ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 13, pp. 3173–3183. © Pleiades Publishing, Ltd., 2021.

A Few Remarks on Asymptotic Stabilities of Markov Operators on L^1-Spaces

Farrukh Mukhamedov*

(Submitted by A. I. Volodin)

Department of Mathematical Sciences, College of Science, United Arab Emirates University, Al-Ain, Abu Dhabi, 15551 UAE

Received March 31, 2021; revised April 10, 2021; accepted April 14, 2021

Abstract—In this paper, the investigation of a generalized Dobrushin ergodicity coefficient to obtain uniform ergodicity and uniform mean ergodicities of positive contractions of L^1-spaces is carried out. Through the introduction of notions such as *mean P-completely mixing* and *P-completely mixing*, the last one being an extension of the complete mixing, several analogues of the Akcoglu and Sucheston theorem are proved. As an applications of these results, we establish mean ergodicities of positive contractions of L^1-spaces. It is vital to note that P stands for a Markov projection of L^1.

DOI: 10.1134/S1995080222010139

Keywords and phrases: *positive contraction; P-completely mixing; Markov operator; uniform ergodicity; Dobrushin ergodicity coefficient.*

1. INTRODUCTION

Markov chains have many applications in natural sciences as they play an essential role which are described by transition probabilities over some measurable space. There are many books on Markov chains yet one that is remarkable to the extent that it stands out is the book of Foguel [14] where probability theory and functional analysis methods are combined (for other monographs the reader is referred to [11, 22, 27, 37]). Therefore, in this paper functional analysis language is going to be used where the where the evolution and asymptotic properties of a Markov process is reflected as an asymptotic limiting behaviour of iterates T^n, where T is a linear, positive operator defined on some Banach function space.

It is well known that Doeblin and Dobrushin [6, 18] characterized the convergence T^n to its invariant distribution in terms of the ergodicty coefficient $\delta(T)$, i.e. if $\delta(P) < 1$. The Dobrushin's condition played a major role as a source of inspiration for many mathematicians to do interesting work on the theory of Markov processes (see for example [18, 27, 35]). Nevertheless, if $\delta(T) = 1$, then such a coefficient is not effective while T^n converges. Hartfiel et al. [15, 16] introduced a generalized coefficient which covers the mentioned type of convergence in the finite-dimensional setting. This type of coefficient has not been thoroughly studied even in the classical L^1-spaces. Recently, in [33] a generalized Dobrushin ergodicity coefficient has been introduced abstract spaces. Therefore, we provide a simple calculation of the generalized Dobrushin ergodicity coefficient [33] in the classical L^1-spaces and it gets applied to the investigation of uniform P-ergodicites of positive contractions on L^1-spaces.

This paper by using methods of [33] we are going to extend the uniform P-ergodicity result for positive contractions of L^1-spaces. Furthermore, another aim of this work is to investigate pointwise asymptotic stabilities of positive contractions of L^1-spaces. To establish such kind of results, we introduce P-complete mixing and mean P-completely mixing (see Section 4) are extended notions of completely mixing are considered. By proving several analogues of well-known results of Akcoglu and Sucheston,

*E-mail: far75m@yandex.ru;farrukh.m@uaeu.ac.ae

and together with smoothing we establish mean ergodicities of positive contractions of L^1-spaces. Note that the smoothness that gets considered is closely related to the same notion given in [20, 21]. In [4, 17, 19] the smoothness in sense of [20] has been investigated in terms of constrictivity of Markov operators. These investigates are related to the asymptotic periodicity of Markov operators which yields that the iterates of Markov operator converges to some projection. In this paper, we are going to investigate the asymptotic stability of Markov operators to some projection without using their periodicity.

It is stressed that there are several notions of mixing (see [1, p. 199], [22, Chapter 8]) (i.e. weak mixing, mixing, completely mixing e.c.t.) of measure preserving transformation on a measure space in the ergodic theory.

Now let us recall some notions and results. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with probability measure μ. Let $L^1(\Omega, \mathcal{F}, \mu)$ be the associated L^1-space. A linear operator $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is called *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $||T|| \leq 1$. Let $L^1_0 = \{f \in L^1(\Omega, \mathcal{F}, \mu) : \int f d\mu = 0\}$. A positive contraction T in $L^1(\Omega, \mathcal{F}, \mu)$ is called *completely mixing* if $||T^n f||$ tends to 0 for all $f \in L^1_0$. Some properties of this mixing was studied in [1]. In [2, Lemma 2.1] Akcoglu and Sucheston (see also [22, Theorem 1.4, Chapter 8]) proved the following result.

Theorem 1.1. *Let $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction. Assume that for $z \in L^1(\Omega, \mathcal{F}, \mu)$ the sequence $(T^n z)$ converges weakly in $L^1(\Omega, \mathcal{F}, \mu)$, then either $\lim\limits_{n\to\infty} ||T^n z|| = 0$ or there exists a positive function $h \in L^1(\Omega, \mathcal{F}, \mu)$, $h \neq 0$ such that $Th = h$.*

This theorem gives an answer to the problem whether K-automorphisms of σ-finite measure space are mixing, and showed that, in fact, invertible mixing measure preserving transformations of σ-finite infinite space do not exist (see [23]) (see for review [22]). Moreover, Theorem 1.1 has a lot of applications, but only a few get mentioned in this paper. Namely, using it in [10, Theorem 1] the existence of an invariant measure for given positive contraction T on $L^1(\Omega, \mathcal{F}, \mu)$ was proved, and in [38, Theorem 8] a criterion of strong asymptotically stability for positive contractions was given by means of Theorem 1.1.

Hence, we bring forth the introduction of notions which are P-complete mixing and P-complete mixing, the last one being an extension of the complete mixing. Here, P stands for a Markov projection of $L^1(\Omega, \mathcal{F}, \mu)$ (see Section 2 for detailed definitions). For these notions, we prove analogue of Theorem 1.1 in Section 4. In Section 3, we establish uniform ergodicity of positive contractions of L^1-spaces in terms of the generalized Dobrushin coefficient [33]. Furthermore, mean ergodocity is also proved in terms of mean P-complete mixing.

2. UNIFORM ERGODICITIES

Throughout the paper, we always assume that $(\Omega, \mathcal{F}, \mu)$ is an arbitrary probability space, i.e. μ is a probability measure. By $L^1(\Omega, \mathcal{F}, \mu)$ we denote the usual L^1-space. A positive operator T is called to be *Markov* if $||Tf|| = ||f||$ for every $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$.

Given $f \in L^1(\Omega, \mathcal{F}, \mu)$ we will denote by $l[f]$ the operator $l[f] : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ defined by $l[f](g) = f \int g d\mu$ for every $g \in L^1(\Omega, \mathcal{F}, \mu)$. If $f \neq 0$, then $l[f]$ is usually called a rank one operator.

Denote

$$\mathcal{D} = \{f \in L^1(\Omega, \mathcal{F}, \mu) : \ f \geq 0, \ ||f|| = 1\}.$$

Example 2.1. Assume that $P(x, A)$ is transition probability on $(\Omega, \mathcal{F}, \mu)$ such that $P(x, \cdot)$ is absolutely continuous with respect to μ. Then it defines a Markov operator T on $L^1(\Omega, \mathcal{F}, \mu)$, whose dual T' acts on $L^\infty(\Omega, \mathcal{F}, \mu)$ as follows

$$(T'f)(x) = \int f(y) P(x, dy), \quad f \in L^\infty.$$

Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive projection (i.e. $P^2 = P$). We notice that such kind of operators can be characterized by conditional expectations [7].

Now, we are going to define a generalized Dobrushin ergodicity coefficient of a Markov operator T with respect to P as follows [33]:

$$\delta_P(T) = \sup_{f \in \ker P,\ f \neq 0} \frac{||Tf||}{||f||}, \tag{2.1}$$

where

$$\ker P = \{f \in L^1(\Omega, \mathcal{F}, \mu) : \ Pf = 0\}. \tag{2.2}$$

If $P = I$, we put $\delta_P(T) = 1$. The quantity $\delta_P(T)$ is called the *generalized Dobrushin ergodicity coefficient of T with respect to P*. We notice that if P is takes as rank one projection, then $\delta_P(T)$ reduces to the well-known Dobrushin's ergodicity coefficient (see [33]).

We notice that if $\Omega = \{1, \dots, n\}$, then there is some formulas to calculate this coefficient [16, Lemma 1] (see also [15]).

Before establishing our main result of this section, we need the following auxiliary fact.

Lemma 2.2. *Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection. Then for every $x \in \ker P$ there exist $u, v \in \mathcal{D}$ with $u \wedge v = 0$, $u - v \in \ker P$ such that $x = ||x||2(u - v)/2$.*

Proof. Given any $x \in \ker P$, we have $Px = 0$. Moreover, there exist $x_+, x_- \in L^1(\Omega, \mathcal{F}, \mu)$ such that $x = x_+ - x_-$, $x_+, x_- \geq 0$, $x_+ \wedge x_- = 0$ with $||x_+|| + ||x_-|| = ||x||$. Clearly $Px_+ = Px_-$. As P a Markov projection $||Px_+|| = ||x_+||$, which yields $||x_+|| = ||x_-||$. Therefore,

$$x = \frac{x_+}{||x_+||}||x_+|| - \frac{x_-}{||x_-||}||x_+|| = ||x_+|| \left(\frac{x_+}{||x_+||} - \frac{x_-}{||x_-||} \right).$$

Letting $u = \frac{x_+}{||x_+||}$ and $v = \frac{x_-}{||x_-||}$, so $u, v \in \mathcal{D}$. Moreover, $Pu = Pv$, and $u - v \in \ker P$. Hence, the lemma is proved. □

Let us denote by Σ the set of all Markov operators defined on $L^1(\Omega, \mathcal{F}, \mu)$, and by Σ_P we denote the set of all Markov operators T on $L^1(\Omega, \mathcal{F}, \mu)$ with $PT = TP$.

Now, using Lemma 2.2 and the argument of [33, Theorem 3.7], one gets.

Theorem 2.3. *Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection. Then for any $T \in \Sigma$, one has*

$$\delta_P(T) = \sup \left\{ \frac{||Tu - Tv||}{2} : \ u, v \in \mathcal{D},\ u \wedge v = 0 \text{ with } u - v \in \ker P \right\}. \tag{2.3}$$

From this result, one finds

$$\begin{aligned} 1 - \delta_P(T) &= \inf \left\{ 1 - \frac{||Tu - Tv||}{2} : \ u, v \in \mathcal{D},\ u \wedge v = 0 \text{ with } u - v \in \ker P \right\} \\ &= \inf \left\{ 1 - \frac{||Tu + Tv - 2(Tu \wedge Tv)||}{2} : \ u, v \in \mathcal{D},\ u \wedge v = 0 \text{ with } u - v \in \ker P \right\} \\ &= \inf \left\{ 1 - \frac{2 - 2||Tu \wedge Tv||}{2} : \ u, v \in \mathcal{D},\ u \wedge v = 0 \text{ with } u - v \in \ker P \right\} \\ &= \inf \left\{ ||Tu \wedge Tv|| : \ u, v \in \mathcal{D},\ u \wedge v = 0 \text{ with } u - v \in \ker P \right\}. \end{aligned} \tag{2.4}$$

Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection. A linear operator $T \in \Sigma$ is called *uniformly P-ergodic* if $\lim_{n\to\infty} ||T^n - P|| = 0$.

We notice if T is uniformly P-ergodic operator on $L^1(\Omega, \mathcal{F}, \mu)$, then $TP = PT = P$. If P is a rank one projection, then the notion of uniformly P-ergodic was called *uniformly ergodic* or *uniform asymptotically stable* (see [3, 28, 30, 32, 34]).

Theorem 2.4. *Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection and $T \in \Sigma$. Then the following conditions are equivalent:*

(i) *T is uniformly P-ergodic;*

(ii) *One has* $PT = TP = P$ *and there exists* $n_0 \in \mathbb{N}$ *such that* $\delta_P(T^{n_0}) < 1$;

(iii) *One has* $PT = TP = P$ *and there exists* $\rho > 0$ *and* $n_0 \in \mathbb{N}$ *such that* $||T^{n_0}u \wedge T^{n_0}v|| \geq \rho$ *for every* $u, v \in \mathcal{D}$, $u \wedge v = 0$ *with* $u - v \in \ker P$.

Moreover, there are constants $C, \alpha \in \mathbb{R}_+$ *and* $n_0 \in \mathbb{N}$ *such that* $||T^n - P|| \leq Ce^{-\alpha n}, \forall n \geq n_0$.

Proof. The implications (i)⇔ (ii) has been proved in [33, Corollary 4.7]. From (2.4) it follows that $\delta_P(T^{n_0}) \leq 1 - \rho$ which implies (ii)⇔ (iii). □

Remark 2.5. In case of P is a rank one projection, analogous results can be found in [3, 8, 28, 31, 38].

Now, we provide an extension of Theorem 2.4 to positive contractions of $L^1(\Omega, \mathcal{F}, \mu)$. Given a linear operator $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ by T' we denote its dual, which acts on $L^\infty(\Omega, \mathcal{F}, \mu)$.

Theorem 2.6. *Let* $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ *be a finite rank Markov projection and* $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ *be a positive contraction such that*

$$TP = PT, \quad TPg = P(gT'\mathbb{1}) = (Pg)T'\mathbb{1}, \tag{2.5}$$

for all $g \in L^1(\Omega, \mathcal{F}, \mu)$. *Assume that there exists* $\rho > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$||T^{n_0}u \wedge T^{n_0}v|| \geq \rho \tag{2.6}$$

for every $u, v \in \mathcal{D}$, $u \wedge v = 0$ *with* $u - v \in \ker P$. *Then* $\{T^n\}$ *converges uniformly to some projection* Q *with* $Q = QPQ$.

Proof. Now, we are going to associate with T a Markov operator $S : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ such that $T \leq S$. Let $T' : L^\infty(\Omega, \mathcal{F}, \mu) \to L^\infty(\Omega, \mathcal{F}, \mu)$ be the dual of T. Let $h := \mathbb{1} - T'\mathbb{1}$. The contractivity of T implies $T'\mathbb{1} \leq \mathbb{1}$ which yields $0 \leq h \leq \mathbb{1}$. Now, we define a linear operator S by

$$S(g) = Tg + gh, \quad g \in L^1(\Omega, \mathcal{F}, \mu).$$

It is clear that $S : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$. According to

$$\int fSgd\mu = \int S'fgd\mu = \int g(T'f + fh)d\mu, \quad f \in L^\infty(\Omega, \mathcal{F}, \mu), g \in L^1(\Omega, \mathcal{F}, \mu),$$

one finds that $S'f = T'f + fh$, for every $f \in L^\infty(\Omega, \mathcal{F}, \mu)$.

It is obvious that $S'\mathbb{1} = \mathbb{1}$, which implies S is a Markov operator.

From $0 \leq T \leq S$, it follows that $0 \leq T^{n_0} \leq S^{n_0}$; therefore, $||S^{n_0}u \wedge S^{n_0}v|| \geq \rho$ for every $u, v \in \mathcal{D}$, $u \wedge v = 0$ with $u - v \in \ker P$. From (2.5), we obtain that $SP = PS = P$. Hence, Theorem 2.4 yields that S is uniformly P-ergodic. Since P is finite rank, then by [34, Theorem 2.2], the sequence $\{T^n\}$ converges uniformly to some projection Q. Due to $0 \leq T^n \leq S^n$, $n \in \mathbb{N}$ and the positivity of T, we have $0 \leq Q \leq P$. This implies $0 \leq QPQ - Q = Q(P - Q)Q \leq P(P - Q)Q = PQ - PQ = 0$, hence $Q = QPQ$. This completes the proof. □

Remark 2.7. From the condition of Theorem 2.6, we infer that TP is compact, and (2.6) due to [33, Corollary 3.15] implies that T is quasi-compact operator. However, the theorem yields T is uniformly Q-ergodic.

Remark 2.8. We note that if T converges uniformly to Q, then clearly one has $TQ = QT$ and $TQ = Q$, but Q is not necessary to be a Markov projection. The first condition in (2.5) means that T has some invariant subspace, however, it is not a'priori known that T converges to the given projection P. It is not known that whether the condition (2.5) is necessary or not.

The second condition in (2.5) is satisfied if P is a conditional expectation and $T'\mathbb{1}$ belongs to the range of P. Let us provide an example for which condition (2.5) holds.

Example 2.9. Let us consider $\mathbb{R}^3$. Define

$$T = \begin{pmatrix} 0 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 < a < 1$. Then it is clear that T is a positive contraction and P is a Markov projection. Moreover, for these operators (2.5) is satisfied. We also have that $T^n \to Q$, where

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. MEAN ERGODICITY AND MEAN COMPLETELY MIXING

Given a bounded linear operator $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$, we set $A_n(T) = \frac{1}{n}\sum_{k=1}^{n} T^k$. Recall that T is said to be

(a) *mean ergodic* if for every $x \in L^1(\Omega, \mathcal{F}, \mu)$

$$\lim_{n\to\infty} A_n(T)x = Qx \text{ (in norm);}$$

(b) *uniformly mean ergodic* if

$$\lim_{n\to\infty} ||A_n(T) - Q|| = 0;$$

for some operator Q on $L^1(\Omega, \mathcal{F}, \mu)$.

In this setting, it is well-known that Q is a projection [22], which is called the *limiting projection of* T, and denoted by Q_T. Moreover, if $T \in \Sigma$, then Q_T is also Markov.

By [33, Theorem 6.3] and (2.4), we obtain the following fact.

Theorem 3.1. *Assume that $T \in \Sigma$ and T is mean ergodic with its limiting projection Q_T. Then the following statements are equivalent:*

(i) *T is uniformly mean ergodic;*

(ii) *there exists an $n_0 \in \mathbb{N}$ such that $\delta_{Q_T}(A_{n_0}(T)) < 1$.*

(iii) *there exists $\rho > 0$ and $n_0 \in \mathbb{N}$ such that $||A_{n_0}(T)u \wedge A_{n_0}(T)v|| \geq \rho$ for every $u, v \in \mathcal{D}$, $u \wedge v = 0$ with $u - v \in \ker Q_T$.*

Moreover,

$$||A_n(T) - Q_T|| \leq \frac{2(n_0 + 1)}{1 - \delta_{Q_T}(A_{n_0}(T))}\,\frac{1}{n}.$$

Remark 3.2. There are a few results in the literature on uniform ergodicities of bounded linear operators on Banach spaces (see, [24–26]).

Definition 3.3. A positive contraction $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is called:

(i) *completely mixing* if

$$\lim_{n\to\infty} ||T^n(u - v)|| = 0 \quad \text{for all} \quad u, v \in L^1(\Omega, \mathcal{F}, \mu), \quad u, v \geq 0, \quad u \neq v, ||u|| = ||v||; \quad (3.1)$$

(ii) *mean completely mixing* if

$$\lim_{n\to\infty} ||A_n(T)(u - v)|| = 0 \quad \text{for all} \quad u, v \in L^1(\Omega, \mathcal{F}, \mu), \quad u, v \geq 0, \quad u \neq v, ||u|| = ||v||. \quad (3.2)$$

We emphasize that mean completely mixing coincides the ergodicity of T [1]. However, for the sake of similarity, in what follows, we will use the mean completely mixing.

Following [38] we introduce a notion of mean smoothing. A positive contraction $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is called *mean smoothing* with respect to $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$, $f \neq 0$, if for every $\varepsilon > 0$ there exists $\tau > 0$ such that

$$\int\limits_E A_n(T) f d\mu < \varepsilon \tag{3.3}$$

for every $E \in \mathcal{F}$ with $\mu(E) < \tau$ and for all $n \in \mathbb{N}$.

We point out that the definition of smoothing is closely related to a notion of ε-*sweeping* given in [20, 21], but in different form.

Next result is an analogue of Theorem 1.1 for averages.

Theorem 3.4. *Let $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction. Assume that there is $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$, $f \neq 0$ such that T is mean smoothing with respect to f. Then $\lim\limits_{n\to\infty} ||T^n f|| = 0$ or T has a non-zero positive invariant element.*

Proof. Due to the positivity of f, we have $||A_n(T)f|| = \frac{1}{n}\sum_{k=1}^n ||T^k f||$. The contractivity of T implies that $\lim_{n\to\infty} ||A_n(T)f|| = \alpha$, $\alpha \geq 0$. Let $\alpha = 0$, then due to the positivity of T and $f \geq 0$, one gets $\lim\limits_{n\to\infty} ||T^n f|| = 0$, which means the statement.

Now, we suppose that $\alpha > 0$. Let L be a Banach limit [9, p. 73], then define $\nu : L^\infty(\Omega, \mathcal{F}\mu) \to \mathbb{R}$ by

$$\nu(x) = L\big(\{\langle A_n(T)f, x\rangle\}_{n\geq 1}\big), \quad x \in L^\infty(\Omega, \mathcal{F}\mu).$$

One can see that

$$\nu(\mathbb{1}) = L\big(\{\langle A_n(T)f, \mathbb{1}\rangle\}_{n\geq 1}\big) = L\big(\{||A_n(T)f||\}_{n\geq 1}\big) = \lim_{n\to\infty} ||A_n(T)f|| \neq 0.$$

Therefore, ν is non-zero. Let us establish that ν is T''-invariant. Indeed, one gets

$$\langle x, T''\nu\rangle = \langle T'x, \nu\rangle = L\big(\{\langle A_n(T)f, T'x\rangle\}_{n\geq 1}\big) = L\big(\{\langle TA_n(T)f, x\rangle\}_{n\geq 1}\big). \tag{3.4}$$

On the other hand, we have

$$TA_n(T) = \frac{1}{n}\sum_{k=1}^n T^{k+1} = \left(1 + \frac{1}{n}\right)A_{n+1}(T) - \frac{1}{n}T$$

and $\left|| \frac{1}{n} A_{n+1}(T)\right|| \leq \frac{1}{n}$. So, $\lim_{n\to\infty} \left||\frac{1}{n}A_{n+1}(T)\right|| = 0$. Hence, the properties of the Banach limit imply that

$$L\big(\{\langle TA_n(T)f, x\rangle\}_{n\geq 1}\big) = L\Big(\{\langle \left(1 + \frac{1}{n}\right)A_{n+1}(T)f, x\rangle\}_{n\geq 1}\Big) - \lim_{n\to\infty}\frac{1}{n}\langle Tf, x\rangle$$
$$= L\big(\{\langle A_{n+1}(T)f, x\rangle\}_{n\geq 1}\big) = L\big(\{\langle A_n(T)f, x\rangle\}_{n\geq 1}\big).$$

Therefore, the last equality together with (3.4) yields

$$\langle x, T''\nu\rangle = L\big(\{\langle A_n(T)f, x\rangle\}_{n\geq 1}\big) = \langle x, \nu\rangle$$

for every $x \in L^\infty(\Omega, \mathcal{F}, \mu)$. This means that ν is T''-invariant.

By means of ν, we define a set function by $\nu(A) = \nu(\chi_A)$, $A \in \mathcal{F}$, here χ_A stands for the indicator function of a set A. It is clear that ν is a finitely additive. Now, let us establish its σ-additivity. It is enough to prove that $\lim\limits_{n\to\infty} \nu(A_k) = 0$, whenever $\{A_k\} \subset \mathcal{F}$ such that

$$A_{k+1} \subset A_k, \quad \bigcap_{k=1}^\infty A_k = \emptyset.$$

Let $\varepsilon > 0$, then due to T is mean smoothing w.r.t. f and $\mu(A_k) \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\int\limits_{A_k} A_n(T) f d\mu < \varepsilon, \quad \forall k \geq n_0, \quad \forall n \geq 1.$$

By means of the definition of the Banach limit, one gets

$$\nu(A_k) = L\big(\{\langle A_n(T)f, \chi_A\rangle\}_{n\geq 1}\big) = L\left(\left\{\int\limits_{A_k} A_n(T)f\right\}_{n\geq 1}\right) \leq \varepsilon, \quad \forall k \geq n_0.$$

Hence, $\lim\limits_{n\to\infty} \nu(A_k) = 0$. This means that ν is a measure on $\mathcal{F}$. Moreover, ν is absolutely continuous with respect to μ. By the Radon–Nykodym Theorem there exists $u \in L^1(\Omega, \mathcal{F}, \mu)$, $u \geq 0$, $u \neq 0$ such that

$$\langle x, \nu\rangle = \int x u d\mu, \quad \forall x \in L^\infty(\Omega, \mathcal{F}, \mu).$$

From

$$\int x u d\mu = \langle x, T''\nu\rangle = \langle T'x, \nu\rangle = \int (T'x) u d\mu = \int x T u d\mu, \quad x \in L^\infty(\Omega, \mathcal{F}, \mu)$$

we obtain $Tu = u$. This completes the proof. □

Remark 3.5. We stress that a similar kind of result has been obtained in [19, Theorem 4] (see also [36]) for the existence of a positive invariant element in L^1.

Remark 3.6. Let $\Omega, \mathcal{F}, \mu)$ be a probability space and $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction. Assume that $\{T^n f\}$ converges weakly for some $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$, $f \neq 0$. Then by [38, Observation 1] T is smoothing w.r.t. f.

Remark 3.7. We stress that Theorem 3.4 is not true, if the measure μ is σ-finite. The corresponding example can be found in [38, Observation 2].

Now, by using of Theorem 3.4, we are ready to prove an analogue of [38, Theorem 8] for averages of positive contractions.

Theorem 3.8. *Let $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction. Then the following statements are equivalent:*

(i) *T is mean ergodic to rank one operator, i.e. there is $h \in L^1(\Omega, \mathcal{F}, \mu)$, $h \geq 0$ such that for every $x \in L^1(\Omega, \mathcal{F}, \mu)$*

$$\lim_{n\to\infty} ||A_n(T)x - l[h]x|| = 0; \tag{3.5}$$

(ii) *T is mean completely mixing and means smoothing with respect to some $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$, $f \neq 0$.*

Proof. (i)⇒(ii). Assume that $u \in L^1(\Omega, \mathcal{F}, \mu)$ such that $\int u d\mu = 0$. It then follows from (3.5) that

$$A_n(T)u \to h \int u d\mu = 0.$$

Hence, T is mean completely mixing.

Now, let us take $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$ with $||f|| = 1$. Then $A_n(T)f \to h$; and $A_n(T)f$ converges weakly to h. By Remark 3.6, we infer that T is mean smoothing w.r.t. f.

(ii)⇒(i). Assume that T is mean smoothing w.r.t. f, here $f \in L^1(\Omega, \mathcal{F}, \mu)$, $f \geq 0$, $f \neq 0$. Then, according to Theorem 3.4 there are two possibilities:

(a) $||A_n(T)f|| \to 0$;

(b) there is $h \in L^1(\Omega, \mathcal{F}, \mu)$, $h \geq 0$, $h \neq 0$ such that $Th = h$.

If, we are in the situation (a), then the mean completely mixing of T yields that $A_n(T)x \to 0$ for any $x \in L^1(\Omega, \mathcal{F}, \mu)$.

Now, if (b) holds, then we may assume that $||h|| = 1$. Again, due to the mean completely mixing, one has

$$A_n(T)(g - h) = A_n(T)g - h \to 0$$

for every $g \in L^1(\Omega, \mathcal{F}, \mu)$, $g \geq 0$, $||g|| = 1$. This implies $A_n(T)x \to l[h]x$ for all $x \in L^1(\Omega, \mathcal{F}, \mu)$. The proof is completed. □

We remark that some sufficient conditions for the mean ergodicity (to rank one operator) of Markov operators have studied in [12, 13, 29].

4. P-COMPLETE MIXING AND P-COMPLETE MEAN MIXING

In this section, we are going to introduce generalizations of complete mixing and complete mean mixing, respectively. The new ones will depend on some Markov projection P. Let, as before, $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a nontrivial (i.e. $P \neq Id$) Markov projection.

Definition 4.1. A positive contraction $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is called:

(i) *P-completely mixing* if

$$\lim_{n\to\infty} ||T^n(u - v)|| = 0 \quad \text{for all} \quad u, v \in \mathcal{D}, \quad u \wedge v = 0, \quad u - v \in \ker P; \tag{4.1}$$

(ii) *mean P-completely mixing* if

$$\lim_{n\to\infty} ||A_n(T)(u - v)|| = 0 \quad \text{for all} \quad u, v \in \mathcal{D}, \quad u \wedge v = 0, \quad u - v \in \ker P. \tag{4.2}$$

We notice that if $f \in \mathcal{D}$ and $P = l[f]$, then one can see P-completely mixing (resp. mean P-completely mixing) coincides with completely mixing (resp. mean completely mixing).

Given a projection P we denote

$$Fix(P) = \{x \in L^1(\Omega, \mathcal{F}, \mu) : \ Px = x\},$$
$$Fix(P)_+ = \{x \in L^1(\Omega, \mathcal{F}, \mu) : \ Px = x, \ x \geq 0\}.$$

We emphasize if P is a Markov projection, then we always have $Fix(P)_+ \neq \emptyset$.

Now, we introduce a notion of P-mean smoothing. Namely, a positive contraction $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ is called *P-smoothing* (resp. *P-mean smoothing*) with respect to $Fix(P)_+$, if for every $f \in Fix(P)_+$ if for every $\varepsilon > 0$ there exists $\tau > 0$ such that

$$\int_E T^n f d\mu < \varepsilon \quad \left(\text{resp.} \ \int_E A_n(T) f d\mu < \varepsilon\right) \tag{4.3}$$

for every $E \in \mathcal{F}$ with $\mu(E) < \tau$ and for all $n \in \mathbb{N}$.

Then using the same argument of the proof of Theorem 3.4 we can establish the following result.

Theorem 4.2. *Let $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction and $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection such that $TP = PT$. Then the following statements hold true:*

(i) *Assume that T is P-smoothing with respect to $Fix(P)_+$. Then either $\lim_{n\to\infty} ||T^n f|| = 0$ for all $f \in Fix(P)_+$ or there is $f \in Fix(P)_+$ with $\lim_{n\to\infty} ||T^n f|| \neq 0$ and $h_f \in Fix(T)_+ \cap Fix(P)_+$, $h_f \neq 0$.*

(ii) *Assume that T is P-mean smoothing with respect to* $Fix(P)_+$. *Then either* $\lim_{n\to\infty} ||A_n(T)f|| = 0$ *for all* $f \in Fix(P)_+$ *or there is* $f \in Fix(P)_+$ *with* $\lim_{n\to\infty} ||A_n(T)f|| \neq 0$ *and* $h_f \in Fix(T)_+ \cap Fix(P)_+$, $h_f \neq 0$.

Proof. Since P is a Markov projection, its range $P(X) = Fix(P)$ is a (closed) sublattice of L^1, and hence, it is isometrically lattice isomorphic to another L^1-space (and the underlying measure can be chosen to be finite, too). Moreover, since P and T commute, T leaves $Fix(P)$ invariant. Hence, by applying Theorem 3.4 to the restriction $T\lceil_{Fix(P)}$ we get the desired assertion. □

Let $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive projection. Then $Fix(P)_+ = P(L^1_+)$, therefore,

$$Fix(P) = P(L^1) = P(L^1_+) - P(L^1_+) = Fix(P)_+ - Fix(P)_+. \tag{4.4}$$

Now, using P-complete mixing, we obtain the asymptotic stability of positive contractions of L^1-spaces.

Theorem 4.3. *Let* $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ *be a positive contraction and* $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ *be a Markov projection such that* $TP = PT$. *Then the following statements are equivalent:*

(i) *T is asymptotical stable to some projection Q with* $Q \leq P$ *(i.e.* $Q = QP = PQ$*), i.e. for every* $x \in L^1(\Omega, \mathcal{F}, \mu)$ *one has*

$$\lim_{n\to\infty} ||T^n x - Qx|| = 0; \tag{4.5}$$

(ii) *T is P-complete mixing and P-smoothing with respect to* $Fix(P)_+$.

Proof. (i)⇒(ii). Assume that $u \in \ker P$. Then by $Q \leq P$, we infer $Qx = 0$. Therefore, from (4.5) it follows that $\lim_{n\to\infty} T^n u = 0$. Hence, T is P-completely mixing.

Now, let us take $f \in Fix(P)_+$ with $||f|| = 1$. Then $T^n f \to Qf$, and hence, $T^n f$ convergence weakly to Qf. By Remark 3.6, we infer that T is smoothing with respect to f. The arbitrariness of f yields that T is P-smoothing with respect to $Fix(P)_+$.

(ii)⇒(i). Assume that T is P-smoothing with respect to $Fix(P)_+$. Then, according to (i) of Theorem 4.2 there are two possibilities:

(a) $\lim_{n\to\infty} ||T^n f|| = 0$ for all $f \in Fix(P)_+$;

(b) there is $f \in Fix(P)_+$ with $\lim_{n\to\infty} ||T^n f|| \neq 0$ and there exists $h_f \in Fix(T)_+ \cap Fix(P)_+$, $h_f \neq 0$.

If, we are in the situation (a), then due to (4.4), we have

$$\lim_{n\to\infty} ||T^n f|| = 0, \quad \text{for all} \quad f \in Fix(P). \tag{4.6}$$

Since, P is a projection, then $L^1(\Omega, \mathcal{F}, \mu)$ is decomposed as follows: $L^1(\Omega, \mathcal{F}, \mu) = X_1 \oplus X_2$, where $X_1 = Fix(P)$ and $X_2 = (I - P)(L^1(\Omega, \mathcal{F}, \mu))$. Therefore, (4.6) with P-completely mixing of T yields that $T^n x \to 0$ for any $x \in L^1(\Omega, \mathcal{F}, \mu)$.

Now, if (b) holds, then denote

$$B = \left\{ h_f \in Fix(T)_+ \cap Fix(P)_+ : \lim_{n\to\infty} ||T^n f|| \neq 0,\ f \in Fix(P)_+ \right\}.$$

Let $\mathfrak{B}$ be the closure of the span of B. It is clear that $\mathfrak{B} \subset Fix(P)$. By $\mathcal{B}$ we denote a σ-subalgebra of $\mathcal{F}$ generated by $\mathfrak{B}$. Let Q be the conditional expectation with respect to $\mathcal{B}$, i.e. $Q(\cdot) = \mathcal{E}(\cdot|\mathcal{B})$. One can see that $Q = QP = PQ$. From the construction of $\mathfrak{B}$, we infer that $T^n x \to Qx$ for all $x \in \mathfrak{B}$. Since

$P(x - Qx) = 0$ and P-complete mixing yields $||T^n(x - Qx)|| \to 0$, which means $T^n x \to Qx$ for all $x \in L^1(\Omega, \mathcal{F}, \mu)$. This completes the proof. □

Then using the same argument of the proof of Theorem 3.8 and Theorem 4.4 we may establish the following result.

Theorem 4.4. *Let $T : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a positive contraction and $P : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{F}, \mu)$ be a Markov projection such that $TP = PT$. Then the following statements are equivalent:*

(i) *T is mean ergodic to some projection Q with $Q \leq P$ (i.e. $Q = QP = PQ$), i.e. for every $x \in L^1(\Omega, \mathcal{F}, \mu)$ one has $\lim_{n\to\infty} ||A_n(T)x - Qx|| = 0$;*

(ii) *T is mean P-complete mixing and P-mean smoothing with respect to $Fix(P)_+$.*

FUNDING

The present work is supported by the UAEU UPAR Grant no. G00003247 (Fund no. 31S391).

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