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On Gradient-Like Flows on Seifert Manifolds

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Abstract—We consider a class of gradient-like flows on three-dimensional closed manifolds whose attractors and repellers belongs to a finite union of embedded surfaces and find conditions when the ambient manifold is Seifert.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

Let M^n be a closed n-manifold. A flow f^t on M^n is called *Morse-Smale* if its non-wandering set consists of a finite set of hyperbolic equilibria and closed trajectories, and invariant manifolds of equilibria and closed trajectories have only transversal intersection. A Morse-Smale flow is called *gradient-like* if its non-wandering set does not contain closed trajectories.

Recall that a *Morse index* of hyperbolic equilibrium p is the number equal to the dimension of the unstable manifold W^u_p of p.

We suppose that $n=3$ and the manifold M^3 is oriented. Let us denote by Ω_{f^t} the set of all equilibria of gradient-like flow f^t on M^3 and by Ω^i, the set of equilibria of Morse index $i\in\{0,1,2,3\}$. Equilibria of Morse indices 0 and 3 are called *nodes* (*sinks and sources*, respectively), equilibria of Morse indices $1,2$ are called *saddles*. Set $\Sigma=\Omega^1\cup\Omega^2$.

The following notation introduced similar to [2].

Definition 1. *A gradient-like flow f^t on M^3 has surface dynamics (belongs to a class $GSD(M^3)$) if the set Σ can be represented as the union of two disjoint subsets Σ_a, Σ_r such that each connected component of the sets $\mathcal{A}_{f^t}=W^u_{\Sigma_a}\cup\Omega^0$, $\mathcal{R}_{f^t}=W^s_{\Sigma_r}\cup\Omega^3$ is an oriented locally flat surface*[1)].

In [1] a topology of manifolds admitting flows from $GSD(M^3)$ was studied. In particular it was proved, that M^3 is *a mapping torus*, that is M^3 is diffeomorphic to a quotient space $M_{g_{f^t},\tau_{f^t}}=\mathbb{S}_{g_{f^t}}\times[0,1]/\sim$ of the direct product of an oriented surface $S_{g_{f^t}}$ of a genus g_{f^t} and the interval $[0,1]$ under an equivalence relation $(z,1)\sim(\tau_{f^t}(z),0)$, where $\tau_{f^t}:S_{f^t}\to S_{f^t}$ is an orientation preserving diffeomorphism.

In this paper we clarify the structure of ambient manifolds for flows from $GSD(M^3)$ under the condition that invariant manifolds of saddle equilibria have simple asymptotic behavior (see definition 2). Main results of the paper are the following.

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1)Let us recall that a surface $S_g\subset M^3$ is locally flat if for any point $x\in S_g$ there exist a neighborhood $U_x\subset M^3$ and a homeomorphism $h_x:U_x\to\mathbb{R}^3$ such that $h_x(S_g\cap U_x)=Oxy$.

Theorem 1. *If invariant manifolds of saddle equilibria of a flow $f^t \in GSD(M^3)$ have simple asymptotic behavior, then the manifold M^3 is Seifert and gluing map τ_{f^t} is periodic.*

Theorem 2. *For any oriented surface $\mathbb{S}_g$ and any orientation preserving periodic diffeomorphism $\tau : \mathbb{S}_g \to \mathbb{S}_g$ there exists a gradient-like flow $f^t \in GSD(M^3)$ with simple asymptotic behavior of invariant manifolds of saddles, and M^3 is the mapping torus M^3_τ.*

2. AUXILIARY FACTS AND BASIC DEFINITIONS

In [2, Theorems 1, 2], [4, Lemma 1, Theorem 1] the topology of manifolds admitting gradient-like cascades with surface dynamics was studied. Since a time one shift along trajectories of gradient-like flow with surface dynamics is such a cascade then results of these papers can be adopted to flows in the following way.

Proposition 1. *Let $f^t \in GSD(M^3)$ then there exist integers $k_{f^t}, g_{f^t} \geq 0$ and an orientation preserving diffeomorphism τ_{f^t} of an oriented surface $\mathbb{S}_{g_{f^t}}$ of a genus g_{f^t} such that:*

1. Sets $\mathcal{A}_{f^t}, \mathcal{R}_{f^t}$ consist of the same number k_{f^t} of connected components, each of which is homeomorphic to $\mathbb{S}_{g_{f^t}}$.

2. Each connected component of the set $\mathcal{A}_{f^t}(\mathcal{R}_{f^t})$ is an attractor (repeller)[2)]

3. The closure of each connected component of the set $M^3 \setminus (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t})$ is homeomorphic to the direct product $\mathbb{S}_{g_{f^t}} \times [0, 1]$.

4. The manufold M^3 is diffeomorphic to the quotient space $M_{g_{f^t},\tau_{f^t}} = \mathbb{S}_{g_{f^t}} \times [0, 1]/\sim$ under the equivalence relation $(z, 1) \sim (\tau_{f^t}(z), 0)$.

Let $\sigma \in \Sigma$. Recall that a connected component of the stable (unstable) manifold $W^s_\sigma \setminus \sigma$ $(W^u_\sigma \setminus \sigma)$ is called *the stable* (*unstable*) separatrix of σ.

Let $\sigma^1 \subset \Omega^1, \sigma^2 \subset \Omega^2$ be points such that $W^u_{\sigma^2} \cap W^s_{\sigma^1} \neq \varnothing$. According to [3] any connected component of the intersection $W^u_{\sigma^2} \cap W^s_{\sigma^1} \neq \varnothing$ is called *heteroclinic trajectory*.

Let V be a connected component of the set $M^3 \setminus (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t})$. Then, due to Proposition 1 there exist connected components A, R of $\mathcal{A}_{f^t}, \mathcal{R}_{f^t}$, respectively, such that $\partial V = A \cup R$. Set $\Omega_A = \Omega_{f^t} \cap A, \Omega^i_A = \Omega^i \cap A,\ i \in \{0, 1, 2\},\ \Omega_R = \Omega_{f^t} \cap R, \Omega^j_R = \Omega^j \cap R,\ j \in \{1, 2, 3\}$. Then the following equalities hold: $A = \bigcup\limits_{p\in\Omega_A} W^u_p, R = \bigcup\limits_{p\in\Omega_R} W^s_p$.

Due to [5, Theorem 2.3] and [2, Lemmas 1, 2] the following statement is true.

Proposition 2. *Let $\sigma^1 \in \Omega^1_A$ and $\omega \in \Omega^0_A$. Then:*

1. $W^u_{\sigma^1} \subset A$ and exist points $\omega_+, \omega_- \in \Omega^0_A$ (it is possible, $\omega_+ = \omega_-$) such that $cl\ W^u_{\sigma^1} \setminus W^u_{\sigma^1} = \omega_+ \cup \omega_-$.

2. there exist points $\sigma^2_+, \sigma^2_- \in \Omega^2_A$ (it is possible, $\sigma^2_+ = \sigma^2_-$) such that the set $W^s_{\sigma^1} \cap (W^u_{\sigma^2_+} \cup W^u_{\sigma^2_-})$ consists of exactly two different heteroclinic trajectories.

3. there exists a point $\sigma^1_* \in \Omega^1_R$ such that $\omega \subset cl\ W^u_{\sigma^1_*}$.

Similar statement is true for points $\sigma^2 \in \Omega^2_R$ and $\alpha \in \Omega^3_R$ with formal change of symbols $A, 0, 1, 2, s, u$ by $R, 3, 2, 1, u, s$, respectively.

Definition 2. *We say that invariant manifolds of saddle equilibria of a flow $f^t \subset GSD(M^3)$ have simple asymptotic behavior if for any triple of connected components $A \subset \mathcal{A}_{f^t}, R \subset \mathcal{R}_{f^t}, V \subset M^3 \setminus (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t})$ such that $\partial V = A \cup R$ the following conditions hold (see Figure 1):*

[2)]Invariant set A is called *an attractor* of a flow f^t if there exists a closed neighborhood (trapping neighborhood) $V \subset M^3$ such that all trajectories of the flow f^t intersect the boundary of V transversally, and $A = \bigcap\limits_{t>0} f^t(V)$. The set R is called *a repeller of f^t if it is an attractor for f^{-t}.*, and the restriction of the flows f^t on this component is topologically equivalent to a gradient-like flow.

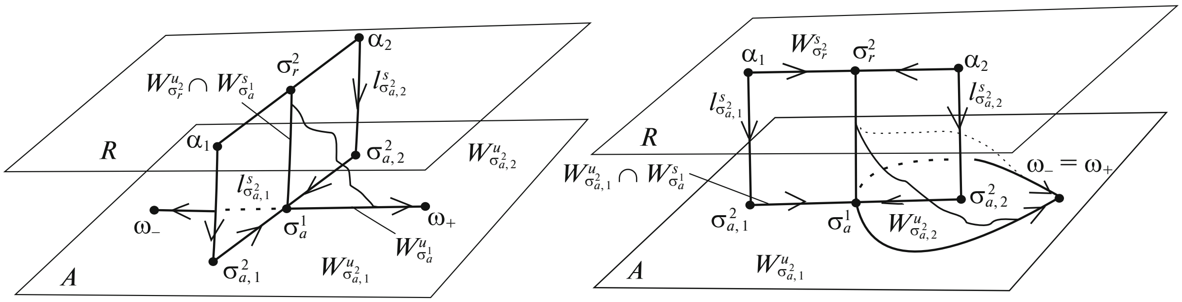

Fig. 1. Simple asymptotic behavior of invariant manifolds of saddle equilibria.

1. for any two different points $\sigma^1_{r,1}, \sigma^1_{r,2} \in \Omega^1_R$ the closures of separatrices $l^u_{\sigma^1_{r,1}}, l^u_{\sigma^1_{r,2}} \subset V$ contain different points $\omega_1, \omega_2 \subset \Omega^0_A$;

2. for any two different points $\sigma^2_{a,1}, \sigma^2_{a,2} \in \Omega^2_A$ the closures of separatrices $l^s_{\sigma^2_{a,1}}, l^s_{\sigma^2_{a,2}} \subset V$ contain different points $\alpha_1, \alpha_2 \subset \Omega^3_R$;

3. for any point $\sigma^1_a \subset \Omega^1_A$ there exists exactly one point $\sigma^2_r \subset \Omega^2_R$ such that the intersection $W^s_{\sigma^1_a} \cap W^u_{\sigma^2_r} \cap V$ is not empty; for any point $\sigma^2_r \subset \Omega^2_R$ there exists exactly one point $\sigma^1_a \subset \Omega^1_A$ such that $W^s_{\sigma^1_a} \cap W^u_{\sigma^2_r} \cap V$ is not empty;

4. for any points $\sigma^1_a \in \Omega^1_A, \sigma^2_r \in \Omega^2_R$ the intersection $W^s_{\sigma^1_a} \cap W^u_{\sigma^2_r} \cap V$ is either empty or consists of exactly one heteroclinic curve.

Let us denote by $GSDS(M^3)$ the subset of $GSD(M^3)$ consisting of flows with the simple asymptotic behavior of separatrices.

3. CELLS OF $GSDS$-FLOWS

For a flows $f^t \in GSDS(M^3)$ let us denote by Γ_A (Γ_R) a union of all equilibria, one-dimensional separatrices and heteroclinic trajectories of f^t that belong to $A(R)$ and by f^t_A (f^t_R) the restriction of f^t on $A(R)$. The set Γ_A is support for a graph whose vertices are equilibria, and edges are one-dimensional separatrices and heteroclinic trajectories. Let us denote by $E(\Gamma_A)$ the set of connected components of the set $\Gamma_A \setminus \Omega_A$.

Definition 3. *A connected component of the set $A \setminus \Gamma_A$ ($R \setminus \Gamma_R$) is called two-dimensional cell* of the flow f^t_A (f^t_R).

It follows from proposition 2 the description of all possible types of two-dimensional cells (see Figure 2).

Proposition 3. *The boundary ∂a of a two-dimensional cell a of the flow f^t_A have one of the following type:*

a_1) ∂a consists of a sink $\omega_a \in \Omega^0_A$, saddles $\sigma^1_{a,+}, \sigma^1_{a,-} \in \Omega^1_A$, separatrices $l^u_{\sigma^1_{a,+}}, l^u_{\sigma^1_{a,-}}$ of $\sigma^1_{a,+}, \sigma^1_{a,-}$ whose closures contain ω_a, a saddle $\sigma^2_a \in \Omega^2_A$, and heteroclinic curves $\gamma_{\sigma^2_a,\sigma^1_{a,+}} \subset W^u_{\sigma^2_a} \cap W^s_{\sigma^1_{a,+}}$, $\gamma_{\sigma^2_a,\sigma^1_{a,-}} \subset W^u_{\sigma^2_a} \cap W^s_{\sigma^1_{a,-}}$;

a_2) ∂a consists of a sink ω_a, an unstable manifold $W^u_{\sigma^1_a}$ of a point $\sigma^1_a \in \Omega^1_A$ whose closure contain ω_a, a saddle $\sigma^2_a \in \Omega^2_A$ and a heteroclinic curve $\gamma_{\sigma^2_a,\sigma^1_a} \subset W^u_{\sigma^2_a} \cap W^s_{\sigma^1_a}$;

a_3) ∂a consists of a sink ω_a, a saddle σ^1_a, a separatrix $l^u_{\sigma^1_a}$ such that $\omega_a \in cl\, l^u_{\sigma^1_a}$, a saddle $\sigma^2_a \in \Omega^2_A$ and heteroclinic curves $\gamma^+_{\sigma^2_a,\sigma^1_a}, \gamma^-_{\sigma^2_a,\sigma^1_a} \subset W^u_{\sigma^2_a} \cap W^s_{\sigma^1_a}$.

The boundary ∂r of any two-dimensional cell r of f^t_R has exactly one of the following three types:

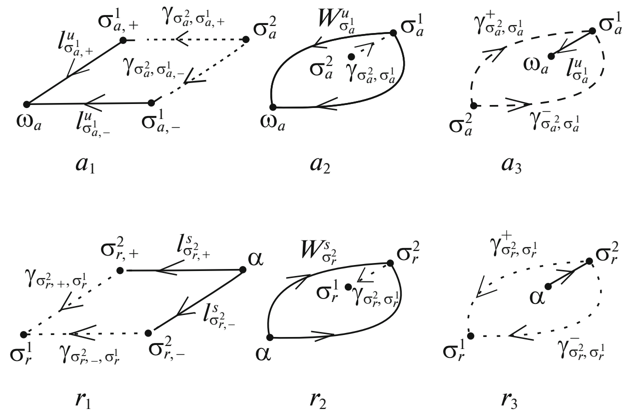

Fig. 2. Two-dimensional cells of GSD-flows.

r_1) ∂r consists of a source $\alpha_r \in \Omega^3_R$, saddles $\sigma^2_{r,+}, \sigma^2_{r,-} \in \Omega^2_R$, separatrices $l^s_{\sigma^2_{r,+}}, l^s_{\sigma^2_{r,-}}$, whose closures contain α_r, a saddle $\sigma^1_r \in \Omega^1_A$ and heteroclinic curves $\gamma_{\sigma^2_{r,+},\sigma^1_r} \subset W^u_{\sigma^2_+} \cap W^s_{\sigma^1_r}$, $\gamma_{\sigma^2_{r,-},\sigma^1_r} \subset W^u_{\sigma^2_{r,-}} \cap W^s_{\sigma^1_r}$;

r_2) ∂r consists of a source α_r, a stable manifold $W^s_{\sigma^2_r}$ of a saddle $\sigma^2_r \subset \Omega^2_R$ whose closure contains α_r, a saddle $\sigma^1_r \subset \Omega^1_R$ and a heteroclinic curve $\gamma_{\sigma^2_r,\sigma^1_r} \subset W^u_{\sigma^2_r} \cap W^s_{\sigma^1_r}$;

r_3) ∂r consists of a source α_r, a saddle $\sigma^2_r \subset \Omega^2_R$, a separatrix $l^u_{\sigma^2_r}$ whose closure contains α_r, a saddle $\sigma^1_r \subset \Omega^1_R$, and heteroclinic curves $\gamma_{\sigma^2_r,\sigma^1_r}, \gamma_{\sigma^2_r,\sigma^1_r} \subset W^u_{\sigma^2_r} \cap W^s_{\sigma^1_r}$.

Let us denote by Γ^u_A a subset of Γ_A that contains all one-dimensional separatrices of saddles from the set Ω^1_A.

Proposition 4. *Γ^u_A is connected.*

Proof. Let us choose a set of pair-wise disjoint disks $\{D_q\}_{q\in\Omega^2_A}$ bounded by circles that meet trajectories of f^t_A transversally and such that $q \in int\ D_q$, $D_q \in W^u_q$ for any $q \in \Omega^2_A$. Set $U = A \setminus \bigcup\limits_{q\in\Omega^2_q} int\ D_q$. By definition the set U is connected, $f^t(U) \subset int\ f^s(U)$ while $t > s$, and $\Gamma^u_A \subset int\ U$. Moreover, $\Gamma^u_A = \bigcap\limits_{t\geq 0} f^t(U)$. Then Γ^u_A is connected as an the intersection of compact connected nested sets. □

Definition 4. *Let $f^t \in GSDS(M^3)$. A connected component of $M^3 \setminus \bigcup\limits_{p\in\Sigma}(cl\, W^u_p \cup cl\, W^s_p)$ is called three-dimensional cell of the flow f^t.*

Proposition below immediately follows from the definition 2 of the class $GSDS(M^3)$.

Proposition 5. *Let $f^t \subset GSDS(M^3)$. Then for any three-dimensional cell C^3 of f^t its boundary ∂C^3 has one of types v_1, v_2, v_3:*

v_1) The intersection $\partial C^3 \cap A$ is a two-dimensional cell with the boundary of type a_1), the set $\partial C^3 \cap R$ is a two-dimensional cell of type r_1), the set $\partial C^3 \cap V$ consists of a separatrix $l^u_{\sigma^1,r}$ such that $\omega_a \subset cl\ l^u_{\sigma^1,r}$, a separatrix $l^s_{\sigma^2,a}$ such that $\alpha_r \subset cl\ l^s_{\sigma^2,a}$, heteroclinic curves $\gamma_{\sigma^2_{r,-},\sigma^1_{a,-}}$, $\gamma_{\sigma^2_{r,+},\sigma^1_{a,+}}$, a subset of the manifold $W^u_{\sigma^2_{r,-}} \cap V$ bounded by curves $\gamma_{\sigma^2_{r,-},\sigma^1_r}, \gamma_{\sigma^2_{r,-},\sigma^1_{a,-}}$, a subset of the manifold $W^u_{\sigma^2_{r,+}} \cap V$ bounded by curves $\gamma_{\sigma^2_{r,+},\sigma^1_r}, \gamma_{\sigma^2_{r,+},\sigma^1_{a,+}}$, a subset of the manifold $W^s_{\sigma^1_{a,-}} \cap V$ bounded by curves $\gamma_{\sigma^2_a,\sigma^1_{a,-}}, \gamma_{\sigma^2_{r,-},\sigma^1_{a,-}}$, and a part of the manifold $W^s_{\sigma^2_{r,+}} \cap V$ bounded by curves $\gamma_{\sigma^2_a,\sigma^1_{a,-}}, \gamma_{\sigma^2_{r,+},\sigma^1_{a,+}}$.

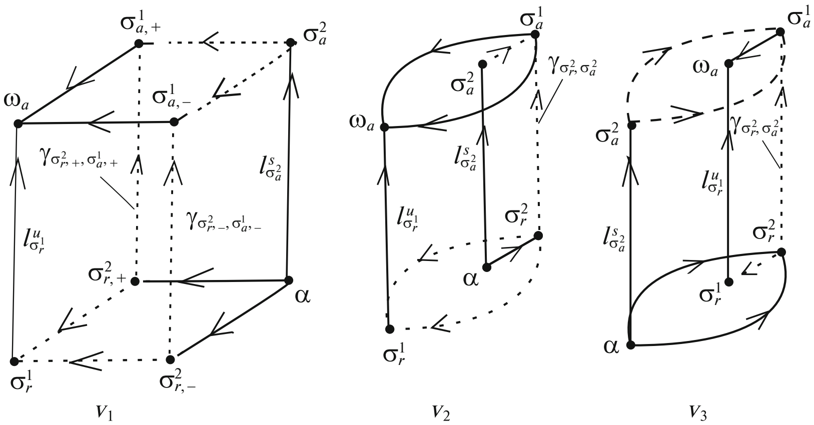

Fig. 3. Three-dimensional cells of GSDS-flows.

v_2) *The intersection $\partial C^3 \cap A$ is a two-dimensional cell of type a_2), the set $\partial C^3 \cap R$ is a two-dimensional cell of type r_3, the set $\partial C^3 \cap V$ is defined similar to the item v_1).*

v_3) *The intersection $\partial C^3 \cap A$ is a two-dimensional cell of type a_3), the set $\partial C^3 \cap R$ is a two-dimensional cell of type r_2, the set $\partial C^3 \cap V$ is defined similar to the item v_1) (see Figure 3).*

For any set of connected components $A \subset \mathcal{A}_{f^t}, R \subset \mathcal{R}_{f^t}, V \subset M^3 \setminus (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t})$ such that $\partial V = A \cup R$ let us denote by $\mathcal{C}^2_A, \mathcal{C}^2_R, \mathcal{C}^3_V$ the set of all cells of dimension two and three that belongs to A, R, V, respectively. Let us choose an arbitrary component $A \subset \mathcal{A}_{f^t}$ and denote by $V_1, \dots, V_{2k_{f^t}}$ all pair-wise disjoint connected components of the set $M^3 \setminus (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t})$. We will suppose that indices are chosen in such a way that $cl(V_i) \cap cl(V_{i+1}) \neq \varnothing$ for any $i \in \{1, \dots, 2k_{f^t} - 1\}$ and $cl(V_{2k_{f^t}}) \cap cl(V_1) \supset A$. Then the following proposition holds.

Proposition 6. *For any two-dimension cell $c^2 \subset A$ of the flow f^t there is a sequence $C^3_1, \dots, C^3_{2k_{f^t}}$ of three-dimensional cells such that $cl(C^3_1) \cap A = cl(c^2)$, $C^3_i \subset V_i$, $i \in \{1, \dots, 2k_{f^t}\}$ and the intersections $cl(C^3_i) \cap cl(C^3_{i+1}) \setminus A$, $i \in \{1, \dots, 2k_{f^t} - 1\}$, $cl(C^3_{2k_{f^t}}) \cap A$ are non-empty and each of them consists of a closure of two-dimensional cell.*

Lemma 1. *There is a well-defined one-to-one map $\mu_A : \mathcal{C}^2_A \to \mathcal{C}^2_A$ which assigns to each cell $c^2 \in \mathcal{C}^2_A$ a cell $\tilde{c}^2$ belonging to the intersection $cl(C^3_{2k_{f^t}}) \cap A$. Moreover, μ_A induces orientation preserving homeomorphism $h_A : A \to A$ with the following properties:*

1. $h_A(\Omega^i_A) = \Omega^i_A$, $i \in \{0, 1, 2\}$;

2. $h_A(cl\ c^2) = cl\ \mu_A(c^2)$ for any cell $c^2 \in \mathcal{C}^2_A$;

3. there exist an integer $m > 0$ such that for ant arc $l \in \Gamma_A \setminus \Omega^i_A$ equalities $h^m_A(l) = l$, $h^i_A(l) \neq l$ hold for any natural $i < m$.

Proof. Let $c^2 \subset A$, $C^3_1, \dots, C^3_{2k_{f^t}}$ is a sequence of three-dimensional cells defined in Proposition 6 for c^2, and $\tilde{c}^2$ is a two-dimensional cell belonging to the intersection $cl\ C^3_{2k_{f^t}} \cap A$. Set $\mu_A(c^2) = \tilde{c}^2$.

Suppose that c^2 has type a_1 (see Proposition 3). Let us choose an orientation on the boundary ∂c^2 of c^2 in such a way that if we going around it in counterclockwise direction (in the positive direction) provided that the observer is in the region V, the cell c^2 remains to the left from ∂c^2. If the cell c^2 has type a_2 (a_3) we choose the similar orientation on the closed curve consisting of closures of unstable separatrices of saddle point $\sigma^1_{c^2} \in \partial c^2$ (closures of heteroclinic curves joining points $\sigma^2_{c^2}, \sigma^1_{c^2} \in \partial c^2$). Thus we obtain a finite set $\mathcal{E}_A$ of oriented simple closed curves cutting the surface A into open disks. Since A is oriented then for any arc $l \subset \partial c^2 \cap \partial \tilde{c}^2$ the orientations of $\partial c^2, \partial \tilde{c}^2$ induce opposite orientations on l.

let us construct an orientation preserving homeomorphism $h_{c^2} : cl\ c^2 \to cl\ (\mu_A(c^2))$ in the following way.

Denote by $\mathbb{B}^2 \subset \mathbb{R}^2$ the standard unit disk with the center in the Origin O and by $\mathbb{S}^1$ the boundary of $\mathbb{B}^2$.

Let $e_{c^2} : \mathbb{B}^2 \to cl\ c^2$ and $e_{\mu_A(c^2)} : \mathbb{B}^2 \to cl\mu_A(c^2)$ be homeomorphisms preserving the orientation of the unit circle and such that $e_{c^2}(\mathbb{S}^1)$ and $e_{\mu_A(c^2)}(\mathbb{S}^1) \in \mathcal{E}_A$. Without lost of generality suppose that if c^2 is a cell of type a_2 (a_3) then $e_{c^2}(O) = \sigma^2_{c^2}$ $(e_{c^2}(O) = \omega_{c^2})$ and $e_{c^2}(r_x) = \gamma_{\sigma^2_{c^2},\sigma^1_{c^2}}(e_{c^2}(r_x) = l^u_{\sigma^1_{c^2}})$, where $r_x \in \mathbb{B}^2$ is the interval joining the center O of $\mathbb{B}^2$ with a point $x \in \mathbb{S}^1$.

Let us denote by $g^1_{c^2,\mu_A(c^2)} : \mathbb{S}^1 \to \mathbb{S}^1$ an orientation preserving homeomorphism with following property: if $p \in \Omega^i_A \cap \partial c^2$, where $i \in \{0, 1, 2\}$, then there is $p' \in \Omega^i_A \cap \partial\mu_A(c^2)$ such that $g^1_{c^2,\mu_A(c^2)}(e^{-1}_{c^2}(p)) = e^{-1}_{\mu_A(c^2)}(p')$.

Let $g^2_{c^2,\mu_A(c^2)} : \mathbb{B}^2 \to \mathbb{B}^2$ be a homeomorphism given in the polar coordinates φ, r by the formula $g^2_{c^2,\mu_A(c^2)}(\varphi, r) = (g^1_{c^2,\mu_A(c^2)}(\varphi), r)$. At last define a homeomorphism $h_{c^2,\mu_A(c^2)} : cl\ c^2 \to cl\ \mu_A(c^2)$ by $h_{c^2,\mu_A(c^2)} = e_{\mu_A(c^2)} g^2_{c^2,\mu_A(c^2)} e^{-1}_{c^2}$. By construction $h_{c^2,\mu_A(c^2)}$ satisfy the following conditions:

1. $h_{c^2,\mu_A(c^2)}$ preserves the orientation of boundaries of cells $c^2, \mu_A(c^2)$;
2. $h_{c^2,\mu_A(c^2)}(\partial c^2 \cap \Omega^i_A) = \partial\mu_A(c^2) \cap \Omega^i_A$.

Let $\hat{c}^2$ be a two-dimensional cell such that $\partial c^2 \cap \partial\hat{c}^2 \neq \emptyset$. Let us define an orientation preserving homeomorphism $g^1_{\hat{c}^2,\mu_A(\hat{c}^2)} : \mathbb{S}^1 \to \mathbb{S}^1$ in such a way that:

1. for any $p \in \Omega^i_A \cap \partial\hat{c}^2$, $i \in \{0, 1, 2\}$, there is $p' \in \Omega^i_A \cap \partial\mu_A(\hat{c}^2)$ such that $g^1_{\hat{c}^2,\mu_A(\hat{c}^2)}(e^{-1}_{\hat{c}^2}(p)) = e^{-1}_{\mu_A(\hat{c}^2)}(p)$;
2. $g_{\hat{c}^2,\mu_A(\hat{c}^2)}(e^{-1}_{\hat{c}^2}(x)) = e^{-1}_{\mu_A(\hat{c}^2)}(h_{c^2,\mu_A(c^2)}(x))$ for any point $x \in \partial c^2 \cap \partial\hat{c}^2$.

At last, let us define a homeomorphism $h_{\hat{c}^2,\mu_A(\hat{c}^2)} : cl\ \hat{c}^2 \to cl\ \mu_A(\hat{c}^2)$ similar to $h_{c^2,\mu_A(c^2)}$ and continue the process until we run out all cells from the set $\mathcal{C}^2$. The agreement of orientations of the boundaries of all cells guarantees that in the finale we get an orientation preserving homeomorphism $h_A : A \to A$. It follows from construction and Proposition 5 that homeomorphism h_A satisfies to items 1, 2 of the Lemma.

Let us prove the item 3.

Since the set Ω_A is finite then for any point $p \in \Omega_A$ there exists an integer $m_p > 0$ such that $h^{m_p}_A(p) = p$ and $h^i_A(p) \neq p$ for any natural i less than m_p. For any curve $l \subset \Gamma_A \setminus \Omega_A$ and a cell $c^2 \in \mathcal{C}^2_A$ denote similar numbers by m_l, m_{c^2}.

Let $p \in \Omega^1_A$ and $\mathcal{L}_p$ be the set of arcs from the set $\Gamma_A \setminus \Omega_A$ the closures of which contain the point p. It follows from Lemma 2 that the set $\mathcal{L}_p$ consists exactly on four arcs and a pair of them belongs to the unstable manifold of p, and the other pair belongs to the intersection of the two-dimensional unstable manifold of p with A. Let $d^2_p \subset A$ be a disk such that $d^2_p \cap \Omega_A = p$, $p \in int\ d^2_p$, and any arc $l^i_p \in \mathcal{L}_p$ intersects the boundary ∂d^2_p of d^2_p at the single point z^i_p. Let us choose an orientation of the curve ∂d^2_p and suppose, without loss of the generality, that when one goes along the curve from the point z^1_p to point z^4_p in positive direction then points z^2_p, z^3_p appears on order of the numbering decreasing. Then the points belonging to the stable and the unstable manifolds of p alternate. Then $m_{l^1_p} = m_{l^3_p}, m_{l^2_p} = m_{l^4_p}$, and $m_{l^1_p}, m_{l^2_p} \in \{m_p, 2m_p\}$. Since h_A is orientation preserving then the order of the points of the intersection of arcs $h^{m_p}_A(l^i_p)$ with $h^{m_p}_A(d^2_p)$ coincides with the order of points $\{z^i_p\}$. Then $m_{l^1_p} = m_{l^2_p}$.

Let us show that for any pair $l, l' \in \Gamma_A \setminus \Omega_A$ the equality $m_l = m_{l'}$ holds. The set $\Gamma_A \setminus \Omega_A$ can be presented as a disjoint unit of the subsets $\mathcal{L}^u_A$ and $\mathcal{L}^s_A$ consisting of one-dimensional separatrices and heteroclinic trajectories of the flow f^t respectively. Let us prove the statement for the arcs from $\mathcal{L}^u_A$. Then from this fact and the previous paragraph one gets that all numbers m_l are equal to each other.

Let $\mathcal{L}_\omega \subset \mathcal{L}^u_A$ be a set of one-dimensional separatrices of the flow f^t_A whose closures contain the sink ω, and k_ω be the number of arcs in the set $\mathcal{L}_\omega$. Denote by $d^2_\omega \subset W^s_\omega$ a disk such that $p \in int\, d^2_\omega$ and ∂d^2_ω intersects each separatrix $l \in \mathcal{L}_\omega$ at the single point. Let us choose an orientation on the curve ∂d^2_ω and numbering of the point of the intersection of the arcs from $\mathcal{L}_\omega$ in order induces by the chosen orientation. Let $j = 1, 2, \dots$. The homeomorphism h^{jm_ω} preserves the set $\mathcal{L}_\omega$ and the orientation of A, hence it sends any three consecutive points from the set $\{z^i_\omega\}$ into points of the intersection of curves from $\mathcal{L}_\omega$ with $h^{jm_\omega}(\partial d^2_\omega)$ following in the same order. Then $m_{l^i_\omega} = m_{l^j_\omega}$ for $i, j \in \{1, \dots, k_\omega\}$.

Now the coinciding of all periods of curves from the set $\mathcal{L}_A$ follows from the connectedness of the set Γ^u_A consisting of closures of all one-dimensional separatrices laying in A, that was proven in Proposition 4. □

3.1. Construction of Seifert Fibration of the Ambient Manifold

Let us recall that a homeomorphism $\tau\colon \mathbb{S}_g \to \mathbb{S}_g$ of the closed oriented surface S_g is called *a periodic homeomorphism of the period* $r > 1$ if $\tau^r(x) = x$ for any point $x \in \mathbb{S}_g$, and $\tau^l \neq Id$ if $l \in (0, r)$. A number $\mu_x > 0$ such that $\tau^{\mu_x}(x) = x$ and $\tau^l(x) \neq x$ form any $l \in (0, \mu_x)$ is called the period of the point x. Due to [6] the set $X_\tau \subset S_g$ of points whose period less that r is finite.

Topological classification of oriented preserving periodic homeomorphisms of surfaces is obtained by Nielsen in [6].

Lemma 2. *There exists a periodic homeomorphism $\tau : A \to A$ such that $\tau(cl\, c^2) = h_A(cl\, c^2)$ for any cell $c^2 \subset \mathcal{C}_A$, where $h_A : A \to A$ is a homeomorphisms defined in Lemma 1, and $X_\tau \subset \Omega_A$.*

Proof. For a point $p \in \Omega_A$ set $\tau(p) = h_A(p)$. Let $\mathcal{L}_0 \subset \Gamma_A \setminus \Omega_A$ be a set of all arcs such that h does not send any arc from $\mathcal{L}_0$ to the arc from $\mathcal{L}_0$. For any arc $l_0 \in \mathcal{L}_0$ set $l_i = h^i_A(l_0)$, $i \in \{1, \dots, m_{l_0} - 1\}$. Remark that the boundary of any arc $l \in \Gamma_A \setminus \Omega_A$ contains exactly one saddle point $\sigma_1(l)$ of the flow f^t whose Morse index equals one. Denote by $e_i\colon [0, 1] \to cl\, l_i$ a homeomorphisms such that $e_i(0) = \sigma_1(l_i)$. For $i \in \{0, \dots, m_{l_0} - 1\}$, $x \in l_i$ set $\tau(x) = e_{i+1}(e^{-1}_i(x))$, for $x \in l_{m_{l_0}-1}$ set $\tau(x) = e_0(e^{-1}_{m_{l_0}-1}(x))$. By construction the map τ is periodic on Γ_A. Then it is possible to extend it into the set $\mathcal{C}^2$ similar to construction of the homeomorphism h_A in the proof of Lemma 1. □

Let ν, μ be co-prime integers, $0 \leq \nu < \mu$, and $\theta : \mathbb{B}^2 \to \mathbb{B}^2$ be a rotation of the disk $\mathbb{B}^2$ by the angle $2\pi\frac{\nu}{\mu}$. Denote by $N^3 = \mathbb{B}^2 \times [0, 1]/_\sim$ the quotient space by means of equivalence relation $(x, 1) \sim (\theta(x), 0)$, $x \in \mathbb{B}^2$.

Recall that a manifold M^3 is called *Seifert manifold* if M^3 is fibered into circles and any fiber has a neighborhood in M^3 fiber by fiber homeomorphic to N^3.

Proof of Theorem 1. Let us show that M^3 is a Seifert manifold. Due to Proposition 6 for any two-dimensional cell $c^2 \subset A$ of the flow f^t_A there exists a sequence $C^3_1, \dots, C^3_{2k_{f^t}}$ of three-dimensional cells such that $cl(C^3_1) \cap A = cl(c^2)$, $C^3_i \subset V_i$, $i \in \{1, \dots, 2k_{f^t}\}$, and intersections $cl(C^3_i) \cap cl(C^3_{i+1}) \setminus A$, $i \in \{1, \dots, 2k_{f^t} - 1\}$, $cl(C^3_{2k_{f^t}}) \cap A$ are non-empty and consist of a closures of two-dimensional cells. Let $\tau : A \to A$ be a periodic homeomorphism constructed in Lemma 2.

Denote by Q_{c^2} a union of closures of all three-dimensional cells belonging to the obtained sequence and set $X_{c^2} = \{x \in cl\, c^2 : x = \tau(x)\}$. It follows from Propositions 3, 5 that there exists a continuous map $g_{c^2} : \mathbb{B}^2 \times [0, 1] \to Q_{c^2}$ such that:

1. $g_{c^2}(\mathbb{B}^2 \times \{0; 1\}) \subset A$;
2. $g_{c^2}(\mathbb{B}^2 \times \{0\}) = cl\, c_2$;
3. $g_{c^2}(z, 1) = \tau(g_{c^2}(z, 0))$ for any point $z \in \mathbb{B}^2$;

4. the restriction of g_{c^2} on $\mathbb{B}^2\times[0,1]\setminus\widetilde{X}$, where $\widetilde{X}=\{(z,1),z\in\mathbb{B}^2,g_{c^2}(z,1)\subset X_{c^2}\}$, is a homeomorphism.

Let $c_0^2\in\mathcal{C}_A^2$ be an arbitrary cell. If $\tau=id$ then the map $g_{c_0^2}$ induces a continuous fibration $\mathcal{F}_{c_0^2}$ of the set $Q_{c_0^2}\setminus X_{c_0^2}$ into circles. If $\tau\neq id$ then the map $g_{c_0^2}$ induces a continuous fibration of the set $Q_{c_0^2}\setminus X_{c_0^2}$ into segments $I_{c_0^2}=\{g_{c_0^2}(z\times[0,1]),z\notin g_{c_0^2}^{-1}(X)\cap\mathbb{B}^2\times\{0\}\}$ supplemented with a finite set of circles $S_{c_0^2}=\{g_{c_0^2}(z\times[0,1]),z\in g_{c_0^2}^{-1}(X)\cap\mathbb{B}^2\times\{0\}\}$. In this case set $\mathcal{F}_{c_0^2}=I_{c_0^2}\cup S_{c_0^2}$.

Suppose that the cell $c_1^2\in\mathcal{C}_A^2\setminus c_0^2$ is such that $cl\ c_0^2\cap cl\ c_1^2\neq\emptyset$ and $Q_{c_1^2}$ is the closure of the sequence of three-dimensional cells of the flow f^t described above, and the set $\widetilde{F}_1\subset(\partial\mathbb{B}^2)\times[0,1]$ is such that $g_{c_0^2}(\widetilde{F}_1)\subset Q_{c_1^2}$. It follows from Proposition 5 that $\widetilde{F}_1$ is homeomorphic to the disk. Therefore there exists a continuous map $g_{c_1^2}:\mathbb{B}^2\times[0;1]\to Q_{c_1^2}$ that consists with $g_{c_0^2}$ on the set $\widetilde{F}_1$ and satisfy the conditions 1–4 above (if to replace c^2 with c_1^2 in the notations). Denote by $\mathcal{F}_{c_1^2}$ a fibration of the set $Q^2_{c_1^2}$ induced by $g_{c_1^2}$ and continue the process of constructing the fibration in a similar way until all two-dimensional cells are exhausted.

Since the homeomorphism τ is periodic then after a finite number of steps we obtain a fibration $\mathcal{F}_A$ of M^3 into circles. Let us show that this fibration is Seifert. Denote by X_τ the set of points from A (possibly, empty) whose period with respect to τ is less than the period m_τ of τ. Due to Lemmas 1, 2 $X_\tau\subset\Omega_A$.

Two cases are possible: 1) X_τ is non-empty, then τ is not identity; 2) the set X_τ is empty.

Let us consider the case 1). Let $q\in X_\tau$ and $u_q\subset A$ be a neighborhood of q wish does not contain any equilibria different from q. Suppose that q is fixed point of τ (if period m_q of q greater then one then τ move to τ^{m_q} and apply the similar reasoning). Denote by L_q a set of all arcs from $\mathcal{L}_A$ whose closures contain q. It follows from the construction of τ that the set L_q contains at least two arcs. Let $l_1,l_2\subset L_q$ be arcs that belongs to a boundary of the same cell c_*^2 and $\nu\in\{1,\dots,m_\tau-1\}$ be such a number that $l_2=\tau^\nu(l_1)$. Let us choose a point $x_1\in l_1$ and join it with the point $x_2=\tau^\nu(x_1)$ by an arc b_q without self-intersections such that the interior of b_q belongs to the set $c_*^2\cap u_q$. Set $S_q=\bigcup\limits_{i=0}^{m-1}\tau^i(b_q)$. By construction, S_q is τ-invariant simple closed curve bounding a disk $D_q\subset u_q$, such that $q\in int\ D_q$. Since $\tau|_{S_q}$ is orientation preserving and periodic then there exists a homeomorphism $h_q:S_q\to\mathbb{S}^1$ which topologically conjugates $\tau|_{S_q}$ with the rotation $\theta|_{\mathbb{S}^1}$ of the circle $\mathbb{S}^1=\partial\mathbb{B}^2$ with an angle $2\pi\frac{n}{m_\tau}$, where n,m_τ are co-prime. A homeomorphism h_q can be extended into a homeomorphism $H_q:D_q\to\mathbb{B}^2$ which conjugates $\tau|_{D_q}$ with the periodic rotation $\theta:\mathbb{B}^2\to\mathbb{B}^2$.

Denote by λ_x a fiber of the fibration $\mathcal{F}_A$ passing through the point $x\in A$, and by N_q a neighborhood of the fiber λ_q generated by the fibers of the fibration $\mathcal{F}_A$ passing through points of D_q. Let us extend the homeomorphism $H_q:D_q\to\mathbb{B}^2$ up to a fiber by fiber homeomorphism $G_q:N_q\to N_\theta$, where $N_\theta=\mathbb{B}^2\times[0,1]/_\sim$ is the quotient space of $\mathbb{B}^2\times[0,1]$ by means of equivalence relation $(x,1)\sim(\theta(x),0)$, $x\in\mathbb{B}^2$. For this, remark that the orientation of any segment $z\times[0,1]$, $z\in\mathbb{B}^2$ induces the orientation of the closed curve $(z\times[0,1])/_\sim$ in N_θ. Similar, the homeomorphisms g_{c^2} induce the orientations of fibers generated the set N_q. Let $x\in D_q,y\in\lambda_x$ be points such that there exists an arc $\hat\lambda_{x,y}\subset\lambda_x$ joining points x,y and such that the movement along this arc from x to y is agreed with the orientation of the fiber λ_x and $\hat\lambda_{x,y}\cap d_p=x$. Denote by $|\hat\lambda_{x,y}|$ the length of the arc $\hat\lambda_{x,y}$ and by $x'\in\mathbb{B}^2,\lambda'_{x'},y',\hat\lambda'_{x',y'},|\hat\lambda'_{x',y'}|$ similar objects for the manifold N_θ. Set $G(y)=y'$, where y' is a point such that $\frac{|\hat\lambda_{x,y}|}{|\hat\lambda_{x,\tau(x)}|}=\frac{|\hat\lambda'_{x',y'}|}{|\hat\lambda'_{H_q(x),\theta(H_q(x))}|}$.

For an arbitrary point $p\in A\setminus X_\tau$ there exists a closed neighborhood $D_p\subset A$ consisting of points of equal periods with respect to the map τ. Denote by N_p the neighborhood of the fiber λ_p generated by the fibers of the fibration $\mathcal{F}_A$ passing though the points of D_p. Similar to the construction of G_q one can construct fiber by fiber homeomorphism $G_p:N_p\to\mathbb{B}^2\times\mathbb{S}^1$.

In the case 2) for any points $p\in A$ reasoning are similar to ones above.

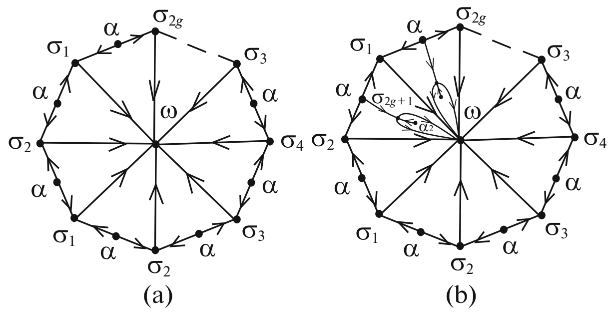

Fig. 4. Construction of an axillary flow on the module surface.

Thus, we have proved that the constructed bundle is a Seifert bundle.

Since A is homeomorphic to the surface $\mathbb{S}_{g_{f^t}}$ we can identify it with $\mathbb{S}_{g_{f^t}}$. Then applying the techniques of the proof of the Theorem 1 one can immediately get the following statement.

Corollary 1. *Manifold M^3 is diffeomorphic to the quotient space $\mathbb{S}_g \times [0,1]/_\sim$ of $\mathbb{S}_g \times [0,1]$ by equivalence relation $(z,1) \sim (\tau_{f^t}(z), 0)$, where $\tau : \mathbb{S}_{g_{f^t}} \to \mathbb{S}_{g_{f^t}}$ is the periodic homeomorphisms defined in Lemma 2.*

4. REALIZATION

In this section we prove Theorem 2. Let $\mathbb{S}_g$ be an orientable surface of genus g and $\tau : \mathbb{S}_g \to \mathbb{S}_g$ be an orientation preserving periodic diffeomorphism. Let us prove that there exists a gradient-like flow $f^t \in GSD(M^3)$ with simple asymptotic behavior of invariant manifolds of saddles given on the mapping torus M^3_τ.

According to [6] the orbit space $\mathbb{S}_g/_\tau$ is homeomorphic to an orientable surface S_{g_τ} of genus g_τ (*a module surface*) and the natural projection $p_\tau : \mathbb{S}_g \to S_{g_\tau}$ is m_τ-branched covering everywhere except points of the set $X_\tau \subset \mathbb{S}_g$ consisting of points whose period is less then m_τ. Any point $x \in X_\tau$ of period m_x is a branch point of the order $\lambda_x = \frac{m_x}{m_\tau}$. It means that there exists a neighborhood $U_x \subset \mathbb{S}_g$ of the point x and homeomorphisms $h_x : U_x \to \mathbb{C}$, $\chi_x : p_\tau(U_x) \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane, such that $h_x(x) = O$, $\chi_x(p_\tau(x)) = O$ and $\chi(p_\tau(h^{-1}(z))) = z^{\lambda_x}$, $z \in \mathbb{C}$.

Let us present the surface S_{g_τ} by an $4g_\tau-$polyhedron unfolding as it shown on the Figure 4, (a) and denote by g^t a gradient-like flow on the surface S_{g_τ} with phase portrait shown on the Figure 4. If the set X_τ contains more than two fixed points then one can modify the flow g^t as in shown on the Figure 4, (b), by adding pairs of a sink and a saddle in such a way that projections of all branch points belongs to equilibria of obtained flow (see [7, Theorem 3.1.2] for more details). We will denote the modified flow also by g^t. Then there exists a gradient-like flow G^t on S_{g_τ} such that $p_\tau G^t = g^t p_\tau$, so trajectories of the flow G^t are τ-invariant.

Define on the segment $[0,1]$ a flow ψ^t by equation $\dot{s} = \sin \pi s$, $s \in [0,1]$ and consider the flow F^t on the direct product $\mathbb{S}_g \times [0,1]$ given by $F^t(z,s) = (G^t(z), \psi^t(s))$, $z \in \mathbb{S}_g$, $s \in [0,1]$. Let $\pi_\tau : \mathbb{S}_g \times [0,1] \to M_\tau$ be the natural projection. Then the flow $f^t = \pi_\tau F^t \pi_\tau^{-1}$ is the desired flow.

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