On Generalized Solutions of the Problems of Electromagnetic Wave Diffraction in the Open Space

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Abstract—In this article we construct a special class of the generalized functions for the rigorous justification of joining method of solving some diffraction problems of electromagnetic waves by thin conducting screens. Linear functionals on a set of linear combinations of Hermit functions are considered as the generalized functions. The traces of the solutions of Helmholtz equation on the plane are interpreted in the generalized sense. The infinite sets of linear algebraic equations are derived directly from the generalized boundary conditions. The results of the computing experiment are presented.

1. INTRODUCTION

Special classes of generalized functions were built in the work [1], it is convenient to use these functions when solving problems of diffraction of electromagnetic waves on transverse screens in cylindrical waveguides with metal walls. The generalized functions were defined as linear functionals on set of linear combinations of functions that form a complete orthogonal set at the cross-section of the waveguide. Similar constructions were used in the theory of φ-distributions in [2].

It was proposed to consider generalized solutions of electromagnetic field equations as mappings, these mappings assign a generalized function (or a set of generalized functions) to each value of the longitudinal coordinates. This approach allows to define correctly the traces of the unknown solutions of partial differential equations on the cross-section of the cylindrical domain in the case when mixed boundary conditions are set on this section. Moreover, the boundary conditions of diffraction problems on the lateral screen in a cylindrical waveguide in the terms of generalized solutions of Helmholtz equation and Maxwell equations are formulated directly as an infinite set of equation for the coefficients of expansion of the field by eigen waves of the waveguide.

In this paper, the generalized functions are defined as linear functionals on set of linear combinations of Hermit functions. The example of the problem of diffraction of electromagnetic wave on a thin conductive ribbon shows that the generalized boundary conditions of this problem are equations of some infinite set of linear algebraic equations. This set of equations can be derived also from the integral equation of the diffraction problem, which we propose to consider as an integral equation of the 4th kind.

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2. HERMIT FUNCTIONS AND GENERALIZED FUNCTIONS

We recall some properties of the Hermit functions. The Hermit polynomials [3], 10.13,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x), \quad m = 1, 2, \ldots$$

are orthogonal with a weight e^{-x^2} set of functions on the axis $(-\infty, +\infty)$. Hermit functions

$$h_m(x) = e^{-x^2/2} H_m(x), \quad m = 0, 1, \ldots$$

are also orthogonal, but are not normalized,

$$\int\limits_{-\infty}^{+\infty} h_k(x)h_m(x)dx = \{k \neq m : 0; \quad k = m : h_m\}, \quad h_m = 2^m m! \sqrt{\pi}.$$

Here and then h_m with brackets are the Hermit functions, and h_m without brackets are squares of norms.

For any $m = 0, 1, 2, \ldots$ we have

$$h'_m(x) = mh_{m-1}(x) - {}^1/_2 h_{m+1}(x), \quad xh_m(x) = mh_{m-1}(x) + {}^1/_2 h_{m+1}(x),$$

further,

$$h''_m(x) = m(m-1)h_{m-2}(x) - (m + {}^1/_2)h_m(x) + {}^1/_4 h_{m+2}(x),$$

$$x^2 h_m(x) = m(m-1)h_{m-2}(x) + (m + {}^1/_2)h_m(x) + {}^1/_4 h_{m+2}(x), \ldots$$

Finally, the Hermit functions are the eigne functions of Fourier integral transformation $\mathcal{F}$:

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} h_m(x)e^{ix\xi}dx = i^m h_m(\xi)$$

or briefly, $(\mathcal{F}h_m)(\xi) = i^m h_m(\xi)$. Here and further i is an imaginary unit.

Simultaneously with the functions $h_m(x)$ we will use the functions $\widetilde{h}_m(x) = h_m(x)/h_m$ (they'll also be called Hermit functions). Functions $h_m(x)$ and $\widetilde{h}_m(x)$ are bi-orthonormal sets of function.

There are also simple formulas for the functions $\widetilde{h}_m(x)$ for differentiation and multiplication by degrees of x:

$$\widetilde{h}'_0(x) = -\widetilde{h}_1(x), \quad \widetilde{h}'_m(x) = {}^1/_2 \widetilde{h}_{m-1}(x) - (m+1)\widetilde{h}_{m+1}(x), \quad m = 1, 2, \ldots$$

$$x\widetilde{h}_0(x) = \widetilde{h}_1(x), \quad x\widetilde{h}_m(x) = {}^1/_2 \widetilde{h}_{m-1}(x) + (m+1)\widetilde{h}_{m+1}(x), \quad m = 1, 2, \ldots$$

$$\widetilde{h}''_0(x) = -{}^1/_2 \widetilde{h}_0(x) + 2\widetilde{h}_2(x), \quad \widetilde{h}''_1(x) = -{}^3/_2 \widetilde{h}_1(x) + 6\widetilde{h}_3(x),$$

$$\widetilde{h}''_m(x) = {}^1/_4 \widetilde{h}_{m-2}(x) - (m + {}^1/_2)\widetilde{h}_m(x) + (m+1)(m+2)\widetilde{h}_{m+2}(x), \quad m = 2, 3, \ldots$$

$$x^2\widetilde{h}_0(x) = {}^1/_2 \widetilde{h}_0(x) + 2\widetilde{h}_2(x), \quad x^2\widetilde{h}_1(x) = {}^3/_2 \widetilde{h}_1(x) + 6\widetilde{h}_3(x),$$

$$x^2\widetilde{h}_m(x) = {}^1/_4 \widetilde{h}_{m-2}(x) + (m + {}^1/_2)\widetilde{h}_m(x) + (m+1)(m+2)\widetilde{h}_{m+2}(x), \quad m = 2, 3, \ldots$$

and et cetera.

As the *generalized functions* (g.f.), we will consider *linear functionals* on a set of linear combinations of the Hermit functions $h_m(\cdot)$. Any such g.f. $f[\cdot]$ can be identified with the set (sequence) of its values $f_0, f_1, \ldots$ on functions $h_0(\cdot), h_1(\cdot), \ldots$

We use round or square brackets to refine type of universal parameter $\cdot$ ("dot" or "joker").

The Hermit functions are functions that quickly decrease on infinity. A set of their linear combinations is a subspace of a space of fast decreasing functions. Therefore, the linear space of the g.f. contains in itself the space of Schwartz distributions [4].

The functional

$$f[\cdot]:\ h_m(\cdot) \mapsto f_m = \int\limits_{-\infty}^{+\infty} f(x)\widetilde{h}_m(x)dx, \quad m = 0, 1, \ldots$$

corresponds to the integrable function $f(\cdot)$. Its values on the Hermit functions are the coefficients of expansion into the Fourier series by the Hermit functions $h_m(\cdot)$.

If a series with coefficients f_m converges, and its sum is equal to the function $f(\cdot)$, that is, the function $f(\cdot)$ expands into a Fourier series by Hermit functions, we will call the appropriate to it g.f. *regular*. For regular g.f. we will use the designations of the form $f[x]$.

In general, the formula $f(\cdot) = \sum_{m=0}^{+\infty} f_m h_m(\cdot)$ (formal series by the Hermit functions) will be considered only as instruction that the appropriate functional accepts values $f_m =$ on functions $h_m(\cdot)$.

3. OPERATIONS ON GENERALIZED FUNCTIONS

It is clear that a set of g.f. is a linear space with obvious addition and multiplication operations by number. Other linear operations on the g.f. are introduced in a similar way as operations over distributions [5].

We define *multiplying the g.f. by the ordinary function* as follows:

$$a(\cdot)f[\cdot] : h_m(\cdot) \mapsto f[a(\cdot)h_m(\cdot)],$$

if $a(\cdot)h_m(\cdot)$ is a linear combination of Hermit functions.

For example, as $xh_m(x) = mh_{m-1}(x) + {}^1\!/_2 h_{m+1}(x)$ and $x^2 h_m(x) = m(m-1)h_{m-2}(x) + (m + {}^1\!/_2)h_m(x) + {}^1\!/_4 h_{m+2}(x)$, then $(xf)_m = mf_{m-1} + {}^1\!/_2 f_{m+1}$ and $(x^2 f)_m = m(m-1)f_{m-2} + (m + {}^1\!/_2)f_m + {}^1\!/_4 f_{m+2}$ (instead of $a(x) = x$ or $a(x) = x^2$ we write x or x^2). More generally, let

$$a(x)h_m(x) = \sum_{n=0}^{+\infty} a_{mn}h_n(x), \quad m = 0, 1, \ldots, \quad a_{mn} = \int\limits_{-\infty}^{+\infty} a(x)h_m(x)\widetilde{h}_n(x)dx, \quad n = 0, 1, \ldots$$

Then by definition

$$a(\cdot)f[\cdot] : h_m(\cdot) \mapsto \sum_{n=0}^{+\infty} a_{mn}f_n$$

(if all these series converge). It is clear that if the sums are finite, this definition is just the same as the previous definition.

Let's define *first derivative* of g.f. $f[\cdot]$ as g.f. $f'[\cdot]$ with values $(f')_m = -{}^1\!/_2 f_{m-1} + (m + 1)f_{m+1}$. We have for an ordinary differentiable function

$$(f')_m = \int\limits_{-\infty}^{+\infty} f'(x)\widetilde{h}_m(x)dx = -\int\limits_{-\infty}^{+\infty} f(x)\widetilde{h}'_m(x)dx$$

$$= -\int\limits_{-\infty}^{+\infty} f(x)[{}^1\!/_2\widetilde{h}_{m-1}(x) - (m + 1)\widetilde{h}_{m+1}(x)]dx = -{}^1\!/_2 f_{m-1} + (m + 1)f_{m+1}.$$

When $m = 0$ and $m = 1$, we set $\widetilde{h}_m(x) \equiv 0$ for $m < 0$.

Second derivative of g.f. $f[\cdot]$ is linear functional with values

$$(f'')_m = {}^1\!/_4 f_{m-2} - (m + {}^1\!/_2)f_m + (m + 1)(m + 2)f_{m+2}.$$

It is easy to see also that the second derivative of the g.f. is a derivative of the first derivative. Any next derivative is calculated as a derivative of the previous derivative.

Fourier's transformation $\mathcal{F}$ of g.f. is a linear functional also, whose values are $(\mathcal{F}f)_m = i^m f_m$. This definition coincides with the definition used in the distribution theory:

$$\mathcal{F}f[\cdot] : h_m(\cdot) \mapsto f[\mathcal{F}h_m(\cdot)] = i^m f[h_m(\cdot)].$$

Let's find the Fourier transform of the first derivative of the g.f.:

$$(\mathcal{F}f')_m = i^m(f')_m = i^m[-{}^1/_2 f_{m-1} + (m+1)f_{m+1}] = -{}^1/_2 i^m f_{m-1} + (m+1)i^m f_{m+1}$$

$$= -{}^1/_2 i(\mathcal{F}f)_{m-1} - (m+1)i(\mathcal{F}f)_{m+1} = (-i\xi(\mathcal{F}f))_m.$$

Similarly, the Fourier transform of the second derivative of the g.f. has the form

$$(\mathcal{F}f'')_m = i^m(f'')_m = i^m[{}^1/_4 f_{m-2} - (m+{}^1/_2)f_m + (m+1)(m+2)f_{m+2}]$$

$$= -{}^1/_4 i^{m-2} f_{m-2} - (m+{}^1/_2)i^m f_m - (m+1)(m+2)i^{m+2} f_{m+2}$$

$$= -{}^1/_4(\mathcal{F}f)_{m-2} - (m+{}^1/_2)(\mathcal{F}f)_m - (m+1)(m+2)(\mathcal{F}f)_{m+2} = (-\xi^2(\mathcal{F}f))_m.$$

here we use the degrees ξ as multipliers at $\mathcal{F}f$ to emphasize that the second multiplier is the Fourier transform.

The sequence of g.f. $f_j[\cdot]$ *converges* to the g.f. $f[\cdot]$ for $j \to +\infty$, if $f_{j,m} = (f_j)_m \to f_m$ when $j \to +\infty$ $\forall m = 0, 1, \ldots$ The parameter that extracts the g.f. in the parametric family is not necessarily integer. For example, the sequence of g.f. $f_z[\cdot]$ converges to the g.f. $f_0[\cdot]$ for $z \to 0$, if $(f_z)_m \to (f_0)_m$ when $z \to 0$ for all $m = 0, 1, \ldots$

The values of the g.f. at the points are not defined. G.f. $f[\cdot]$ and $g[\cdot]$ are equal, if $f_m = g_m$ $\forall m$.

Now let the numerical axis x consist of two parts: $\mathcal{M}$ and $\mathcal{N}$. We will say that $f[\cdot] = 0$ on $\mathcal{M}$ or on $\mathcal{N}$, if

$$\sum_{m=0}^{+\infty} I_{mn} f_m = 0 \quad n = 0, 1, \ldots \quad \text{or} \quad \sum_{m=0}^{+\infty} J_{mn} f_m = 0 \quad n = 0, 1, \ldots,$$

here

$$I_{mn} = \int_{\mathcal{M}} h_m(x)\widetilde{h}_n(x)dx, \quad J_{mn} = \int_{\mathcal{N}} h_m(x)\widetilde{h}_n(x)dx.$$

It's easy to see that $J_{mn} = \delta_{mn} - I_{mn}$. So $f[\cdot] = 0$ on $\mathcal{M}$ or on $\mathcal{N}$ then and only then if

$$f_n = \sum_{m=0}^{+\infty} J_{mn} f_m \quad n = 0, 1, \ldots \quad \text{or} \quad f_n = \sum_{m=0}^{+\infty} I_{mn} f_m \quad n = 0, 1, \ldots$$

If g.f. $f[\cdot]$ coincides with the ordinary function $g(\cdot)$ on $\mathcal{M}$, then

$$\sum_{m=0}^{+\infty} I_{mn} f_m = \int_{\mathcal{M}} g(x)\widetilde{h}_n(x)dx, \quad n = 0, 1, \ldots$$

4. ABSTRACT FUNCTIONS. HELMHOLTZ EQUATION

Let z be a number belonging to a certain interval of the real axis. Mappings $z \mapsto$ of g.f. will be called abstract functions (a.f.). Then the a.f. $u(z)[\cdot]$ is identified with the sequence of functions $u_m(z)$, their values at a fixed z are the values of g.f. on $h_m(\cdot)$.

The properties of a.f. on the argument z are determined through the properties of the functions $u_m(z)$. For example, if all functions $u_m(z)$ are differentiable, then the abstract function $u(z)[\cdot]$ will be called a differentiable by z. Then $u'(z)[\cdot]$ is the a.f. also with values $u'_m(z)$ on $h_m(\cdot)$.

How to solve differential equations in which derivatives of a.f. by z are involved? If $u'(z)[\cdot] = 0[\cdot]$, which is equivalent to $u'_m(z) = 0$, $m = 0, 1, \ldots$, then $u_m(z) = c_m$ or $u(z)[\cdot] = c[\cdot]$, where $c[\cdot]$ is an arbitrary g.f.

If a linear homogeneous equation of an order n is set, and its coefficients do not depend on z (i.e. they are functions of x, and multiplication by function of x is performed by the rule of multiplication of the g.f. by the usual function), we reason like this (by analogy with the Euler method). We build a characteristic polynomial of λ and find its roots. If $\lambda(x)$ is a simple root of characteristic polynomial, then $e^{\lambda(x)z}$ is a particular solution of the differential equation, it can be multiplied by arbitrary g.f. (more precisely, the arbitrary g.f. is multiplied by ordinary function $e^{\lambda(x)z}$ of x). Then the summand $e^{\lambda(x)z}c[\cdot]$ appears in the general solution of the equation.

We'll look for a.f. $u(z)[\cdot]$, satisfying Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u(x, z) = 0.$$

Helmholtz equation in generalized sense is understood as an infinite set of equations

$$u_m''(z) + k^2 u_m(z) + {}^1/_4 u_{m-2}(z) - (m + {}^1/_2)u_m(z) + (m+1)(m+2)u_{m+2}(z) = 0, \quad m = 0, 1, \dots$$

Multiply these equations by i^m, this operation is equivalent to Fourier transformation on the variable x. In front of $u_m(z)$ there will appear a multiplier i^m, and in front of $u_{m-2}(z)$ and $u_{m+2}(z)$ the multipliers $-i^m$ will appear.

For $(\mathcal{F}u)_m(z) = i^m u_m(z)$ we have equations $(m = 0, 1, \dots)$

$$(\mathcal{F}u)_m''(z) + k^2(\mathcal{F}u)_m(z) - {}^1/_4(\mathcal{F}u)_{m-2}(z) - (m + {}^1/_2)(\mathcal{F}u)_m(z)$$
$$- (m+1)(m+2)(\mathcal{F}u)_{m+2}(z) = 0.$$

This is equivalent to the equation

$$(\mathcal{F}u)''(z)[\xi] + (k^2 - \xi^2)(\mathcal{F}u)(z)[\xi] = 0,$$

and general solution of which has the form

$$(\mathcal{F}u)(z)[\xi] = e^{i\gamma(\xi)z}a[\xi] + e^{-i\gamma(\xi)z}b[\xi].$$

Here we write down ξ instead of "dot" in order to suggest that this formula is written down for the Fourier transforms of g.f., at the same time, the variable x moves into a variable ξ when Fourier transformation.

In the terms of formal series we have

$$a[\xi] = \sum_{m=0}^{+\infty} a_m h_m(\xi), \quad b[\xi] = \sum_{m=0}^{+\infty} b_m h_m(\xi)$$

and then

$$e^{i\gamma(\xi)z}a[\xi] + e^{-i\gamma(\xi)z}b[\xi] = \sum_{m=0}^{+\infty} \left[a_m e^{i\gamma(\xi)z} + b_m e^{-i\gamma(\xi)z}\right]h_m(\xi).$$

Also formally we have (Fourier inverse transformation is written down as the integral)

$$u(z)[x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(e^{i\gamma(\xi)z}a[\xi] + e^{-i\gamma(\xi)z}b[\xi]\right)e^{-i\xi x}d\xi.$$

But if $a[\xi], b[\xi]$ are regular g.f., then

$$u(x, z) = \sum_{m=0}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(a_m e^{i\gamma(\xi)z} + b_m e^{-i\gamma(\xi)z}\right)h_m(\xi)e^{-i\xi x}d\xi.$$

Let's call the solutions of Helmholtz equation *positively oriented* when $b[\cdot] = 0$ and *negatively oriented* when $a[\cdot] = 0$.

Let's agree not to put the symbol of Fourier transformation for the Fourier transforms of g.f., we will specify the corresponding variable in brackets: we write $g[\xi]$ instead of $\mathcal{F}g[\cdot]$. Then for positively oriented solutions of the Helmholtz equation we have

$$u_0[\xi] = u(0)[\xi] = a[\xi], \quad u_1[\xi] = u'(0)[\xi] = i\gamma(\xi)a[\xi]$$

and the condition $u_1[\xi] - i\gamma(\xi)u_0[\xi] = 0$ is a necessary and sufficient condition for the g.f. $u_0[\cdot]$ and $u_1[\cdot]$ be the traces of positively oriented solution of Helmholtz equation (for $z > 0$) on the straight line $z = 0$.

Similarly, the condition $u_1[\xi] + i\gamma(\xi)u_0[\xi] = 0$ is a necessary and sufficient condition for the g.f. $u_0[\cdot]$ and $u_1[\cdot]$ be the traces of negatively oriented solution of Helmholtz equations (for $z < 0$) on the straight line $z = 0$.

Note that the values of the Helmholtz equation solutions at $z > 0$ and at $z < 0$ will be regular g.f. Within the framework of the theory of the g.f., their traces ("zero" and "first") on a straight line $z = 0$ always exist, which in general case are not regular functions.

In addition, the limit of the Fourier transform of a.f. for $z \to 0$ and the Fourier transform of its limit for $z \to 0$ are equal.

If you don't care about the mathematical rigor of reasonings, here's what you can formally do. After Fourier transformation by variable $x \to \xi$, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u(x, z) = 0$$

goes into the equation

$$(\mathcal{F}u)''(\xi, z) + (k^2 - \xi^2)(\mathcal{F}u)(\xi, z) = 0.$$

It's general solution has the form

$$(\mathcal{F}u)(\xi, z) = a(\xi)e^{i\gamma(\xi)z} + b(\xi)e^{-i\gamma(\xi)z}$$

and then for, positively oriented solutions we have

$$u(x, z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} a(\xi)e^{i\gamma(\xi)z}e^{-i\xi x} d\xi.$$

Hence

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} a(\xi)e^{-i\xi x} d\xi, \quad u_1(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} a(\xi)i\gamma(\xi)e^{-i\xi x} d\xi,$$

i.e. $u_0(\xi) = a(\xi)$, $u_1(\xi) = i\gamma(\xi)a(\xi)$ and $u_1(\xi) - i\gamma(\xi)u_0(\xi) = 0$.

5. THE PROBLEM OF DIFFRACTION OF ELECTROMAGNETIC WAVE BY RIBBON

We consider the two-dimensional problem of diffraction of the parallel polarized electromagnetic wave on a thin conducting infinite ribbon [6].

Let $\mathcal{M}$ be a part of the axis x corresponding to the ribbon, and $\mathcal{N}$ be a supplement of $\mathcal{M}$ to the whole axis. We'll look for a.f. $u^1(z)[\cdot]$ and $u^2(z)[\cdot]$, satisfying the Helmholtz equation at $z > 0$ and $z < 0$ respectively.

We need to find functions for $z > 0$ and for $z < 0$, satisfying Helmholtz equation, conditions at infinity and boundary conditions for $z = 0$:

$$u_0^0[\cdot] + u_0^1[\cdot] = 0, \quad u_0^0[\cdot] + u_0^2[\cdot] = 0 \quad \text{on} \quad \mathcal{M},$$

$$u_0^1[\cdot] = u_0^2[\cdot], \quad u_1^1[\cdot] = u_1^2[\cdot] \quad \text{on} \quad \mathcal{N}.$$

Here $u_0^0(x) = u^0(x, 0)$ is a zero trace on the axis x (it is a regular function) of potential function $u^0(x, z)$ of the wave from an external source.

The diffraction problem is reduced to the one-side boundary value problem

$$u_0^1[\cdot] = -u_0^0(\cdot) \quad \text{on} \quad \mathcal{M}; \quad u_1^1[\cdot] = 0 \quad \text{on} \quad \mathcal{N}.$$

Really, $u_0^1[x] = u_0^2[x]$ all over the axis. Therefore $u_0^1[\xi] = u_0^2[\xi]$, $u_1^1[\xi] = -u_1^2[\xi]$ and $u_0^1[x] = -u_0^2[x]$. But then $u_0^1[x] = u_0^2[x] = 0$ on $\mathcal{N}$.

The conditions of the boundary value problem in terms of the g.f. have the form

$$\sum_{m=0}^{+\infty} I_{mn} u_{0m} = -u_{0n}^0, \quad n = 0, 1, \ldots, \tag{1}$$

$$\sum_{m=0}^{+\infty} J_{mn} u_{1m} = 0, \quad n = 0, 1, \ldots, \tag{2}$$

here u_{0m} and u_{1m} are the values of the functionals $u_0^1[\cdot]$ and $u_1^1[\cdot]$, the same values are Fourier coefficients of traces on the axis x of solution of the Helmholtz equation when $z > 0$. As before, these integrals have the form

$$I_{mn} = \int_{\mathcal{M}} h_m(x)\widetilde{h}_n(x)dx, \quad J_{mn} = \delta_{mn} - I_{mn}.$$

For the plane wave $u^0(x, z) = e^{-i\kappa \sin\theta \cdot x - i\kappa\cos\theta \cdot z}$ we have

$$u_{0n}^0 = \int_{\mathcal{M}} u_0^0(x)\widetilde{h}_n(x)dx = \int_{\mathcal{M}} e^{-i\kappa\sin\theta\cdot x}\widetilde{h}_n(x)dx.$$

We add to the equations (1) and (2) the connection between the traces of the solution in the terms of the Fourier transforms: $u_1^1[\xi] - i\gamma(\xi)u_0^1[\xi] = 0$, that is

$$(\mathcal{F}u_1)_m - i\sum_{n=0}^{+\infty}\gamma_{mn}(\mathcal{F}u_0)_n = 0, \quad \gamma_{mn} = \int_{-\infty}^{+\infty}\gamma(\xi)h_m(\xi)\widetilde{h}_n(\xi)d\xi, \quad \gamma(\xi) = \sqrt{\kappa^2 - \xi^2}.$$

Since $(\mathcal{F}u_1)_m = i^m u_{1m}$, $(\mathcal{F}u_0)_n = i^n u_{0n}$, then

$$u_{1n} + \sum_{m=0}^{+\infty} i^{m-n-1}\gamma_{mn}u_{0m} = 0, \quad n = 0, 1, \ldots \tag{3}$$

So, the conditions of the diffraction problem are formulated as ISLAE, composed of three groups of equations (1), (2), (3).

If the solvability conditions of the over-determined problem are written down in the form of $u_0^1[\xi] + \dfrac{i}{\gamma(\xi)}u_1^1[\xi] = 0$, then instead of (3) we will have

$$u_{0n} + \sum_{m=0}^{+\infty} u_{1m} i^{m-n+1}\widetilde{\gamma}_{mn} = 0, \quad \widetilde{\gamma}_{mn} = \int_{-\infty}^{+\infty}\frac{1}{\gamma(\xi)}h_m(\xi)\widetilde{h}_n(\xi)d\xi. \tag{4}$$

We can also leave only one group of unknowns in the final ISLAE: we express u_{1n} through u_{0m} out of (3) and substitute it into (2) or we express u_{0n} through u_{1m} out of (4) and substitute it in (1).

The formal construction of the ISLAE of the diffraction problem on the ribbon can be carried out as follows. We'll look for zero trace and the first trace in the form

$$u_0^1(x) = \sum_{m=0}^{+\infty} u_{0m}h_m(x), \quad u_1^1(x) = \sum_{m=0}^{+\infty} u_{1m}h_m(x).$$

Let's start directly with the conditions of the one-side boundary value problem

$$u_0^1(x) = -u_0^0(x) \quad \text{on} \quad \mathcal{M}; \quad u_1^1(x) = 0 \quad \text{on} \quad \mathcal{N}.$$

Multiply these equalities by $\widetilde{h}_n(x)$ and integrate them over the appropriate part of the axis x. We'll get (1) and (2).

We substitute Fourier transforms of traces of the solution of the Helmholtz equation into the solvability condition of the over-determined problem, (multipliers i^m appear in sums), then we multiply this condition by $\widetilde{h}_n(x)$ and integrate it all over the axis x. We get (3) (or (4)).

6. INTEGRAL EQUATIONS OF THE 4TH KIND

The problem of diffraction of the electromagnetic wave on the ribbon is also reduced to integral equations of different kinds [7]. For example, you can think as follows:

$$u_0^1(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u_0^1(\xi) e^{-i\xi x} d\xi = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \left(\frac{1}{i\gamma(\xi)} u_1^1(\xi) \right) e^{-i\xi x} d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{-i}{\gamma(\xi)} \left(\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u_1^1(x_1) e^{ix_1\xi} dx_1 \right) e^{-i\xi x} d\xi = \int\limits_{-\infty}^{+\infty} u_1^1(x_1) K_{-1}(x_1, x) dx_1,$$

where

$$K_{-1}(x_1, x) = \frac{-i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{\gamma(\xi)} e^{ix_1\xi} e^{-i\xi x} d\xi = -\frac{i}{2} H_0^{(1)}(k|x - x_1|).$$

Since $u_1^1(x) = 0$ at $\mathcal{N}$, we have an integral equation of the 1st kind

$$\int\limits_{\mathcal{M}} u_1^1(x_1) K_{-1}(x_1, x) dx_1 = -u_0^0(x), \quad x \in \mathcal{M}.$$

Its kernel contains a logarithmic singularity (on a diagonal), therefore, it is appropriate for a numerical solving such an equation to use the Galerkin method with the expansion of the unknown function by Chebyshev polynomials of the 1st kind with the weight.

We will call the integral equation of the form

$$a(x)\varphi(x) + \int\limits_{-\infty}^{+\infty} \varphi(x_1) K(x_1, x) dx_1 = f(x), \quad x \in (-\infty, +\infty),$$

the integral equation of the 4th kind, if $K(t, x) = 0$ for x, belonging to some subset $(-\infty, +\infty)$.

Let, as in our case, $(-\infty, +\infty) = \mathcal{M} \cup \mathcal{N}$, $K(t, x) = 0$ and $a(x) \neq 0$ when $x \in \mathcal{N}$. Then the function $\varphi(x)$ directly is found on $\mathcal{N}$, and the 4th kind equation turns into the equation of the form

$$a(x)\varphi(x) + \int\limits_{\mathcal{M}} \varphi(x_1) K(x_1, x) dx_1 = g(x), \quad x \in \mathcal{M}.$$

Its right-hand side contains an integral by $\mathcal{N}$ of the already known function $\varphi(x)$. Inverse procedure is also possible: integral equation on $\mathcal{M}$ can be easily extended to $\mathcal{N}$, if you extend the unknown function on $\mathcal{N}$.

Thus, from the diffraction problem, we move to the integral equation of the 4th kind on $(-\infty, +\infty)$

$$\int\limits_{-\infty}^{+\infty} u_1^1(x_1) K_{-1}(x_1, x) dx_1 = -u_0^0(x), \quad x \in \mathcal{M}, \quad u_1^1(x) = 0, \quad x \in \mathcal{N}.$$

We can now use orthogonal at $(-\infty, +\infty)$ Hermit function set. We will look for a solution of the integral equation in the following form: $u_1(x_1) = \sum_{m=0}^{+\infty} u_{1m} h_m(x_1)$. Then

$$\int\limits_{-\infty}^{+\infty} u_1(x_1) K(x_1, x) dx_1 = \sum_{m=0}^{+\infty} u_{1m} \frac{-i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{\gamma(\xi)} e^{-i\xi x} \left(\int\limits_{-\infty}^{+\infty} h_m(x_1) e^{i\xi x_1} dx_1 \right) d\xi$$

$$= \sum_{m=0}^{+\infty} u_{1m} \frac{-i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{\gamma(\xi)} e^{-i\xi x} \sqrt{2\pi} i^m h_m(\xi) d\xi.$$

Let's take into account the decomposition $e^{-i\xi x} = \sqrt{2\pi} \sum_{k=0}^{+\infty} i^{-k} \widetilde{h}_k(\xi) h_k(x)$. Therefore,

$$\int_{-\infty}^{+\infty} u_1(x_1) K(x_1, x) dx_1 = -\sum_{m=0}^{+\infty} u_{1m} \sum_{k=0}^{+\infty} i^{m-k+1} h_k(x) \widetilde{\gamma}_{mk}, \quad \widetilde{\gamma}_{mk} = \int_{-\infty}^{+\infty} \frac{1}{\gamma(\xi)} h_m(\xi) \widetilde{h}_k(\xi) d\xi.$$

So,

$$\sum_{m=0}^{+\infty} u_{1m} \sum_{k=0}^{+\infty} i^{m-k+1} \gamma'_{mk} h_k(x) = u_0^0(x), \quad x \in \mathcal{M}.$$

After projecting on $\widetilde{h}_n(x)$, we have ISLAE

$$\sum_{m=0}^{+\infty} u_{1m} \sum_{k=0}^{+\infty} i^{m-k+1} \gamma'_{mk} I_{kn} = u_{0n}^0, \quad n = 0, 1, \dots$$

$$u_{1n} - \sum_{m=0}^{+\infty} u_{1m} I_{mn}, \quad n = 0, 1, \dots$$

Here, as before, we have

$$I_{kn} = \int_{\mathcal{M}} h_k(x) \widetilde{h}_n(x) dx, \quad u_{0n}^0 = \int_{\mathcal{M}} u_0^0(x) \widetilde{h}_n(x) dx.$$

7. COMPUTING EXPERIMENT

To get an approximate solution of the diffraction problem, we will carry out a truncation of ISLAE (1), (2), (3) as follows. We leave M_0 unknowns u_{0m}, M_1 unknowns u_{1m}, and N_0 equations in the group (1), N_1 equations in the group (2) and M_1 equations in the group (3):

$$\sum_{m=0}^{M_0-1} I_{mn} u_{0m} = -u_{0n}^0, \quad n = 0..N_0 - 1,$$

$$\sum_{m=0}^{M_1-1} J_{mn} u_{1m} = 0, \quad n = 0..N_1 - 1,$$

$$u_{1n} + \sum_{m=0}^{M_0-1} i^{m-n-1} \gamma_{mn} u_{0m} = 0, \quad n = 0..M_1 - 1.$$

This SLAE will be presented in the following form

$$\sum_{m=0}^{P} a_{nm} v_m = b_n, \quad n = 0..Q,$$

where $P = M_0 + M_1 - 1$, $Q = N_0 + N_1 + M_1 - 1$. Here, by v_m the unknowns $u_{00}, \dots, u_{0,M_0-1}$, $u_{10}, \dots, u_{1,M_1-1}$ are denoted.

Since the number of equations in the general case is greater than the number of unknowns (for example, if $M_0 = M_1 = N_0 = N_1 = M$), then we will look for a minimum of residual all over the SLAE equations. Then we'll get a SLAE with a square matrix

$$\sum_{m=0}^{P} \left(\sum_{n=0}^{Q} a_{nk}^* a_{nm} \right) v_m = \sum_{n=0}^{Q} a_{nk}^* b_n, \quad k = 0..P.$$
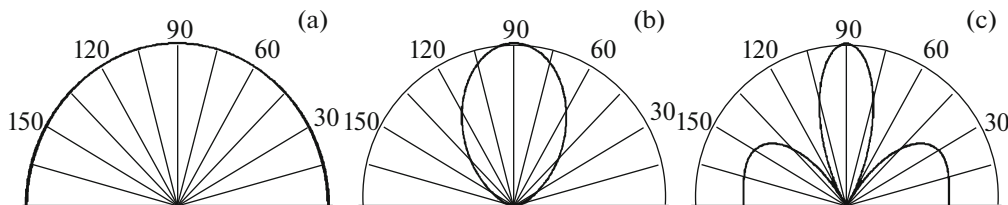
Fig. 1. Scattering diagram at $\chi = $ (a) 0.1, (b) 1.0, and (c) 2.0.

We can exclude the unknowns u_{1m} and decrease twice the size of the final SLAE. Let $M_0 = M_1 = N_0 = N_1 = M$. Then we have

$$\sum_{m=0}^{M-1} I_{mn}u_{0m} = -u_{0n}^0, \quad n = 0..M-1; \quad \sum_{m=0}^{M-1} J_{mk}u_{1m} = 0, \quad k = 0..M-1. \tag{5}$$

We substitute here $u_{1m} = \sum_{n=0}^{M-1} i^{n-m+1}\gamma_{nm}u_{0n}$ and we get

$$\sum_{m=0}^{M-1} J_{mk}\left(\sum_{n=0}^{M-1} i^{n-m+1}\gamma_{nm}u_{0n}\right) = 0, \quad k = 0..M-1$$

or

$$\sum_{m=0}^{M-1}\left(\sum_{k=0}^{M-1} i^{m-k+1}J_{kn}\gamma_{mk}\right)u_{0m} = 0, \quad n = 0..M-1. \tag{6}$$

Double SLAE (5), (6) will be written in the form

$$\sum_{m=0}^{M-1} a_{nm}u_{0m} = b_n, \quad n = 0..2M-1.$$

Here at $n = 0..M-1$ we have

$$m = 0..M-1: \quad a_{nm} = I_{mn}, \quad a_{n+M,m} = \sum_{k=0}^{M-1} i^{m-k+1}J_{kn}\gamma_{mk};$$

$$b_n = -u_{0n}^0, \quad b_{n+M} = 0.$$

The minimum condition of residual all over the equations of the SLAE gives a new SLAE

$$\sum_{m=0}^{M-1}\left(\sum_{n=0}^{2M-1} a_{nk}^* a_{nm}\right)u_m = \sum_{n=0}^{2M-1} a_{nk}^* b_n, \quad k = 0..M-1. \tag{7}$$

As above, the solution of the Helmholtz equation at the $z > 0$ has the form

$$u(x,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u_0(\xi)e^{i\gamma(\xi)z}e^{-i\xi x}d\xi.$$

If $\xi \in (-k, k)$, then the elementary plane wave $e^{i\gamma(\xi)z}e^{-i\xi x}$ transfers energy in the direction of $(\cos\varphi, 0, \sin\varphi)$, where the angle φ is counted from the axis of x. It's easy to see that $\xi = -k\cos\varphi$, $\gamma(\xi) = k\sin\varphi$. Dependencies between $|u_0(\xi)| = \left|\sum_{m=0}^{+\infty} u_{0m}i^m h_m(\xi)\right|$ and angle φ are built on polar diagrams, these diagrams show the distribution of the energy of the scattered field in the far zone.

Let $\chi = \lambda/h$ be the ratio of wavelength to the half-wide of the ribbon. At very small χ (wide ribbon) the energy is distributed almost evenly in all directions. There is one lobe on the diagram at $\chi = 1.0$. With the increase of χ the number of lobes increases. On Figs. 1 and 2 the case of the normal fall of the on the diagram on the ribbon is shown, the angle $\theta = 0$.
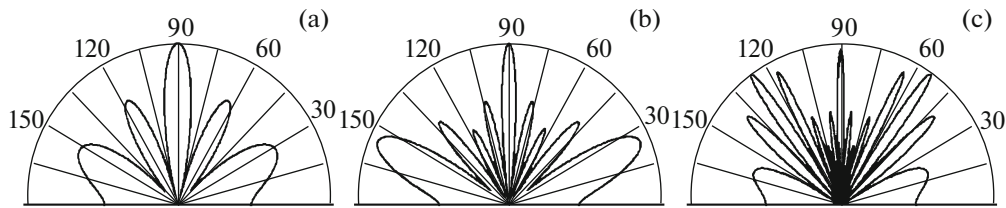
Fig. 2. Scattering diagram at $\chi = $ (a) 4.0, (b) 8.0, and (c) 16.0.
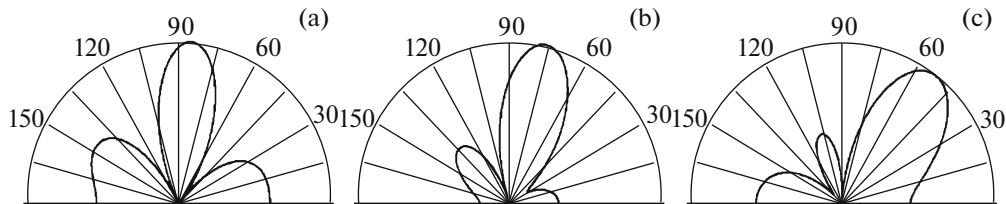

Fig. 3. Scattering diagram at $\theta = $ (a) $\pi/6$, (b) $\pi/4$, and (c) $\pi/3$.

Figure 3 shows how the scattering diagram changes when a wave falls obliquely on a ribbon $\chi = 2.0$.

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