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On the Existence of Periodic Solutions to One Class of Systems of Nonlinear Differential Equations

G. V. Demidenko[1,2*]

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[1]*Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia*

[2]*Novosibirsk State University, Novosibirsk, 630090 Russia*

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Abstract—A class of systems of nonlinear differential equations is considered. It is assumed that the linear part of the system has constant coefficients and is exponentially dichotomous. Conditions for the existence of periodic solutions are established and their stability is proved for small perturbations of the coefficients of the linear part and nonlinear terms.

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1. INTRODUCTION

In the present paper, we consider systems of nonlinear differential equations

$$\frac{dy}{dt} = Ay + f(t, y), \quad -\infty < t < \infty, \tag{1}$$

where A is $(n \times n)$-matrix with constant elements, continuous vector-function $f(t, y)$ satisfies the Lipschitz condition locally with respect to y

$$||f(t, y^1) - f(t, y^2)|| \le L||y^1 - y^2||$$

and the following conditions

$$f(t + T, y) \equiv f(t, y), \quad ||f(t, y)|| \le q(1 + ||y||)^{\omega}, \tag{2}$$

where $q > 0$ and $\omega \ge 0$ are constants. We assume that the linear system

$$\frac{dy}{dt} = Ay, \quad -\infty < t < \infty, \tag{3}$$

is exponentially dichotomous (see, for example, [1]). According to the spectral criterion, this is equivalent to the fact that the spectrum of matrix A does not intersect with the imaginary axis. Our aim is to study conditions for the existence of T-periodic solutions to system (1) and their stability for small perturbations of the coefficients of the linear part and nonlinear terms.

We rely on the criterion of the exponential dichotomy for autonomous systems of differential equations, established in the works by M.G. Krein (see [1], chapter 1), which is formulated in terms of the solvability of a special problem for the Lyapunov matrix equation. Namely, the exponential dichotomy of

*E-mail: demidenk@math.nsc.ru

system (3) is equivalent to the existence of Hermitian positive definite matrix H and matrix P, which are the solution to the problem

$$\begin{cases} HA + A^*H = -P^*CP + (I-P)^*C(I-P), \quad C = C^* > 0, \\ H = P^*HP + (I-P)^*H(I-P), \\ P^2 = P, \quad PA = AP. \end{cases} \tag{4}$$

Various proofs of theorems from [1] (chapter 1) are contained in [2–4]. Note that using M.G. Krein's theorems, in the works by S.K. Godunov and A.Ya. Bulgakov, algorithms with guaranteed accuracy was developed for solving the dichotomy problem for autonomous systems of linear differential equations (see [2, 3] and references therein).

Using M.G. Krein's criterion of exponential dichotomy, it is not difficult to obtain estimates for all dichotomy parameters (see [2–4]). Based on this criterion, we will show how to establish the existence of T-periodic solutions to systems of nonlinear equations (1) and to prove their stability under small perturbations of the right-hand sides.

As shown in [1] (chapter 1), the exponential dichotomy of system (3) implies the existence of a unique solution $H = H^* > 0$, P to the system of matrix equations (4), while matrix H is an analog of the Lyapunov matrix integral

$$H = \int_0^\infty P^* e^{tA^*} C e^{tA} P ds + \int_{-\infty}^0 (I-P)^* e^{tA^*} C e^{tA} (I-P) ds = H^+ + H^- \tag{5}$$

and P is a projector on the maximal invariant subspace of matrix A, corresponding to the eigenvalues lying in the left half-plane $\{\lambda : Re\lambda < 0\}$.

Hereinafter, for simplicity, we assume that $C = I$ and using the matrices from (5) we introduce the following notation

$$\nu_H = ||H||\, ||H^{-1}||$$

is the number of conditionality of matrix H,

$$p_H = 2\sqrt{\nu_H ||H||}\left(\sqrt{||H^+||} + \sqrt{||H^-||}\right). \tag{6}$$

We formulate the main results.

Theorem 1. *Let $L > 0$ be the Lipschitz constant with respect to y of vector-function $f(t, y)$ in the ball $B(0, Y) = \{y : ||y|| \le Y\}$ and the conditions be valid*

$$p_H q (1+Y)^\omega < Y, \quad p_H L < 1. \tag{7}$$

Then the system of equations (1) has a unique T-periodic solution in the ball $B(0, Y)$.

Theorem 2. *Let matrix A_1 satisfy the condition*

$$||A_1|| p_H < 1, \tag{8}$$

then the system with perturbations in the coefficients

$$\frac{dx}{dt} = (A + A_1)x, \quad -\infty < t < \infty, \tag{9}$$

is exponentially dichotomous, herewith matrices A and $A + A_1$ have the same number of eigenvalues lying in the left half-plane $\{\lambda : Re\lambda < 0\}$.

Now we consider the question of stability of T-periodic solutions to system (1) with respect to small perturbations of the coefficients and nonlinear terms.

Let matrix function $\mathcal{A}(\mu) \in C([-\mu_0, \mu_0])$ be such that $\mathcal{A}(0) = A$. We assume that the condition is valid

$$||\mathcal{A}(\mu) - A|| p_H < 1, \quad |\mu| \le \mu_0. \tag{10}$$

Then, by Theorem 2, the system of differential equations with a parameter $\mu \in [-\mu_0, \mu_0]$

$$\frac{dx}{dt} = \mathcal{A}(\mu)x, \quad -\infty < t < \infty,$$

is exponentially dichotomous. For this system, we write out an analog of problem (4):

$$\begin{cases} \mathcal{H}\mathcal{A}(\mu) + \mathcal{A}^*(\mu)\mathcal{H} = -\mathcal{P}^*\mathcal{P} + (I-\mathcal{P})^*(I-\mathcal{P}), \\ \mathcal{H} = \mathcal{P}^*\mathcal{H}\mathcal{P} + (I-\mathcal{P})^*\mathcal{H}(I-\mathcal{P}), \\ \mathcal{P}^2 = \mathcal{P}, \quad \mathcal{P}\mathcal{A}(\mu) = \mathcal{A}(\mu)\mathcal{P}. \end{cases} \tag{11}$$

This problem, like (4), has a unique solution $\mathcal{H}(\mu) = \mathcal{H}^*(\mu) > 0$, $\mathcal{P}(\mu)$. Obviously, its solution has the same properties as the solution to problem (4) and, in particular, for matrix $\mathcal{H}(\mu)$, an analog of formula (5) is valid for $C = I$

$$\mathcal{H}(\mu) = \int_0^\infty \mathcal{P}^*(\mu)e^{t\mathcal{A}^*(\mu)}e^{t\mathcal{A}(\mu)}\mathcal{P}(\mu)ds$$

$$+ \int_{-\infty}^0 (I-\mathcal{P}(\mu))^* e^{t\mathcal{A}^*(\mu)}e^{t\mathcal{A}(\mu)}(I-\mathcal{P}(\mu))\,ds = \mathcal{H}^+(\mu) + \mathcal{H}^-(\mu). \tag{12}$$

Theorem 3. *Under condition (10), the convergence takes place*

$$||\mathcal{H}(\mu) - H|| \to 0, \quad ||\mathcal{P}(\mu) - P|| \to 0, \quad \mu \to 0. \tag{13}$$

By analogy with (6), we introduce the notation

$$\nu(\mathcal{H}(\mu)) = ||\mathcal{H}(\mu)||\,||\mathcal{H}^{-1}(\mu)||,$$

$$p(\mathcal{H}(\mu)) = 2\sqrt{\nu(\mathcal{H}(\mu))||\mathcal{H}(\mu)||}\left(\sqrt{||\mathcal{H}^+(\mu)||} + \sqrt{||\mathcal{H}^-(\mu)||}\right). \tag{14}$$

We consider the system of nonlinear differential equations with a parameter $\mu \in [-\mu_0, \mu_0]$

$$\frac{dx}{dt} = \mathcal{A}(\mu)x + \hat{f}(t,x,\mu), \quad -\infty < t < \infty, \tag{15}$$

where matrix function $\mathcal{A}(\mu)$ is specified above, continuous vector-function $\hat{f}(t,y,\mu)$ satisfies the Lipschitz condition locally with respect to y

$$||\hat{f}(t,y^1,\mu) - \hat{f}(t,y^2,\mu)|| \le L||y^1 - y^2||, \quad -\infty < t < \infty, \quad \mu \in [-\mu_0, \mu_0],$$

the Lipschitz constant $L > 0$ is specified in Theorem 1, while

$$\hat{f}(t,y,0) \equiv f(t,y), \quad \hat{f}(t+T,y,\mu) \equiv \hat{f}(t,y,\mu), \quad ||\hat{f}(t,y,\mu)|| \le q(1+||y||)^\omega,$$

and constants $q > 0$, $\omega \ge 0$ are specified in (2).

By virtue of the definition of constant (15), it follows from Theorem 3 that there exists $\mu_1 \le \mu_0$ such that, for $\mu \in [-\mu_1, \mu_1]$, the conditions of the form (7) are valid

$$p(\mathcal{H})q(1+Y)^\omega < Y, \quad p(\mathcal{H})L < 1.$$

Then, as in Theorem 1, the system of equations (15) has a unique T-periodic solution $x(t,\mu)$ in the ball $B(0,Y)$.

Theorem 4. *Let the conditions of Theorem 1 be satisfied and $y(t)$ be T-periodic solution to system (1) in the ball $B(0,Y)$. Then the convergence takes place*

$$\max_{t\in[0,T]} ||x(t,\mu) - y(t)|| \to 0, \quad \mu \to 0.$$

2. ESTIMATES OF SOLUTIONS TO LINEAR SYSTEMS

This section contains auxiliary statements that will be used in the proofs of theorems.

We consider the boundary value problem on the whole axis for the system of linear differential equations

$$\begin{cases} \dfrac{dy}{dt} = Ay + \varphi(t), \quad -\infty < t < \infty, \\ \sup\limits_{-\infty<t<\infty} ||y(t)|| < \infty. \end{cases} \tag{16}$$

Due to the exponential dichotomy of system (3), this problem has a unique solution $y(t) \in C^1(R)$ for any bounded continuous vector-function $\varphi(t)$, herewith it can be written in the integral form

$$y(t) = \int\limits_{-\infty}^{t} e^{(t-s)A} P\varphi(s)ds - \int\limits_{t}^{\infty} e^{(t-s)A}(I-P)\varphi(s)ds. \tag{17}$$

The following lemma is valid.

Lemma 1. *The estimate holds*

$$||y(t)|| \le p_H \sup_{\xi\in R} ||\varphi(\xi)||, \quad -\infty < t < \infty. \tag{18}$$

Proof. First, by analogy with [5], we obtain the following inequalities

$$||e^{tA}P||^2 \le ||H^{-1}||\,||H^+|| \exp\left(-\frac{t}{||H||}\right), \quad t \ge 0, \tag{19}$$

$$||e^{-tA}(I-P)||^2 \le ||H^{-1}||\,||H^-|| \exp\left(-\frac{t}{||H||}\right), \quad t \ge 0. \tag{20}$$

Using the solution H, P to system (4), obviously, we have the identity

$$\frac{d}{dt}\langle He^{tA}Pv, e^{tA}Pv\rangle + \langle e^{tA}Pv, e^{tA}Pv\rangle \equiv 0, \quad v \in \mathbb{C}^n, \quad -\infty < t < \infty.$$

Since

$$\langle He^{tA}Pv, e^{tA}Pv\rangle \le ||H||\,||e^{tA}Pv||^2,$$

then

$$\frac{d}{dt}\langle He^{tA}Pv, e^{tA}Pv\rangle + \frac{1}{||H||}\langle He^{tA}Pv, e^{tA}Pv\rangle \le 0.$$

Therefore,

$$\langle He^{tA}Pv, e^{tA}Pv\rangle \le \exp\left(-\frac{t}{||H||}\right)\langle HPv, Pv\rangle, \quad t \ge 0,$$

and taking into account the equality $\langle HPv, Pv\rangle = \langle H^+Pv, Pv\rangle$, we obtain (19). Estimate (20) is proved in the same way.

Using inequalities (19), (20) and formula (17), we have

$$||y(t)|| \le \sqrt{||H^{-1}||\,||H^+||}\left(\int\limits_{-\infty}^{t} \exp\left(-\frac{t-s}{2||H||}\right)ds\right)\sup_{\xi\in R}||\varphi(\xi)||$$

$$+ \sqrt{||H^{-1}||\,||H^-||}\left(\int\limits_{t}^{\infty} \exp\left(\frac{t-s}{2||H||}\right)ds\right)\sup_{\xi\in R}||\varphi(\xi)||$$

$$= 2\sqrt{||H^{-1}||}||H||\left(\sqrt{||H^+||} + \sqrt{||H^-||}\right)\sup_{\xi\in R}||\varphi(\xi)||.$$

By virtue of the definition of constant (6), from this inequality, estimate (18) follows.

Lemma is proved. □

When proving Theorem 2, we will use the theorem on the unique solvability of the system

$$\frac{dy}{dt} = Ay + \varphi(t), \quad -\infty < t < \infty, \tag{21}$$

in the Sobolev space $W_2^1(R)$ for any $\varphi(t) \in L_2(R)$ (see, for example, [4, 6]). In the following lemma, we will give estimates for the L_2-norm of the solution $y(t) \in W_2^1(R)$ to equation (21).

Lemma 2. *The estimate holds*

$$||y(t), L_2(R)|| \le p_H ||\varphi(t), L_2(R)||. \tag{22}$$

Proof. By virtue of the embedding theorem of the Sobolev space $W_2^1(R)$ in $C(R)$, the solution to system (21) from $W_2^1(R)$ is the solution to problem (16). Therefore, for $\varphi(t) \in L_2(R)$, the solution has the form (17).

Using the Heaviside function $\theta(t)$, formula (17) can be written in the following form

$$y(t) = \int_{-\infty}^{\infty} \theta(t-s) e^{(t-s)A} P\varphi(s)ds - \int_{-\infty}^{\infty} \theta(s-t) e^{(t-s)A} (I-P)\varphi(s)ds.$$

Applying the Minkowski and Young inequalities, we have

$$||y(t), L_2(R)|| \le \left|\left| \int_{-\infty}^{\infty} ||\theta(t-s) e^{(t-s)A} P|| \, ||\varphi(s)||ds, L_2(R) \right|\right|$$

$$+ \left|\left| \int_{-\infty}^{\infty} ||\theta(s-t) e^{(t-s)A} (I-P)|| \, ||\varphi(s)||ds, L_2(R) \right|\right|$$

$$\le \left(\int_0^{\infty} ||e^{tA}P||dt + \int_{-\infty}^{0} ||e^{tA}(I-P)||dt \right) ||\varphi(s), L_2(R)||.$$

Using estimates (19) and (20), we obtain

$$||y(t), L_2(R)|| \le \left(\sqrt{||H^{-1}|| \, ||H^+||} \int_0^{\infty} \exp\left(-\frac{t}{2||H||}\right) dt \right.$$

$$\left. + \sqrt{||H^{-1}|| \, ||H^-||} \int_{-\infty}^{0} \exp\left(\frac{t}{2||H||}\right) dt \right) ||\varphi(t), L_2(R)||.$$

By virtue of the definition of constant (6), from this inequality, estimate (22) follows.

Lemma is proved. □

3. PROPERTIES OF SOLUTIONS TO NONLINEAR SYSTEMS

In this section, the main theorems will be proved.

3.1. Proof of Theorem 1

When proving Theorem 1, we use a well-known technique, consisting in the application of the contraction mapping principle (see, for example, [1]).

We consider the system of nonlinear equations (1). It follows from the exponential dichotomy of the homogeneous system (3) and conditions on vector-function $f(t, y)$ that, by virtue of formula (17), finding bounded solutions to system (1) on R is equivalent to constructing a solution to the system of integral equations

$$y(t) = \int_{-\infty}^{t} e^{(t-s)A} P f(s, y(s)) ds - \int_{t}^{\infty} e^{(t-s)A} (I - P) f(s, y(s)) ds \tag{23}$$

or, in the operator form, $y(t) = (\mathcal{G}y)(t)$. Obviously, from the definition of operator $\mathcal{G}$ and conditions on $f(t, y)$, it follows that this operator maps T-periodic vector-functions to T-periodic ones. Therefore, to prove the existence and uniqueness of T-periodic solution to system (1) in the ball $B(0, Y)$, it is sufficient to prove the unique solvability of system (23) in the space of continuous T-periodic vector-functions C_T such that $||y(t)|| \leq Y < \infty$.

If $y(t)$ is bounded solution to system (1), therefore, the solution to the operator equation (23), then, taking into account condition (2), the definition of constant (6), and inequality (18), we have

$$||y(t)|| \leq p_H q \sup_{\xi \in R} (1 + ||y(\xi)||)^{\omega}.$$

It follows from this inequality that for any solution $y(t)$ from C_T such that $y(t) \in B(0, Y)$, the estimate holds

$$||y(t)|| \leq p_H q \, (1 + Y)^{\omega}.$$

Therefore, if condition (7) is fulfilled, then, by virtue of definition (23), the operator $\mathcal{G} : C_T \to C_T$ maps the closed ball $B(0, Y)$ to itself.

We show that under condition (7), the operator $\mathcal{G} : B(0, Y) \to C_T$ is a contraction operator.

Indeed, for any vector-functions $y^1(t),\ y^2(t) \in C_T$ such that $||y^i(t)|| \leq Y$, $i = 1, 2$, by virtue of the definition of operator $\mathcal{G}$, we have

$$(\mathcal{G}y^1)(t) - (\mathcal{G}y^2)(t) \equiv \int_{-\infty}^{t} e^{(t-s)A} P \left(f(s, y^1(s)) - f(s, y^2(s)) \right) ds$$

$$- \int_{t}^{\infty} e^{(t-s)A} (I - P) \left(f(s, y^1(s)) - f(s, y^2(s)) \right) ds.$$

Then, taking into account the local Lipschitz condition and inequality (18), we obtain the estimate

$$\max_{t \in [0,T]} ||(\mathcal{G}y^1)(t) - (\mathcal{G}y^2)(t)|| \leq p_H L \max_{s \in [0,T]} ||y^1(s) - y^2(s)||,$$

where the Lipschitz constant L is defined in Theorem 1 and, by the condition on $f(t, y)$, depends on Y. Therefore, if $p_H L < 1$, then the mapping $\mathcal{G} : B(0, Y) \to C_T$ is a contraction mapping.

It follows from the conducted reasoning that if vector-function $f(t, y)$ is such that the constants q and L satisfy inequalities (7), then, by virtue of the contraction mapping principle, the operator equation (23) is uniquely solvable in the space C_T, herewith the solution lies in the ball $B(0, Y)$. Therefore, the system of equations (1) has a unique T-periodic solution, $||y(t)|| \leq Y$.

Theorem 1 is proved.

3.2. Proof of Theorem 2

Obviously, to prove Theorem 2, it is sufficient to show that the system of linear equations with perturbations in the coefficients

$$\frac{dv}{dt} = (A + A_1)v + F(t), \quad -\infty < t < \infty, \tag{24}$$

has a solution $v(t) \in W_2^1(R)$ for any vector-function $F(t) \in L_2(R)$. To do this, as in [4, 6], we find a solution to system (24) in the form

$$v(t) = Bf(t) = \int_{-\infty}^{t} e^{(t-s)A} P f(s) ds - \int_{t}^{\infty} e^{(t-s)A} (I - P) f(s) ds, \quad f(t) \in L_2(R). \tag{25}$$

It is clear that vector-function $f(t)$ must be a solution to the integral equation

$$f(t) - A_1 B f(t) = F(t). \tag{26}$$

From the explicit form of the operator B and Lemma 2, it follows that this operator is linear and continuous in $L_2(R)$. Then the operator

$$A_1 B : L_2(R) \to L_2(R)$$

is also linear and continuous, while due to inequality (22), the estimate is valid

$$||A_1 B f(t), L_2(R)|| \leq ||A_1|| p_H ||f(t), L_2(R)||.$$

Due to the arbitrariness of $f(t) \in L_2(R)$ and condition (8), the norm of the operator $A_1 B$ is strictly less than 1. Therefore, according to the Neumann theorem, equation (26) has a unique solution

$$f(t) = (I - A_1 B)^{-1} F(t) \in L_2(R)$$

for any vector-function $F(t) \in L_2(R)$. Consequently, substituting $f(t)$ in (25), we obtain the formula for the solution to equation (24)

$$v(t) = B(I - A_1 B)^{-1} F(t) \in W_2^1(R)$$

for any $F(t) \in L_2(R)$. Using properties of the Fourier operator in $L_2(R)$ and repeating the reasoning from [4] (chapter 2), it can be shown the uniqueness of a solution to equation (24). Hence we obtain that matrix $(A + A_1)$ has no eigenvalues on the imaginary axis, i.e., system (9) is exponentially dichotomous.

It is not difficult to show that the number of eigenvalues of the perturbed matrix $(A + A_1)$ lying in the left half-plane is equal to the number of eigenvalues of matrix A lying in the left half-plane.

Theorem 2 is proved.

3.3. Proof of Theorem 3

We prove Theorem 3. As already noted, problem (11) has a unique solution $\mathcal{H}(\mu)$, $\mathcal{P}(\mu)$, and by analogy with problem (4), for these matrix functions, we can write explicit formulas. In particular, for matrix $\mathcal{H}(\mu)$, formula (12) is valid, herewith projector $\mathcal{P}(\mu)$ can be written as the Riesz integral

$$\mathcal{P}(\mu) = \frac{1}{2\pi i} \int_{\gamma} (\lambda I - \mathcal{A}(\mu))^{-1} \, d\lambda,$$

where the contour γ covers only the eigenvalues of matrix $\mathcal{A}(\mu)$ lying in the left half-plane (see, for example, [1], chapter 1; [4], chapter 2).

It is not difficult to show that the integral formula (12) for the representation of $\mathcal{H}(\mu)$ can be rewritten as

$$\mathcal{H}(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ((i\xi I - \mathcal{A}(\mu))^*)^{-1} (i\xi I - \mathcal{A}(\mu))^{-1} \, d\xi$$

(see [4], chapter 2).

Note that the solution to problem (4) can be written using similar formulas

$$P = \frac{1}{2\pi i} \int\limits_{\gamma} (\lambda I - A)^{-1}\, d\lambda, \quad H = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} ((i\xi I - A)^*)^{-1} (i\xi I - A)^{-1}\, d\xi.$$

Then by virtue of the continuity of matrix function $\mathcal{A}(\mu)$ on the segment $[-\mu_0, \mu_0]$ and the equality $\mathcal{A}(0) = A$, we obtain the convergence (13). Theorem 3 is proved.

3.4. Proof of Theorem 4

We prove Theorem 4. Since $y(t)$, $x(t, \mu)$ are T-periodic solutions to systems (1), (15), respectively, then for vector-function $u(t, \mu) = y(t) - x(t, \mu)$, the identity is valid

$$\frac{du(t,\mu)}{dt} \equiv Au(t,\mu) + (A - \mathcal{A}(\mu))x(t,\mu) + f(t, y(t)) - \hat{f}(t, x(t,\mu), \mu), \quad -\infty < t < \infty. \tag{27}$$

We introduce notation for the last two terms

$$F(t,\mu) = f(t, y(t)) - \hat{f}(t, x(t,\mu), \mu)$$

and rewrite it in the following form

$$F(t,\mu) = (f(t, (u(t,\mu) + x(t,\mu))) - f(t, x(t,\mu))) + \left(\hat{f}(t, x(t,\mu), 0) - \hat{f}(t, x(t,\mu), \mu)\right).$$

Taking into account the Lipschitz condition, we have the estimate

$$||F(t,\mu)|| \le L||u(t,\mu)|| + \max_{\xi \in [0,T], ||v|| \le Y} ||f(\xi, v) - \hat{f}(\xi, v, \mu)||. \tag{28}$$

By virtue of (27), vector-function $u(t, \mu)$ is T-periodic solution to the system

$$\frac{du}{dt} = Au + (A - \mathcal{A}(\mu))x(t,\mu) + F(t,\mu),$$

therefore, using Lemma 1 and estimate (28), we obtain the inequality

$$(1 - p_H L) \max_{t \in [0,T]} ||u(t,\mu)|| \le p_H Y ||A - \mathcal{A}(\mu)|| + p_H \max_{\xi \in [0,T], ||v|| \le Y} ||f(\xi, v) - \hat{f}(\xi, v, \mu)||.$$

By condition (7), we have $p_H L < 1$, therefore, due to the conditions on matrix functions and nonlinear terms in systems (1) and (15), from this estimate, the convergence follows

$$\max_{t \in [0,T]} ||u(t,\mu)|| \to 0, \quad \mu \to 0.$$

Theorem 4 is proved.

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