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# The λ-Statistical Convergence in Riesz Spaces

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**Abstract**—We introduce the $\lambda$-statistical monotone and the $\lambda$-statistical order convergent sequences in Riesz spaces. We also give some relations between the lattice operations and the $\lambda$-statistical convergence in Riesz spaces, and also, some relations between the order convergence and $\lambda$-statistical order convergence.

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## 1. INTRODUCTION AND PRELIMINARIES

Statistical convergence is a generalization of the ordinary convergence of a real sequence. The idea of statistical convergence was firstly introduced by Zygmund [24] in the first edition of his monograph in 1935. Fast [10] and Steinhaus [20] independently improved this idea in the same year 1951. Several generalizations and applications of this concept have been investigated by several authors in series of papers (c.f. [3, 7, 9, 12, 13, 15]). But, statistical convergence on Riesz spaces has not been studied extensively. A few studies have been conducted on this recently; see for example [4, 5, 8, 21]. They show some relations between the order convergence and the statistical convergence on Riesz spaces. We aim to introduce a concept of the statistical convergence on Riesz spaces by using the $\lambda$-density property which is a useful and classical tool of statistical convergence (cf. [16]).

Natural density plays an important role in statistical convergence. Recall that if the limit $\lim_{n\to+\infty}\frac{1}{n}\left|\{k\le n : k\in A\}\right|$ exists then this unique limit is called *the natural density of* subset $A$ of $\mathbb{N}$, and it is mostly abbreviated by $\delta(A)$, where $|\{k\le n : k\in A\}|$ is the number of members of $A$. Also, a sequence $(x_k)$ *statistically converges* to $L$ provided that

$$\lim_{n\to+\infty}\frac{1}{n}\left|\{k\le n : |x_k-L|\ge\varepsilon\}\right|=0$$

for each $\varepsilon>0$. Then it is written by $S-\lim x_k=L$. If $L=0$ then $(x_k)$ is said to be a statistically null sequence. Throughout this paper, the vertical bar of sets will stand for *the cardinality* of sets.

Let consider a non-decreasing sequence $(\lambda_n)$ of positive scalars such that $\lambda_1=1$ and $\lambda_{n+1}\le\lambda_n+1$. Then we can construct a new sequence of intervals $I_n:=[n-\lambda_n+1,n]$. A sequence $(x_n)$ is said to be *$\lambda$-statistically convergent* to $L$ if, for every $\varepsilon>0$, we have

$$\lim_{n\to+\infty}\frac{1}{\lambda_n}\left|\{k\in I_n : |x_k-L|\ge\varepsilon\}\right|=0.$$

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Thus, we abbreviate the limit as $S_\lambda - \lim x_n = L$. Moreover, the *λ-density* of a subset $M$ of $\mathbb{N}$ is denoted by $\delta_\lambda(M) := \lim\limits_{n\to+\infty} \frac{1}{\lambda_n}\big|\{k \in I_n : k \in M\}\big|$ (cf. [16]).

Now, we turn our attention to Riesz space another concept of functional analysis introduced by F. Riesz in [18]. It has many applications in measure theory, operator theory, and optimization (cf. [1, 2, 14, 17, 22, 23]). A real-valued vector space $E$ with an order relation "$\leq$" is called *ordered vector space* whenever

(1) $x + z \leq y + z$ for all $z \in E$,

(2) $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$

for every $x, y \in E$ with $x \leq y$. An ordered vector space $E$ is called *Riesz space* or *vector lattice* if the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}$$

exist in $E$ for every vectors $x, y \in E$, respectively. For an element $x$ in a Riesz space $E$, *the positive part*, *the negative part*, and the *module* of $x$ are

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x),$$

respectively. In the present paper, the vertical bar $|\cdot|$ of elements in Riesz spaces will stand for the module of elements. It is clear that the positive and negative parts of vectors are positive. On the other hand, order convergence is crucial for the concept of Riesz spaces. Thus, we continue with its definition.

**Definition 1.** A sequence $(x_n)$ in a Riesz space $E$ is called *order convergent to* $x \in E$ whenever there exists another sequence $(y_n) \downarrow 0$, i.e., $\inf y_n = 0$ and $y_n \downarrow$, such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$.

To introduce the statistical convergence in Riesz spaces, the notion of statistical monotonic sequences was introduced and studied (cf. [4, 6, 8, 19]. We take the following notion from [21].

**Definition 2.** A sequence $(x_n)$ in a Riesz space $E$ is called statistically monotone decreasing if there exists a set $K = \{n_1 < n_2 < \cdots\}$ in $\mathbb{N}$ such that $\delta(K) = 1$ and $(x_{n_k})$ is decreasing. In this case, we write $x_n \downarrow^{st}$. Moreover, if $\inf(x_{n_k}) = x$ for some $x \in E$ then $(x_n)$ is said to be statistically monotone convergent to $x$, and abbreviated as $x_n \downarrow^{st} x$.

## 2. λ-STATISTICAL MONOTONE SEQUENCES

We begin the section with the notion of λ-statistical monotone sequence in Riesz spaces with respect to the order convergence and the λ-density.

**Definition 3.** A sequence $(x_n)$ in a Riesz space $E$ is said to be *λ-statistically decreasing* if there exists a subset $M = \{n_1 < n_2, \cdots\}$ of the natural numbers $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the sequence $(x_{n_m})_{n\in M}$ is monotone decreasing. Moreover, if $\inf(x_n) = x$ on $M$ for some $x \in E$ then $(x_n) \downarrow^{\lambda_{st}} x$.

One can define the notion of *λ-statistically increasing sequence*. Therefore, if $(x_{n_m}) \downarrow x$ or $(x_{n_m}) \uparrow x$ in $E$ then $(x_n)$ is called *λ-statistically monotone convergent* to $x$.

**Proposition 1.** *Every monotone sequence is λ-statistical monotone in Riesz spaces.*

**Proof.** Suppose that $(x_n)$ is a monotone decreasing sequence in a Riesz space $E$. Then take the subset $M = \{n_1 < n_n, \cdots\}$ in Definition 3 as $\mathbb{N}$. So, we obtain that $\delta_\lambda(M) = 1$ and $(x_n)$ is decreasing on $M$, and so, $(x_n)$ is λ-statistically decreasing. □

The converse of Proposition 1 does not hold in general. To see this, we consider the following example.

**Example 1.** Consider the Riesz space $E := \mathbb{N}$. Assume that $(x_k)$ is a sequence in $E$ defined by

$$x_k := \begin{cases} 1, & n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ k, & \text{otherwise}. \end{cases}$$

So, $(x_k)$ is a λ-statistical increasing sequence, but it is not monotone increasing.

It is clear that the order convergence does not imply $\lambda$-statistically monotone convergence because the order convergent sequence does not need to be monotone. But, we have an observation of the following result.

**Proposition 2.** *Every order convergent decreasing sequence is $\lambda_{st}$-convergent to its order limit in Riesz spaces.*

**Proof.** Assume that $x_n x$ in a Riesz space $E$. Thus, there exists another sequence $(y_n) \downarrow 0$ such that $|x_n - x| \leq y_n$ for all $n \in \mathbb{N}$. Since $(y_n)$ is decreasing, it follows from Proposition 1 that $(y_n)$ is $\lambda$-statistical monotone sequence. So, we have $(y_n) \downarrow^{\lambda_{st}} 0$. Hence, there exists a subset $M$ of the natural numbers with $\delta_\lambda(M) = 1$ and $(y_{n_m}) \downarrow 0$. Therefore, we have $(x_{n_m}) \downarrow x$. □

**Proposition 3.** *If $(x_n)$ is a $\lambda$-statistically increasing sequence in a Riesz space then the $\lambda$-density of the set $\{n \in \mathbb{N} : x_n \nleq x_{n+1}\}$ is equal to zero.*

**Proof.** Suppose that $(x_n)$ is a $\lambda$-statistically increasing sequence in a Riesz space $E$. Then there is a subset $M = \{n_1 < n_2, \cdots\}$ of $\mathbb{N}$ such that $\delta_\lambda(M) = 1$ and $(x_n)$ is monotone increasing on $M$, i.e., $x_n \leq x_n + 1$ for all $n \in M$. Thus, we have

$$\{n \in \mathbb{N} : x_n \nleq x_{n+1}\} \subseteq \mathbb{N} - M.$$

Therefore, $\delta_\lambda\big(\{n \in \mathbb{N} : x_n \nleq x_{n+1}\}\big) = 0$ because of $\delta_\lambda(\mathbb{N} - M) = 0$. □

**Corollary 1.** *If $(x_n)$ is a $\lambda$-statistically decreasing sequence then $\delta_\lambda\big(\{n \in \mathbb{N} : x_{n+1} \nleq x_n\}\big) = 0$.*

In the next result, we prove that the lattice operators are $\lambda$-statistically continuous.

**Theorem 1.** *$(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$ implies $(x_n \vee y_n) \downarrow^{\lambda_{st}} x \vee y$ in Riesz spaces.*

**Proof.** Assume that $(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$ in a Riesz space $E$. Then there exist subsets $M_1$ and $M_2$ of $\mathbb{N}$ such that $\delta_\lambda(M_1) = \delta_\lambda(M_2) = 1$, and also, $(x_{n_i})_{i \in M_1} \downarrow x$ and $(y_{n_j})_{j \in M_2} \downarrow x$ for some $x, y \in E$. Let consider the set $M = M_1 \cap M_2$. Then following from the inequality $\delta_\lambda(M_1) + \delta_\lambda(M_2) \leq 1 + \delta_\lambda(M_1 \cap M_2)$, we have $\delta_\lambda(M_1 \cap M_2) = 1$. On the other hand, $(x_n \vee y_n)$ is monotone decreasing on $M$ because both $(x_n)$ and $(y_n)$ are monotone decreasing on $M$. Now, by applying [14, Thm.12.4], we can obtain

$$\begin{aligned}|x_n \vee y_n - x \vee y| &\leq |x_n \vee y_n - y_n \vee x| + |x \vee y_n - x \vee y| \\ &\leq |x_n - x| + |y_n - y|.\end{aligned}$$

Thus, $\inf(x_n \vee y_n) = x \vee y$ on $M$ because of $\inf(x_n - x) = 0$ and $\inf(y_n - y) = 0$ on $M$. Hence, we get the desired result, $(x_n \vee y_n) \downarrow^{\lambda_{st}} x \vee y$. □

**Corollary 2.** *If $(x_n) \downarrow^{\lambda_{st}} x$ hold then we have the following facts:*

(i) $(x_n)^+ \downarrow^{\lambda_{st}} x^+$;

(ii) $(x_n)^- \downarrow^{\lambda_{st}} x^-$;

(iii) $|x_n| \downarrow^{\lambda_{st}} |x|$.

**Theorem 2.** *Let $(x_n) \uparrow^{\lambda_{st}} x$ and $(y_n) \uparrow^{\lambda_{st}} y$. Then $(x_n \vee y_n) \uparrow^{\lambda_{st}} x \vee y$.*

**Proof.** Modify Theorem 1. □

From now on, we only focus on $\lambda$-statistically decreasing sequences. Similarly, one can prove the other case. We continue with several basic and useful results that are motivated by their analogies from the Riesz space theory.

**Proposition 4.** *Let $(x_n)$ and $(y_n)$ be two sequences in a Riesz space $E$. Then, for any $x, y \in E$, the following statements hold:*

(i) $x_n \downarrow^{\lambda_{st}} x$ *if and only if* $(x_n - x) \downarrow^{\lambda_{st}} 0$;

(ii) $(x_n) \downarrow^{\lambda_{st}} x$ *and* $(y_n) \downarrow^{\lambda_{st}} y$ implies $(x_n \wedge y_n) \downarrow^{\lambda_{st}} x \wedge y$;

(iii) $(x_n) \downarrow^{\lambda_{st}} x$ *implies* $(\alpha x_n) \downarrow^{\lambda_{st}} \alpha x$ *for every* $\alpha \in \mathbb{R}$;

(iv) $(x_n) \downarrow^{\lambda_{st}} x$ *and* $(y_n) \downarrow^{\lambda_{st}} y$ *implies* $(x_n + y_n) \downarrow^{\lambda_{st}} (x + y)$;

(v) $(x_{n_k}) \downarrow^{\lambda_{st}} x$ *is hold for any subsequence* $(x_{n_k})$ *of* $(x_n) \downarrow^{\lambda_{st}} x$ *whenever* $(x_{n_k})$ *is decreasing and* $\delta_\lambda(\{n_1, n_2, n_3, \cdots\}) = 1$;

(vi) $(x_n) \downarrow^{\lambda_{st}} x$ *and* $(x_n) \downarrow^{\lambda_{st}} y$ implies $x = y$;

(vii) $0 \leq (x_n) \downarrow^{\lambda_{st}} x$ *implies* $x \in E_+$;

(viii) *if* $0 \leq y_n \leq x_n$ *for all* $n \in \mathbb{N}$, $(x_n) \downarrow^{\lambda_{st}} 0$, *and* $(y_n)$ *is decreasing then* $(y_n) \downarrow^{\lambda_{st}} 0$;

(ix) *if* $(x_n) \downarrow^{\lambda_{st}} x$, $(y_n) \downarrow^{\lambda_{st}} y$, and $x_n \geq y_n$ *for all* $n \in \mathbb{N}$ *then* $x \geq y$.

**Proof.** The axioms $(i)$ and $(iii)$ follow immediately from Definition 3. Also, by using [2, Thm.1.3(1)], we obtain $(ii)$.

$(iv)$ Suppose that $(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$ in $E$. Then there exist subsets $M_1$ and $M_2$ of $\mathbb{N}$ such that $\delta_\lambda(M_1) = \delta_\lambda(M_2) = 1$ and the sequences $(x_n)$ and $(y_n)$ are monotone decreasing to $x$ and $y$ on $M_1$ and $M_2$, respectively. Take a new subset $M = M_1 \cap M_2$ of $\mathbb{N}$. Then it is clear that $\delta_\lambda(M) = 1$ and $(x_n + y_n)$ is a decreasing sequence on $M$. It follows from the inequality $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y|$ that $(x_n + y_n) \downarrow (x + y)$ on $M$. Therefore, $(x_n + y_n) \downarrow^{\lambda_{st}} (x + y)$.

$(v)$ Assume that $(x_n) \downarrow^{\lambda_{st}} x$ in $E$. Then there exists a subset $M$ of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the sequence $(x_{n_m})$ is monotone decreasing to $x$. Thus, it follows from Proposition 2 that $(x_{n_m}) \downarrow^{\lambda_{st}} x$. However, we should show the argument for arbitrary subsequences. By the way, consider a decreasing subsequence $(x_{n_k})$ of $(x_n)$ such that $\delta_\lambda(K) = 1$ for $K = \{n_1, n_2, n_3, \cdots\}$. Assume $K \neq M$. Otherwise, the proof is obvious. Also, if $K$ does not exist then there is nothing to prove. Now, we prove $(x_{n_k}) \downarrow^{\lambda_{st}} x$. Since $(x_{n_m})$ is monotone decreasing to $x$, we have $(x_{n_m}) \geq x$ for all $m \in M$. Also, we can see that $M$ and $L$ are almost equal because the $\lambda$-density of the set $J = M \cap K$ is equal to one. Hence, we can find a subsequence $(x_{n_{k_j}})$ of $(x_{n_k})$ such that $x$ is the lower bound of it. Also, it is clear that $(x_{n_{k_j}})$ is monotone decreasing and the $\lambda$-density of its index set is equal to one. Take another lower bound $w \in E$ of $(x_{n_{k_j}})$, i.e. $x_{n_{k_j}} \geq w$ for all $j \in \mathbb{N}$. Fix an index $j$. Then since $M$ and $L$ are almost equal, one can find an index $m_j \in M$ so that $x_{m_j} = x_{n_{k_j}} \geq w$. Thus, we get $f \geq w$ because $x$ is the infimum of $(x_{n_m})$. As a result, we see that $x$ is the infimum of $(x_{n_{k_j}})$, i.e., $(x_{n_k}) \downarrow^{\lambda_{st}} x$.

$(vi)$ Suppose that $(x_n) \downarrow^{\lambda_{st}} x$ and $(x_n) \downarrow^{\lambda_{st}} y$ in $E$. Then there exist subsets $M$ and $K$ of $\mathbb{N}$ with $\delta_\lambda(M) = \delta_\lambda(K) = 1$ such that the subsequences $(x_{n_m})$ and $(x_{n_k})$ are monotone decreasing to $x$ and $y$, respectively. Now, if we choose $J = M \cap K$ then we have $\delta_\lambda(J) = 1$. Thus, we can consider a subsequence $(x_{n_j})$. So, $(x_{n_j})$ is monotone decreasing to both $x$ and $y$ because $(x_{n_j})$ is a subsequence of both $(x_{n_m})$ and $(x_{n_k})$. Therefore, we obtain $x = y$ because the order limits are uniquely determined.

$(vii)$ Assume $0 \leq (x_n) \downarrow^{\lambda_{st}} x$. Then, by using Corollary 2, we have $(x_n) = (x_n)^+ \downarrow^{\lambda_{st}} = x^+$. So, it follows from $(vi)$ that $x = x^+ \in E_+$.

$(viii)$ Suppose $0 \leq y_n \downarrow \leq x_n$ for all $n \in \mathbb{N}$ and $(x_n) \downarrow^{\lambda_{st}} 0$ in $E$. Then there is a subset $M$ of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the subsequence $(x_{n_m})$ is monotone decreasing to $x$. Now, consider the subsequence $(y_{n_m})$ of $(y_n)$. Then we have $0 \leq (y_{n_m}) \leq (x_{n_m})$ for all $n_m \in M$. Hence, we get $(y_{n_m}) \downarrow 0$ because of $(x_n) \downarrow 0$ on $M$.

$(ix)$ By applying $(iii)$ and $(iv)$, we can obtain $(x_n - y_n) \downarrow^{\lambda_{st}} (x - y)$. Next, by using $(vii)$, one can see that $x - y \in E_+$, i.e., $x \geq y$ because of $0 \leq (x_n - y_n) = (x_n - y_n)^+$. □

## 3. THE λ-STATISTICAL ORDER CONVERGENCE

We begin with the following definition which is crucial for the present paper.

**Definition 4.** Let $E$ be a Riesz space and $(x_n)$ be a sequence in $E$. Then $(x_n)$ is called $\lambda$-*statistical order convergent* to $x \in E$ if there exist another sequence $(y_n) \downarrow^{\lambda_{st}} 0$ in $E$ and a subset $M$ of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that $|x_{n_m} - x| \leq y_{n_m}$ holds for each $n_m \in M$. We abbreviate it as $x_n \xrightarrow{\lambda_{st}o} x$.

For a given sequence $(x_n)$ in a Riesz space, one can observe that if there exists another sequence $(y_n) \downarrow^{\lambda_{st}} 0$ such that the $\delta_\lambda$-density of the set $\{n \in \mathbb{N} : |x_n - x| \nleq y_n\}$ is equal to zero then $x_n \xrightarrow{\lambda_{st}o} x$.

**Remark 1.** It is clear that the $\lambda$-statistically monotone convergence implies the $\lambda$-statistical order convergence. Indeed, suppose that a sequence $(x_n)$ is $\lambda$-statistically decreasing to $x$ in a Riesz space $E$. Hence, there exists a subset $M$ of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the sequence $(x_{n_m})$ is monotone decreasing to $x$. Thus, we have $(w_{n_m}) := (x_{n_m} - x) \downarrow 0$. Now, by applying Proposition 2, we obtain $(w_{n_m}) \downarrow^{\lambda_{st}} 0$. Therefore, we get $x_n \xrightarrow{\lambda_{st}o} x$ because of $|x_{n_m} - x| \leq w_{n_m}$.

**Proposition 5.** *The order convergence implies the $\lambda$-statistical order convergence in Riesz spaces.*

**Proof.** Suppose $x_n x$ in a Riesz space $E$. Then there exists another sequence $(y_n) \downarrow 0$ in $E$ such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$. Now, by using Proposition 2, we can get $(y_n) \downarrow^{\lambda_{st}} 0$. So, there is a subset $M$ such that $\delta_\lambda(M) = 1$ and $(y_{n_m}) \downarrow 0$. Moreover, we have $|x_{n_m} - x| \leq y_{n_m}$, and so, we get $x_n \xrightarrow{\lambda_{st}o} x$. □

Now, we give several basic and useful results.

**Theorem 3.** *Let $E$ be Riesz spaces. Then the following conditions hold*:

(i) $x_n \xrightarrow{\lambda_{st}o} x$ *if and only if* $(x_n - x) \xrightarrow{\lambda_{st}o} 0$ *if and only if* $|x_n - x| \xrightarrow{\lambda_{st}o} 0$;

(ii) *the lattice operations are $\lambda$-statistically order continuous;*

(iii) *the $\lambda_{st}o$-limit is linear;*

(iv) *the $\lambda_{st}o$-convergence has an unique limit;*

(v) *the positive cone $E_+$ is closed under the $\lambda_{st}o$-convergence in $E$.*

**Proof.** $(i)$ It can be observed from Definition 4.

$(ii)$ Assume that $(x_n) \xrightarrow{\lambda_{st}o} x$ and $(y_n) \xrightarrow{\lambda_{st}o} y$ hold in $E$. Then there exist some sequences $(u_n) \downarrow^{\lambda_{st}} 0$ and $(v_n) \downarrow^{\lambda_{st}} 0$, and subsets $M$ an $K$ of $\mathbb{N}$ such that $\delta_\lambda(M) = \delta_\lambda(K) = 1$, and $|x_{n_m} - x| \leq u_{n_m}$ and $|y_{n_k} - x| \leq v_{n_k}$ for all $n_m \in M$ and $n_k \in K$. It is enough to show that $x_n \vee y_n \xrightarrow{\lambda_{st}o} x \vee y$. Now, by applying [2, Thm.1.9(2)], we obtain the inequality

$$|x_n \vee y_n - x \vee y| \leq |x_n - x| + |y_n - y|.$$

Thus, one can get the assertion from the following fact $\{n \in \mathbb{N} : |x_n \vee y_n - x \vee y| \nleq v_n + u_n\} \subseteq \{n \in \mathbb{N} : |x_n \vee y_n - x \vee y| \nleq v_n\} \cup \{n \in \mathbb{N} : |x_n \vee y_n - x \vee y| \nleq u_n\}$. The other cases of lattice operations are analogous.

$(iii)$ The part of the scalar multiplication is clear. Thus, we show the additive part. Consider two sequences $(x_n) \xrightarrow{\lambda_{st}o} x$ and $(y_n) \xrightarrow{\lambda_{st}o} y$ in $E$. Then there exist some sequences $(u_n) \downarrow^{\lambda_{st}} 0$ and $(v_n) \downarrow^{\lambda_{st}} 0$ such that $\delta_\lambda\big(\{n \in \mathbb{N} : |x_n - x| \nleq u_n\}\big) = 0$ and $\delta_\lambda\big(\{n \in \mathbb{N} : |y_n - y| \nleq v_n\}\big) = 0$. Also, one can obtain

$$\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \nleq v_n + u_n\} \subseteq \{n \in \mathbb{N} : |x_n - x| \nleq u_n\} \cup \{n \in \mathbb{N} : |y_n - y| \nleq v_n\}.$$

Following from Proposition 4$(iv)$ and $\delta_\lambda\big(\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \nleq v_n + u_n\}\big) = 0$, we get the desired result.

$(iv)$ Suppose that $x_n \xrightarrow{\lambda_{st}o} x$ and $x_n \xrightarrow{\lambda_{st}o} y$ hold in $E$. Then we have sequences $(u_n) \downarrow^{\lambda_{st}} 0$ and $(v_n) \downarrow^{\lambda_{st}} 0$ and a subset $M$ of $\mathbb{N}$ such that $\delta_\lambda(M) = 0$, $|x_{n_m} - x| \leq u_{n_m}$ and $|x_{n_m} - y| \leq v_{n_m}$. Following from the inequality

$$0 \leq |x - y| \leq |x - x_{n_m}| + |x_{n_m} - y| \leq u_{n_m} + v_{n_m},$$

we obtain $x = y$ because of $u_{n_m} \downarrow 0$ and $v_{n_m} \downarrow 0$.

$(v)$ Suppose that $(x_n)$ is a non-negative sequence, and it is $\lambda_{st}o$-convergent to $x \in E$. It follows from $(ii)$ and $(iv)$ that $x_n = x_n^+ \xrightarrow{\lambda_{st}o} x^+ = x$. So, we get the desired result, $x \in E_+$. □

**Proposition 6.** *Let $(x_n), (y_n)$ and $(z_n)$ be sequences in a Riesz space $E$ such that $z_n \leq y_n \leq x_n$ holds for all $n \in M \subseteq \mathbb{N}$ with $\delta_\lambda(M) = 1$. If $z_n \xrightarrow{\lambda_{st}o} 0$ and $x_n \xrightarrow{\lambda_{st}o} 0$ in $E$ then $y_n \xrightarrow{\lambda_{st}o} 0$.*

**Proof.** Modify [21, Thm.7.]. □

For the converse of Proposition 5, we give the next result.

**Proposition 7.** *Every monotone λ-statistical order convergent sequence is order convergent to its $\lambda_{st}o$-limit in Riesz spaces.*

**Proof.** It is enough to show that if $E \ni x_n \uparrow$ and $x_n \xrightarrow{\lambda_{st}o} x$ then $x_n \uparrow x$. Take an arbitrary index $n_0$. Then $x_n - x_{n_0} \in X_+$ for $n \geq n_0$. By using $(iii)$ and $(v)$ of Theorem 3, we have $x_n - x_{n_0} \xrightarrow{\lambda_{st}o} x - x_{n_0} \in E_+$. Thus, we get $x \geq x_{n_0}$ for any $n$. Since $n_0$ is arbitrary, $x$ is an upper bound of $x_{n_0}$. Now, assume that $y \geq x_n$ for all $n$. Then, by using Theorem 3, we have $y - x_n \xrightarrow{\lambda_{st}o} y - x \in E_+$, or $y \geq x$. Thus, $x_n \uparrow x$. □

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