{array}\right\|$$

$$= \frac{\sqrt{\tau}}{\pi t\sqrt{t-\tau}}\exp\left\{-\frac{\lambda^2 t}{\tau(t-\tau)}\right\}-\frac{\sqrt{\tau}}{2\pi t\sqrt{t-\tau}}\exp\left\{-\frac{\lambda^2 t}{\tau(t-\tau)}\right\}$$

$$\times\left[\operatorname{erf}\left(-\frac{i\lambda}{\sqrt{t-\tau}}\right)+\operatorname{erf}\left(\frac{i\lambda}{\sqrt{t-\tau}}\right)\right]=\frac{\sqrt{\tau}}{\pi t\sqrt{t-\tau}}\exp\left\{-\frac{\lambda^2 t}{\tau(t-\tau)}\right\}.$$

Hence (according to the replacement $z_1 = iz$) the following equality is true:

$$\operatorname{erf}(-iz)+\operatorname{erf}(iz)=\frac{2i}{\sqrt{\pi}}\left[\int\limits_0^{-z}\exp\left\{z_1^2\right\}dz_1+\int\limits_0^{z}\exp\left\{z_1^2\right\}dz_1\right]=0.$$

To obtain the resolvent of the studied equation, we write the above formulas for inverting the images of the Laplace transform:

1^0. $\exp\{-\tau p\} \div \delta(t-\tau)$,

2^0. $\exp\left\{-2\lambda\sqrt{p}\right\} \div \dfrac{\lambda\exp\left\{-\lambda^2/t\right\}}{\sqrt{\pi}t^{3/2}}$,

3^0. $\dfrac{\lambda\sqrt{\pi}}{\tau^{3/2}}\exp\left\{\lambda^2/\tau\right\}\operatorname{erfc}\left(\sqrt{\tau p}-\lambda/\tau^{3/2}\right) \div \dfrac{\lambda}{\sqrt{\pi}t\tau\sqrt{t-\tau}}\exp\left\{-\dfrac{\lambda^2}{t-\tau}\right\}$.

In addition, from (18), (19) and 1^0–3^0, we obtain

$$R^*(t,\tau)=\frac{\delta(t-\tau)}{\tau}+R(t,\tau), \tag{20}$$

where

$$R(t,\tau)=\frac{\lambda^2}{\pi\tau}\int\limits_\tau^t\frac{\exp\left\{-\frac{\lambda^2}{t-\varsigma}\right\}}{(t-\varsigma)^{3/2}}\cdot\frac{\exp\left\{-\frac{\lambda^2}{\varsigma-\tau}\right\}}{\varsigma\sqrt{\varsigma-\tau}}d\varsigma=\left\|\text{a replacement} z=\sqrt{\frac{\varsigma-\tau}{t-\varsigma}}\right\|$$

$$=\frac{1}{t\tau}\left\{\frac{2\lambda^2}{\pi(t-\tau)}\exp\left\{-\frac{2\lambda^2}{t-\tau}\right\}\int\limits_0^\infty\left[1+\left(1-\frac{\tau}{t}\right)\left(z^2+\frac{\tau}{t}\right)^{-1}\right]\exp\left\{-\frac{\lambda^2}{t-\tau}\left(z^2+\frac{1}{z^2}\right)\right\}dz\right\}$$

$$=\frac{1}{t\tau}\left[\frac{\lambda}{\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{4\lambda^2}{t-\tau}\right\}+\frac{2\lambda^2}{\pi t}\exp\left\{-\frac{2\lambda^2}{t-\tau}\right\}g(t,\tau)\right]. \tag{21}$$

We calculate in (21) the function $g(t,\tau)$:

$$g(t,\tau)=\int_0^\infty \frac{1}{z^2+\frac{\tau}{t}}\exp\left\{-\frac{\lambda^2}{t-\tau}\left(z^2+\frac{1}{z^2}\right)\right\}dz=\left|\left|\text{a replacement} x=z^2\right|\right|$$

$$=\int_0^\infty \frac{1}{2\sqrt{x}\left(x+\frac{\tau}{t}\right)}\exp\left\{-\frac{\lambda^2}{t-\tau}\left(x+\frac{1}{x}\right)\right\}dx$$

$$=\frac{1}{2}\frac{\pi\sqrt{t}}{\sqrt{\tau}}\exp\left\{\frac{\lambda^2}{t-\tau}\left(\frac{\tau}{t}+\frac{t}{\tau}\right)\right\}\operatorname{erfc}\left[\frac{\lambda}{\sqrt{t-\tau}}\left(\sqrt{\frac{\tau}{t}}+\sqrt{\frac{t}{\tau}}\right)\right]. \tag{22}$$

Here, we have used formula 2.3.16.4 from [30, Vol. 1, p. 277]. Substituting (22) into formula (21), from (20) we obtain

$$R^*(t,\tau)=\frac{1}{t}\left[\delta(t-\tau)+\frac{\lambda}{t\tau\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{4\lambda^2}{t-\tau}\right\}\right.$$
$$\left.+\frac{\lambda^2}{\sqrt{t}\tau^{3/2}}\exp\left\{\frac{\lambda^2}{t-\tau}\left(\sqrt{\frac{\tau}{t}}-\sqrt{\frac{t}{\tau}}\right)^2\right\}\operatorname{erfc}\left\{\frac{\lambda}{\sqrt{t-\tau}}\left(\sqrt{\frac{\tau}{t}}+\sqrt{\frac{t}{\tau}}\right)\right\}\right]. \tag{23}$$

According to relations (15), (23), we obtain the following result.

A particular solution $\psi_{part}(t)$ of the nonhomogeneous integral Abel equation (7) is determined by the relation

$$\psi_{part}(t)=\frac{F(t)}{t}+\int_0^t[\tau R(t,\tau)]\frac{F(\tau)}{\tau}d\tau, \tag{24}$$

where $R(t,\tau)$ and a resolvent $R^*(t,\tau)$ are defined according to the relations (20)–(23).

2.3.3. Estimate of the resolvent. According to the formulas (21) and (23)–(24), we represent the resolvent in the form

$$\tau R(t,\tau)=\tau R_1(t,\tau)+\tau R_2(t,\tau),\quad 0<\tau<t<\infty, \tag{25}$$

where

$$\tau R_1(t,\tau)=\frac{\lambda}{\sqrt{\pi}t^2\sqrt{t-\tau}}\exp\left\{-\frac{4\lambda^2}{t-\tau}\right\};$$

$$\tau R_2(t,\tau)=\frac{\lambda^2}{t^{3/2}\tau^{1/2}}\exp\left\{\frac{\lambda^2}{t-\tau}\left(\sqrt{\frac{\tau}{t}}-\sqrt{\frac{t}{\tau}}\right)^2\right\}\operatorname{erfc}\left\{\frac{\lambda}{\sqrt{t-\tau}}\left(\sqrt{\frac{\tau}{t}}+\sqrt{\frac{t}{\tau}}\right)\right\}.$$

The following result takes place.

For all $\{(t,\tau):0<\tau<t<\infty\}$, the following estimates hold:

$$\tau R_1(t,\tau)\le\frac{C\lambda}{\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{2\lambda^2}{t-\tau}\right\},\quad 0<\tau<t<\infty,$$

$$\tau R_2(t,\tau)\le\frac{C\lambda}{\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{2\lambda^2}{t-\tau}\right\},\quad 0<\tau<t<\infty,$$

for resolvent $\tau R(t,\tau)$ (25). Indeed,

$$\tau R_1(t,\tau)=\frac{\lambda}{\sqrt{\pi}\cdot t^2\sqrt{t-\tau}}\exp\left\{-\frac{4\lambda^2}{t-\tau}\right\}\le\frac{\lambda}{\sqrt{\pi}\cdot(t-\tau)^2\sqrt{t-\tau}}$$
$$\times\exp\left\{-\frac{2\lambda^2}{t-\tau}-\frac{2\lambda^2}{t-\tau}\right\}\le\frac{C_1\lambda}{\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{2\lambda^2}{t-\tau}\right\},$$

where $C_1 = const$ and

$$\tau R_2(t,\tau) = \frac{\lambda^2}{t^{3/2}\tau^{1/2}} \exp\left\{\frac{\lambda^2}{t-\tau}\left(\sqrt{\frac{\tau}{t}} - \sqrt{\frac{t}{\tau}}\right)^2\right\} \operatorname{erfc}\left\{\frac{\lambda}{\sqrt{t-\tau}}\left(\sqrt{\frac{\tau}{t}} + \sqrt{\frac{t}{\tau}}\right)\right\}$$

$$= \frac{\lambda^2}{t^{3/2}\tau^{1/2}} \exp\left\{\frac{\lambda^2}{t-\tau}\left(\frac{(t+\tau)^2}{t\tau} - 4\right)\right\} \operatorname{erfc}\left\{\frac{\lambda(t+\tau)}{\sqrt{t\tau(t-\tau)}}\right\}$$

$$= \frac{\lambda^2}{t^{3/2}\tau^{1/2}} \exp\left\{-\frac{4\lambda^2}{t-\tau}\right\} \exp\left\{\frac{\lambda^2}{t-\tau}\frac{(t+\tau)^2}{t\tau}\right\} \operatorname{erfc}\left\{\frac{\lambda(t+\tau)}{\sqrt{t\tau(t-\tau)}}\right\}$$

$$= \frac{\lambda^2}{t^{3/2}\tau^{1/2}\sqrt{\pi}} \exp\left\{-\frac{4\lambda^2}{t-\tau}\right\} \frac{\sqrt{t\tau(t-\tau)}}{\lambda(t+\tau)} E\left(\frac{\lambda(t+\tau)}{\sqrt{t\tau(t-\tau)}}\right),$$

where $E(\alpha) = \sqrt{\pi}\alpha \operatorname{erfc}(\alpha)\exp(\alpha^2)$.

Here, we need to consider two kinds of features of the function $R_2(t,\tau)$: first, as $\tau \to t$ and $t > 0$, and second, as $t \to 0$. In both cases, we use the asymptotics ([7, p. 718]:

$$\operatorname{erfc}(\alpha) \approx \frac{\exp\{-\alpha^2\}}{\sqrt{\pi}\alpha} \quad \text{as} \quad \alpha \to \infty. \tag{26}$$

Since by virtue of (26), $E(\alpha) \to 1$, $\alpha \to +\infty$, and $|E(\alpha)| < M = const\, \forall\tau > 0, t > 0$. Then,

$$\tau R_2(t,\tau) \le \frac{M\lambda}{\sqrt{\pi}} \frac{1}{t(t+\tau)} \exp\left\{-\frac{2\lambda^2}{t-\tau} - \frac{2\lambda^2}{t-\tau}\right\}$$

$$\le \frac{M\lambda}{\sqrt{\pi}} \frac{1}{(t-\tau)^2} \exp\left\{-\frac{2\lambda^2}{t-\tau} - \frac{2\lambda^2}{t-\tau}\right\} \le \frac{C_2\lambda}{\sqrt{\pi}(t-\tau)^{3/2}} \exp\left\{-\frac{2\lambda^2}{t-\tau}\right\},$$

where $C_2 = const$.

The application of Lemmas 2.1–2.4 from [16] completes the proof of Theorem 3.

3. FURTHER RESULTS: BVP (1), (2) IN A DEGENERATING DOMAIN IN THE CASE OF ANISOTROPY OF THE HEAT CONDUCTION ALONG AN ANGULAR COORDINATE

In this case, boundary value problem (3)–(5) takes the following form: in the domain $\Omega_1 = \{(r,\theta,t) : 0 < r < t,\ 0 \le \theta < 2\pi,\ 0 < t < T\}$, find a solution to the equation

$$\frac{\partial u(r,\theta,t)}{\partial t} = a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,\theta,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u(r,\theta,t)}{\partial\theta^2}\right] \tag{27}$$

satisfying the boundary conditions

$$\lim_{r\to 0}\frac{u(r,\theta,t)}{\ln(1/r)} = u_0(t), \quad 0 \le \theta < 2\pi, 0 < t < T, \tag{28}$$

$$\lim_{r\to t} u(r,\theta,t) = u_1(\theta,t) \equiv u_c(x,y,t)\big|_{\sqrt{x^2+y^2}=t}, \quad \{\theta,t\} \in \partial\Omega_1, \tag{29}$$

where $\partial\Omega_1$ is a lateral surface of the cone.

We apply the Fourier series method (variable separation method) to boundary value problem (27)–(29). So, we seek the required solution $u(r,\theta,t)$ in the form

$$u(r,\theta,t) = U(r,t)\Theta(\theta). \tag{30}$$

Substituting (30) into (27)–(29), we obtain

$$\frac{r^2}{a^2}\left[\frac{\partial U(r,t)}{\partial t} - \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U(r,t)}{\partial r}\right)\right]\Big/U(r,t) = \frac{d^2\Theta(\theta)}{d\theta^2}\Big/\Theta(\theta) = -\lambda. \tag{31}$$

The solution to the spectral problem:

$$\frac{d^2\Theta(\theta)}{d\theta^2} = -\lambda\Theta(\theta), \quad \Theta^j(0) = \Theta^j(2\pi), \quad j = 0, 1 \tag{32}$$

is a system of eigenfunctions and eigenvalues:

$$\Theta_n(\theta) = \exp\{in\theta\}, \quad \lambda_n = n^2, \quad n \in \mathbf{Z} \equiv \{0, \pm 1, \pm 2, ...\}. \tag{33}$$

As a result, we obtain

$$u(r, \theta, t) = \sum_{n\in\mathbf{Z}} U_n(r, t)\Theta_n(\theta). \tag{34}$$

Furthermore, taking into account (31)–(34) from (27)–(29), we obtain a family of boundary value problems for the heat equation:

$$\frac{\partial v_n(r,t)}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_n(r,t)}{\partial r}\right), \quad n \in \mathbf{Z}, \tag{35}$$

$$\lim_{r\to 0}\frac{v_0(r,t)}{\ln(1/r)} = u_0(t), \quad \frac{v_n(r,t)}{\ln(1/r)} = 0, \quad n \in \mathbf{Z}\setminus\{0\}, \quad 0 < t < T, \tag{36}$$

$$\lim_{r\to t} v_n(r,t) = v_{1n}(t), \quad n \in \mathbf{Z}, \quad 0 < t < T, \tag{37}$$

where

$$v_n(r,t) = U_n(r,t)\exp\left\{\frac{a^2n^2}{r^2}t\right\},$$

$$v_{1n}(t) = u_{1n}(t)\exp\left\{\frac{a^2n^2}{t}\right\}, \quad u_{1n}(t) = \int_0^{2\pi} u_1(\theta, t)\exp\{in\theta\}d\theta.$$

Thus, we obtain a family of boundary value problems (35)–(37), each of which is a boundary value problem of the form (3)–(5). The solvability of these boundary value problems has already been considered (Theorems 2 and 3). According to the assertion of Theorem 2, we find their solutions $\{v_n(r,t),\ n \in \mathbf{Z}\}$, and further, using (33)–(34), we formally construct the series

$$u(r,\theta,t) = \sum_{n\in\mathbf{Z}} U_n(r,t)\exp\{in\theta\} = \sum_{n\in\mathbf{Z}} v_n(r,t)\exp\left\{-\frac{a^2n^2}{r^2}t + in\theta\right\}. \tag{38}$$

According to Theorem 2, the solutions of boundary value problems (35)–(37) satisfy the estimate

$$v_n(r,t) \le Cr^{-1/2}, \quad \text{where the constant} \quad C \quad \text{is independent of} \quad n,$$

and a series (formula 5.4.11.2 from [30, p. 585]):

$$\sum_{n\in\mathbf{Z}} \exp\left\{-\frac{a^2n^2}{r^2}t + in\theta\right\}$$

converges for $\forall t > 0$. Thus, due to the convergence of series (38) and according to the assertion of Theorem 2, we obtain the following estimate:

$$|u(r,\theta,t)| \le Cr^{-1/2} \text{for} \forall \{r,\theta,t\} \in \Omega_1.$$

Remark. For equation (27), the fundamental solution is a function [19, p. 181, Problem 2.2.2-1]:

$$G_1(r,\xi,\theta,t) = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2-2r\xi\cos\theta}{4a^2t}\right\}.$$

Then, for its averaging over the angular coordinate with weights

$$\{\sin(n\theta), \quad n \in \{1,2,3,...\}\} \quad \text{and} \quad \{\cos(n\theta), \quad n \in \{0,1,2,3,...\}\},$$

the following relations are valid:

$$\frac{1}{2\pi}\int\limits_0^{2\pi} G_1(r,\xi,\theta,t)\sin(n\theta)d\theta = 0, \quad n\in\{1,2,3,...\}, \tag{39}$$

$$\frac{1}{\pi}\int\limits_0^{\pi} G_1(r,\xi,\theta,t)\cos(n\theta)d\theta = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2}{4a^2t}\right\} I_n\left(\frac{r\xi}{2a^2t}\right), \tag{40}$$

$n\in\{0,1,2,3,...\}$.

Equalities (39) follow from a property of evenness in a variable θ of the function G_1, and equalities (40) are obtained using formula 2.5.40.3 from [30, p. 370] (in addition to the property of evenness). When $n=0$, this formula follows from the following remarkable identity: $\forall\eta\in[0,2\pi]$

$$\frac{1}{\pi}\int\limits_0^{\pi}\exp\left\{\frac{r\xi\cos(\theta-\eta)}{2a^2t}\right\}d\theta = \frac{1}{\pi}\int\limits_0^{\pi}\exp\left\{\frac{r\xi\cos\theta}{2a^2t}\right\}d\theta = I_0\left(\frac{r\xi}{2a^2t}\right).$$

Furthermore, from (39)–(40), we obtain for all $n\in\mathbf{Z}$

$$\frac{1}{2\pi}\int\limits_0^{2\pi} G_1(r,\xi,\theta,t)\exp\{in\theta\}d\theta = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2}{4a^2t}\right\} I_n\left(\frac{r\xi}{2a^2t}\right). \tag{41}$$

Thus, we find that each of the boundary problems (35)–(37) corresponds to its own fundamental solution, defined by formulas (41). Using the corresponding fundamental solutions (41), we can solve each of the BVP (35)–(37). Note that this way differs from the proof of Theorems 2 and 3. As a result, taking into account the fact that modified Bessel functions $\{I_n(x),\ n=0,1,2,...\}$ for large values of the argument x have the same asymptotics, we can obtain the same result that is formulated below theorem. Note that these asymptotics determine the characteristic parts of the integral equations that are usually defined in the Carleman–Vekua regularization method.

Thus, the following theorems are proved for the case where there is no condition of axial symmetry in the angular coordinate θ.

Theorem 4. *Let $t^{-1}u_0(t)$, $t^{-1/2}v_{1n}(t)\in L_\infty(0,T)$, $n\in\mathbf{Z}$. Then, each of the BVP (35)–(37) has a general solution $v_n(r,t)=Cv_{n,hom}(r,t)+v_{n,part}(r,t)\in L_\infty(\Omega;r^{1/2})$, i.e. $r^{1/2}v_n(r,t)\in L_\infty(\Omega)$, $n\in\mathbf{Z}$, where $C=const$, $v_{n,hom}(r,t)$ and $v_{n,part}(r,t)$ are solutions to (homogeneous, when $u_0(t)\equiv 0$, $v_{n,1}(t)\equiv 0$) nonhomogeneous boundary value problems (35)–(37).*

Theorem 5. *Let $t^{-1/2}u_1(\arctan(y/x),t)\big|_{\sqrt{x^2+y^2}=t} \equiv t^{-1/2}u_c(x,y,t)\big|_{\sqrt{x^2+y^2}=t}\in L_\infty(\partial\Omega_1)$. Then, BVP (1), (2) has a general solution*

$$u(x,y,t)=Cu_{hom}(x,y,t)+u_{part}(x,y,t)\in L_\infty(G;(x^2+y^2)^{1/4}),$$

i.e. $(x^2+y^2)^{1/4}u(x,y,t)\in L_\infty(G)$, where $C=const$, $u_{hom}(x,y,t)$ and $u_{part}(x,y,t)$ are solutions to (homogeneous, when $u_c(x,y,t)\equiv 0$) nonhomogeneous BVP (1)–(2).

4. CONCLUSION

By the axial symmetry, the initial BVP (1), (2) was reduced to the BVP (3)–(5). Furthermore, the last problem was reduced to solving the degenerating Abel integral equation of the second kind and the solvability of which was studied. We also noted that boundary condition (4) agrees with the found class of solutions $u(r,t)\le Cr^{-1/2}$, $\{r,t\}\in\Omega$.

We studied a general case where there is no property of axial symmetry. In this case, we proved that Theorems 4 and 5 establish the solvability of boundary value problem (1), (2). The results of this work are a continuation of the research from the work [16].

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