ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 3, pp. 663–675. © Pleiades Publishing, Ltd., 2021.

On Solvability of a Poincare–Tricomi Type Problem for an Elliptic–Hyperbolic Equation of the Second Kind

T. K. Yuldashev[1*], B. I. Islomov[2], and A. A. Abdullaev[3***]**

(Submitted by A. M. Elizarov)

[1]*Uzbek–Israel Joint Faculty of High Technology and Engineering Mathematics, National University of Uzbekistan, Tashkent, 100174 Uzbekistan*

[2]*Department of Differential Equations and Mathematical Physics, National University of Uzbekistan, Tashkent, 100174 Uzbekistan*

[3]*Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, Tashkent, 100000 Uzbekistan*

Received June 26, 2020; revised July 15, 2020; accepted August 7, 2020

Abstract—In this paper we study a boundary value problem with the Poincare–Tricomi condition for a degenerate partial differential equation of elliptic-hyperbolic type of the second kind. In the hyperbolic part of a degenerate mixed differential equation of the second kind the line of degeneracy is a characteristic. For this type of differential equations a class of generalized solutions is introduced in the characteristic triangle. Using the properties of generalized solutions, the modified Cauchy and Dirichlet problems are studied. The solutions of these problems are found in the convenient form for further investigations. A new method has been developed for a differential equation of mixed type of the second kind, based on energy integrals. Using this method, the uniqueness of the considering problem is proved. The existence of a solution of the considering problem reduces to investigation of a singular integral equation and the unique solvability of this problem is proved by the Carleman–Vekua regularization method.

DOI: 10.1134/S1995080221030239

Keywords and phrases: *generalized solution, Poincare–Tricomi type problem, degenerate equation of the second kind, energy integral method, Green function.*

1. INTRODUCTION

Degenerate partial differential equations occupy one of the central places in the theory of general partial differential equations and have numerous applications in various branches of science and technology. Partial differential equations of mixed type with degenerations have been systematically studied since the middle of the last century after the well-known works of F. I. Frankl, which are reflected in [1]. He showed applications of degenerate mixed type differential equations in solving problems of transonic and supersonic gas dynamics. Later, another applications were find of degenerate differential equations of mixed type in other fields of science and technology.

I. N. Vekua in [2] showed the importance of studying mixed-type differential equations in solving problems of the theory of infinitesimal bending of surfaces. The problem of the outflow of a supersonic jet from a vessel with flat walls reduces to the Tricomi problem for the Chaplygin equation. There are a number of works in which the problems of Tricomi, Gellerstedt, and Bitsadze are studied. It is easy to meet works, where new correct problems with bias are posed for equations of elliptic-hyperbolic and parabolic-hyperbolic types of the first kind, for which the line of degeneration is not a characteristic [3–19].

*E-mail: tursun.k.yuldashev@gmail.com
**E-mail: islomovbozor@yandex.com
***E-mail: akmal09.07.85@mail.ru

A feature of degenerate hyperbolic differential equations is that for this kind of equations the Cauchy problem with initial condition on the line of parabolic degeneracy does not always hold. For example, the Cauchy problem in the usual formulation may turn out to be unsolvable, if the hyperbolic equations degenerate along a line that is simultaneously a characteristic. Such type of differential equations are called degenerate equations of the second kind.

Note that the solution of the Cauchy problem with initial condition

$$\lim_{y\to 0-} u(x,y)=\tau(x),\quad 0\leq x\leq 1,\quad \lim_{y\to 0-}\frac{\partial u(x,y)}{\partial y}=\nu(x),\quad 0<x<1$$

for a differential equation

$$(-y)^m u_{xx}-u_{yy}=0,\quad -1<m<0 \tag{1}$$

on the negative half-axis $y<0$ has a representation [19, pages 259–260]:

$$u(x,y)=\gamma_2\int\limits_0^1 \tau(z)\,t^\beta(1-t)^\beta\,dt+\frac{2\gamma_2}{(1+2\beta)(m+2)}(-y)^{\frac{m+2}{2}}\int\limits_0^1 \tau'(z)\,t^\beta(1-t)^\beta(2t-1)\,dt$$

$$+\left[2(1-2\beta)\right]^{1-2\beta}\gamma_1 y\int\limits_0^1 \nu(z)\,t^{-\beta}(1-t)^{-\beta}\,dt, \tag{2}$$

where $-1<2\beta<0$ for $-1<m<0$ and

$$2\beta=\frac{m}{m+2},\quad \gamma_1=\left[2(1-2\beta)\right]^{2\beta-1}\frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)},$$

$$\gamma_2=\frac{\Gamma(2+2\beta)}{\Gamma^2(1+\beta)},\quad z=x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1).$$

Definition 1. *The function $u(x,y)$ (represented by (2)) on the negative half-axis $y<0$ is called a generalized solution from the class $\mathbb{R}_2$ to the Cauchy problem for differential equation (1), if $\tau(z)$ is representable in the following integral form $\tau(z)=\int\limits_0^z (z-t)^{-2\beta}T(t)\,dt$, where $\nu(z)$ and $T(z)$ are continuous and integrable functions on the interval $(0,1)$.*

In [20] a generalized solution was obtained from the class $\mathbb{R}_2$ to the Cauchy problem for a hyperbolic differential equation of the second kind. This generalized solution has a form

$$u(\xi,\eta)=\int\limits_0^\xi (\eta-t)^{-\beta}(\xi-t)^{-\beta}T(t)\,dt+\int\limits_\xi^\eta (\eta-t)^{-\beta}(t-\xi)^{-\beta}N(t)\,dt,$$

where

$$\xi=x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}},\quad \eta=x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}},\quad N(t)=\frac{1}{2\cos\pi\beta}T(t)-\gamma_1\nu(t).$$

Using this solution representation form, in [21–27] local and nonlocal boundary value problems for mixed differential equations of the second kind were studied. It is proved that such type of problems arise in studying some problems of mathematical biology [28] and physics [29].

In [30] for the differential equation $u_{xx}+yu_{yy}+\left(\alpha+\frac{1}{2}\right)u_y=0$ built a special class of solutions in the case, when $y<0,\quad \alpha\in\left(-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right)$.

In the hyperbolic part of the mixed domain, solvability of local and nonlocal boundary value problems are studied in the class $\mathbb{R}_2$, and in the elliptic part of the domain the questions of classical solvability are studied. These problems are equivalently reduced to singular integral equations. Then by the regularization method they are reduced to study Fredholm integral equations of the second kind.

Note that boundary value problems for mixed differential and integro-differential equations in rectangular domains were studied in [31–34]. Boundary value problems with the Poincare–Tricomi condition for degenerate differential equations of elliptic and elliptic-hyperbolic types of the second kind have been studied, relatively little. We indicate here only the papers [35–36].

In this presented our paper by the aid of properties of generalized functions we study a boundary value problem with the Poincare–Tricomi condition for an elliptic-hyperbolic differential equation of the second kind.

2. FORMULATION OF THE PROBLEM

We consider the following mixed differential equation

$$\operatorname{sgn} y\,|y|^m\,u_{xx}+u_{yy}=0,\quad -1<m<0 \tag{3}$$

in the domain $D=D_1\cup D_2$, where D_1 is domain bounded by a curve σ for $y>0$ with the end points $A(0,0)$, $B(1,0)$ and with segment $AB(y=0)$, D_2 is domain bounded by segment AB and by characteristics

$$AC:x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=0,\quad BC:x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1.$$

We introduce the notations

$$J=\{(x,y):\ 0<x<1,\ y=0\}\,,\quad \partial D=\vec{\sigma}\cup\overline{AB},\quad 2\beta=\frac{m}{m+2}$$

for

$$-1<2\beta<0. \tag{4}$$

Problem *PT*. Find the function $u(x,y)$, for which are true the following properties:

1) $u(x,y)\in C\left(\overline{D}\right)\cup C^1\left(D\cup\sigma\cup J\right)$, there u_x can goes to infinity of order less than one units at the point $A\,(0,0)$ and u_y can goes to infinity of order less than -2β at the point $B\,(1,0)$;

2) function $u(x,y)\in C^2\left(\overline{D}_1\right)$ is a regular solution of the differential equation (3) in the domain D_1, and is generalized solution from the class $\mathbb{R}_2$ in the domain D_2;

3) is fulfilled the gluing condition $u_y(x,-0)=-u_y(x,+0)$;

4) $u(x,y)$ satisfies the boundary value conditions

$$\{\delta(s)A_s[u]+\rho(s)u\}|_\sigma=\varphi(s),\quad 0<s<l,\quad A_s\,[u]=y^m\frac{dy}{ds}\frac{\partial u}{\partial x}-\frac{dx}{ds}\frac{\partial u}{\partial y}, \tag{5}$$

$$u(x,y)|_{AC}=\psi(x),\quad 0\le x\le\frac{1}{2}, \tag{6}$$

where $\frac{dx}{ds}=-\cos(n,y)$, $\frac{dy}{ds}=\cos(n,x)$, n is external normal to the curve σ, l is a length of the entire curve σ, s is the arc length of the curve σ, measured from point $B\,(1,0)$; $\delta(s)$, $\rho(s)$, $\varphi(s)$, $\psi(x)$ are given sufficiently smooth functions and $\psi(x)\in C^1\left[0,\frac{1}{2}\right]\cap C^2\left(0;\frac{1}{2}\right)$, $\varphi(l)=\psi(0)=0$.

We note that if we set $\delta(s)=0$ ($\rho(s)=0$), then the problem PTcoincided with the problem $T\,(T_N)$ for elliptic-hyperbolic equation of the first kind (see [37, pp. 177–185]). Therefore, we will assume in our work that $\delta(s)\neq0$, $\rho(s)\neq0$.

We assume that the curve σ satisfies the following conditions:

1) functions $x(s)$ and $y(s)$ are parametric equations of curve σ, have continuous derivatives $x'(s)$, $y'(s)$ (these derivatives are nonzero simultaneously) and have second derivatives, which satisfy the Hölder condition of the order $\kappa\,(0<\kappa<1)$ on the segment $0\le s\le l$;

2) in the neighborhood of the end points of the curve σ the inequality is true:

$$\left|\frac{dx}{ds}\right|\le Cy^{m+1}(s),\quad x(l)=y(0)=0,\quad x(0)=1,\quad y(l)=0,\quad C=\text{const}. \tag{7}$$

3. UNIQUENESS OF THE SOLUTION OF THE PROBLEM PT

Theorem 1. *If (4) and the following conditions are fulfilled*

$$\delta(s)\rho(s) \geq 0, \quad 0 \leq s \leq l, \tag{8}$$

$$\lim_{y\to 0}(-y)^{\frac{m}{2}}u^2(1,y) = 0. \tag{9}$$

Then the solution of the problem PT is unique in the domain D.

Proof. We prove the theorem by the method of energy integrals. Let $u(x,y)$ be a twice continuously differentiable solution of equation (3) in the domain

$$\overline{D}^{\varepsilon_1,\varepsilon_2} \subset D, \quad D^{\varepsilon_1,\varepsilon_2} = D_1^{\varepsilon_1,\varepsilon_2} \cup D_2^{\varepsilon_1,\varepsilon_2},$$

where $D_1^{\varepsilon_1,\varepsilon_2}$ is domain with border

$$\partial D_1^{\varepsilon_1,\varepsilon_2} = A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1} \cup \sigma_{\varepsilon_1} \quad \left(A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1}:\ y=\varepsilon_2\right)$$

strictly lying in the domain D_1, while $D_2^{\varepsilon_1,\varepsilon_2}$ is domain bounded with lines

$$A_{\varepsilon_2}^{\varepsilon_1}B_{\varepsilon_2}^{\varepsilon_1}: y=-\varepsilon_2, \quad A_{\varepsilon_2}^{\varepsilon_1}C_{\varepsilon_1}: x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=\varepsilon_1, \quad B_{\varepsilon_2}^{\varepsilon_1}C_{\varepsilon_1}: x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1-\varepsilon_1,$$

where $\varepsilon_1, \varepsilon_2$ are small positive numbers.

In the domain D_2 differential equation (3) takes the form $(-y)^m u_{xx} - u_{yy} = 0$. It is easy to check that the following identity holds:

$$u\left[(-y)^m u_{xx} - u_{yy}\right] = \frac{\partial}{\partial x}\left[(-y)^m u u_x\right] - \frac{\partial}{\partial y}\left[u u_y\right] - (-y)^m u_x^2 + u_y^2.$$

Integrating this identity over the domain $D_2^{\varepsilon_1,\varepsilon_2}$, we derive

$$0 = \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} u\left[(-y)^m u_{xx} - u_{yy}\right]dxdy = \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} \left\{\frac{\partial}{\partial x}\left[(-y)^m u u_x\right] - \frac{\partial}{\partial y}\left[u u_y\right]\right\}dxdy$$
$$+ \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} \left[u_y^2 - (-y)^m u_x^2\right]dxdy. \tag{10}$$

Applying Green formula [35] to the first integral on the right-hand side of (10), we obtain

$$0 = \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} u\left[(-y)^m u_{xx} - u_{yy}\right]dxdy = \int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1} \cup CB_{\varepsilon_2}^{\varepsilon_1} \cup B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right]$$
$$+ \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} \left[u_y^2 - (-y)^m u_x^2\right]dxdy.$$

Calculating the first integral on the right-hand side of the last equality, taking into account the condition (6) on the characteristic AC, we obtain

$$\int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1} \cup CB_{\varepsilon_2}^{\varepsilon_1} \cup B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] = \int\limits_{A_{\varepsilon_2}^{\varepsilon_1}C^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right]$$
$$+ \int\limits_{CB_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{B_{\varepsilon_2}^{\varepsilon_1}A_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right]$$
$$= \int\limits_{C^{\varepsilon_1}}^{B_{\varepsilon_2}^{\varepsilon_1}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{1-\varepsilon_1}^{\varepsilon_1} u(x,-\varepsilon_2)u_y(x,\varepsilon_2)dx.$$

Consequently, we have

$$\int\limits_{C^{\varepsilon_1}}^{B^{\varepsilon_1}_{\varepsilon_2}} u\left[(-y)^m u_x dy + u_y dx\right] + \int\limits_{1-\varepsilon_1}^{\varepsilon_1} u(x,-\varepsilon_2)u_y(x,\varepsilon_2)dx + \iint\limits_{D_2^{\varepsilon_1,\varepsilon_2}} \left[(-y)^m u_x^2 + u_y^2\right] dxdy = 0.$$

Passing to the limits as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, taking the gluing condition $u_y(x,-0) = -u_y(x,+0)$ into account we obtain

$$\int\limits_0^1 \tau(x)\nu(x)dx = -\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy - \int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right].$$

On the characteristic BC we have $dx = (-y)^{\frac{m}{2}} dy$. Therefore we have

$$\int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u\left[u_x dx + u_y dy\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u dy. \tag{11}$$

Integrating in parts the last integral of (11) with conditions $u|_{AC} = 0$ and (9), we obtain

$$\int\limits_C^B (-y)^{\frac{m}{2}} u du = \frac{m}{4}\int\limits_C^B (-y)^{\frac{m-2}{2}} u^2 dy.$$

For $-1 < m < 0$ from (11) yields

$$\int\limits_C^B u\left[(-y)^m u_x dy + u_y dx\right] = \int\limits_C^B (-y)^{\frac{m}{2}} u du = \frac{m}{4}\int\limits_C^B (-y)^{\frac{m-2}{2}} u^2 du \le 0. \tag{12}$$

Now we show that the first integral on the right-hand side of equality (10) is not positive. Passing to characteristic variables

$$\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}},$$

we obtain

$$\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy = -2\iint\limits_{\Delta_1} \left(\frac{m+2}{4}\right)^{\frac{m}{2}} (\eta-\xi)^{\frac{m}{m+2}} u_\xi u_\eta d\xi d\eta, \tag{13}$$

where $\Delta_1 = \{(\xi,\eta) : 0 < \xi < 1,\ \xi < \eta < 1\}$ is image of the domain D_2 in coordinates (ξ,η).

In domain Δ_1 the differential equation (3) takes the form $u_{\xi\eta} - \frac{\beta}{\eta-\xi}(u_\eta - u_\xi) = 0$. Multiplying both sides of this equation by u_η, we have

$$u_\xi u_\eta = u_\eta^2 - \frac{\eta-\xi}{\beta} u_\eta u_{\xi\eta}. \tag{14}$$

Substitute (14) in (13). As a result yields

$$\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy = -2\left(\frac{m+2}{4}\right)^{\frac{m}{m+2}}$$

$$\times\left[\iint\limits_{\Delta_1} (\eta-\xi)^{\frac{m}{m+2}} u_\eta^2 d\xi d\eta - \frac{1}{\beta}\iint\limits_{\Delta_1} (\eta-\xi)^{\frac{2(m+1)}{m+2}} u_\eta u_{\xi\eta} d\xi d\eta\right].$$

Integrating the last integral in parts, we have

$$\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy = \frac{2(m+2)}{m} \left(\frac{m+2}{4}\right)^{\frac{m}{m+2}}$$

$$\times \left[\iint\limits_{\Delta_1} (\eta - \xi)^{\frac{m}{m+2}} u_\eta^2 d\xi d\eta + (\eta - \xi)^{\frac{2(m+1)}{m+2}} u_\eta^2 \Big|_{\eta=\xi} \right].$$

Since the last term in the square bracket is zero for $\eta = \xi$, then we obtain

$$\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy = \frac{2(m+2)}{m} \left(\frac{m+2}{4}\right)^{\frac{m}{m+2}} \iint\limits_{\Delta_1} (\eta - \xi)^{\frac{m}{m+2}} u_\eta^2 d\xi d\eta.$$

Here by virtue of

$$\iint\limits_{\Delta_1} (\eta - \xi)^{\frac{m}{m+2}} u_\eta^2 d\xi d\eta \geq 0$$

for $-1 < m < 0$ yields

$$\iint\limits_{D_2} \left[u_y^2 - (-y)^m u_x^2\right] dxdy \leq 0. \tag{15}$$

By virtue of inequalities (12) and (15), from (10) we obtain

$$\int\limits_0^1 \tau(x)\nu(x)dx \geq 0. \tag{16}$$

Differential equation (3) in domain D_1 has the form $y^m u_{xx} + u_{yy} = 0$. Integrating the following identity

$$u\left[y^m u_{xx} + u_{yy}\right] = \frac{\partial}{\partial x}\left[y^m u u_x\right] + \frac{\partial}{\partial y}\left[u u_y\right] - y^m u_x^2 - u_y^2$$

over the domain $D_1^{\varepsilon_1,\varepsilon_2} \subset D_1$, we derive

$$0 = \iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} u\left[y^m u_{xx} + u_{yy}\right] dxdy = \iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} \left\{\frac{\partial}{\partial x}\left[y^m u u_x\right] + \frac{\partial}{\partial y}\left[u u_y\right]\right\} dxdy - \iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} \left[y^m u_x^2 + u_y^2\right] dxdy.$$

Applying Green formula [35] to the first integral on the right-hand side of the last equality, we obtain

$$0 = \iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} u\left[y^m u_{xx} + u_{yy}\right] dxdy = -\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} \left[y^m u_x^2 + u_y^2\right] dxdy + \int\limits_{\partial D_1^{\varepsilon_1,\varepsilon_2}} u\left[y^m u_x dy - u_y dx\right].$$

Hence by virtue of $dy = 0$ on AB and $dy = \cos(n,x)ds$, $dx = -\cos(n,y)ds$, we obtain

$$0 = -\iint\limits_{D_1^{\varepsilon_1,\varepsilon_2}} \left[y^m u_x^2 + u_y^2\right] dxdy - \int\limits_{x_1}^{x_2} u(x,\varepsilon_2)u_y(x,\varepsilon_2)dx + \int\limits_{\sigma_{\varepsilon_1}} u A_s[u] ds, \tag{17}$$

where x_1, x_2 are abscissas of the intersection points of line $y = \varepsilon_2$ with curve σ_{ε_1}.

By virtue of condition 1) of the problem PT, zero values $\varphi(s) \equiv b(x) \equiv 0$ and (5), from (17) as $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ we obtain

$$\iint\limits_{D_1} \left[y^m u_x^2 + u_y^2\right] dxdy + \int\limits_0^1 \tau(x)\nu(x)dx + \int\limits_\sigma \frac{\delta(s)\rho(s)}{\delta^2(s)} u^2 ds = 0. \tag{18}$$

By virtue of inequalities (8) and (16), from (18) implies that $u_x = u_y = 0$ in D_1, i.e. $u(x, y) = \text{const}$ for all $(x, y) \in D_1$. Since, each term of equality (18) is non-negative, then $u(x, y) = 0$ on $\overline{\sigma}$. By the aid of the Hopf principle [19, pages 44–48], we conclude that $u(x, y) \equiv 0$ in $\bar{D}_1$ for $\delta(s) \neq 0$. Now from the uniqueness of the solution of Cauchy problem it follows that $u(x, y) \equiv 0$ in $\bar{D}_2$. Consequently, $u(x, y) \equiv 0$ in $\overline{D}$. This is ended the proof of uniqueness of the solution of the problem PT. Theorem 1 is proved. □

Remark. Uniqueness of the solution of the problem PT for $\rho(s) \neq 0$, $\forall s \in [0, l]$ be proved by the principle of extremum.

4. BASIC FUNCTIONAL RELATIONSHIPS

We introduce the following notations:

$$u\left(x, 0\right) = \tau\left(x\right), \quad \left(x, 0\right) \in \overline{J}, \tag{19}$$

$$\lim_{y\to -0} \frac{\partial u\left(x, y\right)}{\partial y} = \nu^{-}\left(x\right), \quad \lim_{y\to +0} \frac{\partial u\left(x, y\right)}{\partial y} = \nu^{+}\left(x\right), \quad \left(x, 0\right) \in J. \tag{20}$$

In studying this problem PT an important role are played functional relations between $\nu^{\pm}(x)$ and $\tau(x)$, which were bring from elliptical and hyperbolic parts of the domain D.

A generalized solution from class $\mathbb{R}_2$ to the Cauchy problem with conditions (19), (20) for differential equation (3) in the domain D_2 is given by the formula [37, p. 230, form. 27.5]:

$$u\left(\xi, \eta\right) = \int_0^{\xi} \left(\eta - t\right)^{-\beta} \left(\xi - t\right)^{-\beta} T\left(t\right) dt + \int_{\xi}^{\eta} \left(\eta - t\right)^{-\beta} \left(t - \xi\right)^{-\beta} N\left(t\right) dt, \tag{21}$$

where

$$\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}, \quad \gamma_2 = [2(1-2\beta)]^{2\beta-1} \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}, \tag{22}$$

$$N\left(t\right) = \frac{T\left(t\right)}{2\cos\pi\beta} - \gamma_2 \nu^{-}\left(t\right), \tag{23}$$

$$\tau\left(x\right) = \int_0^{x} \left(x - t\right)^{-2\beta} T\left(t\right) dt, \tag{24}$$

$T(x)$ and $\nu^{-}(x)$ are functions of continuous in $(0, 1)$ and integrable on $[0, 1]$, $\tau(x)$ is zero of the order not less then -2β as $x \to 0$. Putting $\xi = 0$ in (21) and taking into account (6), (22), we obtain

$$\psi(\eta) = \int_0^{\eta} N(\zeta)\zeta^{-\beta}(\eta - \zeta)^{-\beta} d\zeta.$$

This is first kind Volterra integral equation with respect to $N(\zeta)$. We set

$$\Phi(\zeta) = N(\zeta)\zeta^{-\beta}. \tag{25}$$

Then we have

$$\int_0^{\eta} \Phi(\zeta)(\eta - \zeta)^{-\beta} d\zeta = \psi(\eta). \tag{26}$$

We apply a fractional order differential operator $D_{0x}^{\alpha} f(x)$ to (26):

$$D_{0\eta}^{\beta-1}\Phi(\eta) = \frac{1}{\Gamma(1-\beta)}\psi(\eta). \tag{27}$$

We apply the differential operator $D_{0\eta}^{1-\beta}$ to both sides of equation (27) taking into account $D_{0\eta}^{1-\beta}D_{0\eta}^{\beta-1}\Phi(\eta)=\Phi(\eta)$. Then we have

$$\Phi(\eta)=\frac{1}{\Gamma(1-\beta)}D_{0\eta}^{1-\beta}\psi(\eta). \tag{28}$$

By direct verification it is easy to check that function (28) is a solution of the equation (26). Taking into account (25) and (28) from (23) we obtain the first functional relation between $T(x)$ and $\nu^-(x)$, which are bringing from the D_2 to the domain J:

$$T(\zeta)=\gamma_3\nu^-(\zeta)+\frac{2\cos\pi\beta}{\Gamma(1-\beta)}\zeta^\beta D_{0\eta}^{1-\beta}\psi(\eta), \tag{29}$$

where $\gamma_3=2\gamma_2\cos\pi\beta$.

In the positive half-plane $y>0$ the differential equation (3) takes the form

$$y^m u_{xx}+u_{yy}=0, \quad -1<m<0. \tag{30}$$

Consider the following auxiliary problem.

Problem PT^+. Find in domain D_1 a solution $u(x,y)\in C(\overline{D}_1)\cap C^1\big(D_1\cup\sigma\cup J\big)\cap C^2\big(D_1\big)$ of the equation (30), satisfying to the boundary value conditions (5) and (19).

Solution of the problem PT^+ with conditions (5) and (19) for differential equation (3) in domain D_1 exists, unique and has the form (see [37, page 179]):

$$u(x,y)=\int_0^1\tau(\xi)\frac{\partial}{\partial\eta}G_2(\xi,0;x,y)d\xi+\int_0^l\frac{\varphi(s)}{\delta(s)}G_2(\xi,\eta;x,y)ds, \tag{31}$$

where $G_2(\xi,\eta;x,y)$ is Green function of the problem PT^+ for equation (3) and

$$G_2(\xi,\eta;x,y)=G_{02}(\xi,\eta;x,y)+H_2(\xi,\eta;x,y),$$

$G_{02}(\xi,\eta;x,y)$ is the Green function of the problem PT^+ for equation (3) in normal domain D_0, which bounded with segment $\overline{AB}$ and normal curve

$$\sigma_0:\left(x-\frac{1}{2}\right)^2+\frac{4}{(m+2)^2}y^{m+2}=\frac{1}{4},$$

$$H_2(\xi,\eta;x,y)=G_2(\xi,\eta;x,y)-G_{02}(\xi,\eta;x,y)$$

$$=\int_0^l\lambda_2(s;\xi,\eta)\left\{A_s\left[G_{02}(\xi(s),\eta(s);x,y)\right]+\frac{\rho(s)}{\delta(s)}G_{02}(\xi(s),\eta(s);x,y)\right\}ds, \tag{32}$$

$\lambda_2(s;\xi,\eta)$ is solution of the integral equation

$$\lambda_2(s;\xi,\eta)+2\int_0^l\lambda_2(t;\xi,\eta)\left\{A_s\left[q_2(\xi(t),\eta(t);x(s),y(s))\right]\right.$$

$$\left.+\frac{\rho(s)}{\delta(s)}q_2(\xi(t),\eta(t);x(s),y(s))\right\}dt=-2q_2(\xi(s),\eta(s);\xi,\eta),$$

$q_2(x,y,x_0,y_0)$ is fundamental solution of differential equation (3) and

$$q_2(x,y,x_0,y_0)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}(1-\sigma)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta;1-\sigma), \tag{33}$$

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\}=(x-x_0)^2+\frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}}\mp y_0^{\frac{m+2}{2}}\right)^2,$$

$$\sigma=\frac{r^2}{r_1^2},\quad \beta=\frac{m}{2(m+2)}<0,\quad k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2\left(1-\beta\right)}{\Gamma\left(2-2\beta\right)},$$

$F\left(a,b,c;z\right)$ is Gauss hypergeometric function.

We differentiate (31) with respect to y. Then taking (32) and (33) into account as $y\to 0$ we obtain the functional relation between $\tau(x)$ and $\nu(x)$, which are bringing from the domain D_1 to J:

$$\nu(x)=k_2\int\limits_0^1|t-x|^{2\beta-2}\tau(t)dt-k_2\int\limits_0^1\frac{\tau(t)dt}{(t+x-2tx)^{2-2\beta}}$$
$$+\int\limits_0^1\tau(t)\frac{\partial^2H_2(t,0;x,0)}{\partial\eta\partial y}dt+\int\limits_0^l\chi(s)\frac{\partial q_2(t,\eta;x,0)}{\partial y}ds, \tag{34}$$

where $\chi(s)$ is solution of the integral equation

$$\chi(s)+2\int\limits_0^l\chi(t)\left\{A_s\left[q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\right]+\frac{\rho(s)}{\delta(s)}q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\right\}dt=\frac{2\varphi(s)}{\delta(s)}.$$

Substituting (24) into 34 and taking into account the identities:

$$\int\limits_0^x(x-t)^{2\beta-2}\tau(t)dt=\frac{\Gamma\left(1+2\beta\right)\Gamma\left(1-2\beta\right)}{2\beta\left(2\beta-1\right)}D_{0x}^{1-2\beta}D_{0x}^{2\beta-1}T(x)=\frac{\pi T(x)}{(2\beta-1)\sin 2\pi\beta},$$

$$\int\limits_x^1(t-x)^{2\beta-2}\tau(t)dt=\frac{\Gamma\left(1+2\beta\right)\Gamma\left(1-2\beta\right)}{2\beta\left(2\beta-1\right)}D_{x1}^{1-2\beta}D_{0x}^{2\beta-1}T(x)$$

$$=\frac{\pi\cot 2\beta\pi}{1-2\beta}T\left(x\right)+\frac{1}{1-2\beta}\int\limits_0^1\left(\frac{1-t}{1-x}\right)^{1-2\beta}\frac{T\left(t\right)}{t-x}dt,$$

$$\int\limits_0^1\left(t+x-2tx\right)^{2\beta-2}\tau(t)dt=\frac{1}{1-2\beta}\int\limits_0^1\left(\frac{1-t}{1-x}\right)^{1-2\beta}\frac{T(t)dt}{x+t-2xt},$$

we obtain the functional relationship between $T(x)$ and $\nu^+(x)$, which are bringing from D_1 to domain J:

$$\nu^+(x)=-\frac{\pi k_2\tan\beta\pi}{1-2\beta}T(x)+\frac{k_2}{1-2\beta}\int\limits_0^1\left(\frac{1}{x}\right)^{-2\beta}T(t)\left[\frac{1}{x-t}+\frac{1-2t}{x+t-2xt}\right]dt$$
$$+\int\limits_0^1T(t)dt\int\limits_0^t\left(t-z\right)^{-2\beta}\frac{\partial^2H_2(z,0;x,0)}{\partial\eta\partial y}dz+\int\limits_0^l\frac{\partial q_2(t,\eta;x,0)}{\partial y}\chi(s)ds,\quad (x,0)\in J. \tag{35}$$

5. EXISTENCE OF THE SOLUTION OF PROBLEM PT

Definition 2 [38, pages 255–259]. *We say that the solution $\omega(z)$ of a singular integral equation*

$$\omega(z)+\lambda\int\limits_0^1\frac{\omega(\zeta)d\zeta}{\zeta-z}-\int\limits_0^1K(z,\zeta)\omega(\zeta)d\zeta=F(z)$$

belongs to class $h(0)$, if this function $\omega(z)$ is bounded as $z\to 0$ and unbounded as $z\to 1$.

Theorem 2. *If conditions (4) and (7) are satisfied, then a solution of problem PT exists in the domain D.*

Proof. Eliminating $\nu^-(x)$ and $\nu^+(x)$ from relations (29) and (35), taking into account (20) and the gluing condition $\nu^-(x) = -\nu^+(x)$, we have

$$\tilde{T}(x) - \gamma_4 \int\limits_0^1 \left[\frac{1}{x-t} - \frac{1}{x+t-2xt}\right] \tilde{T}(t)dt - \int\limits_0^1 K(x,t)\tilde{T}(t)dt = F(x), \tag{36}$$

where $\gamma_4 = \frac{\cos\beta\pi}{\pi(\sin\beta\pi-1)}$, $\tilde{T}(x) = x^{1-2\beta}T(x)$,

$$K\,(x,t) = \frac{\gamma_3}{\sin\beta\pi - 1}\left(\frac{x}{t}\right)^{1-2\beta} \int\limits_0^t (t-z)^{-2\beta}\, \frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz, \tag{37}$$

$$F(x) = \frac{2\cos\pi\beta}{(\sin\pi\beta - 1)\,\Gamma(1-\beta)} x^{1-\beta} D_{0\eta}^{1-\beta}\psi(\eta) + \frac{\gamma_3 x^{1-2\beta}}{(\sin\pi\beta-1)} \int\limits_0^l \frac{\partial q_2(t,\eta;x,0)}{\partial y}\chi(s)ds. \tag{38}$$

We study the kernel and the right side of the singular integral equation (36). For $0 < x < 1$ and $0 < z < 1$ it is true the inequality [37, page 181]:

$$\left|\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}\right| < C_1(x+z-2xz)^{2\beta-1}, \tag{39}$$

where $C_1 = \text{const}$ in D_1. By virtue of (39), from (37) we obtain

$$|K(x,t)| \le C_1\frac{\gamma_3}{\sin\beta\pi - 1}\left(\frac{x}{t}\right)^{1-2\beta} \left|\int\limits_0^t (t-z)^{-2\beta}\,(x+z-2xz)^{2\beta-1}dz\right|. \tag{40}$$

By changing the variable $z = t\,(1-\sigma)$ and using the integral representation of the hypergeometric function [37, § 2, form. 2.10], from (40) we obtain

$$|K(x,t)| \le C_1\frac{\gamma_3}{\sin\beta\pi - 1}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta} \left|\int\limits_0^1 \sigma^{-2\beta}\left[1 - \frac{(1-2x)t}{x+t-2xt}\sigma\right]^{2\beta-1} d\sigma\right|$$

$$\le C_1\frac{\gamma_3}{\sin\beta\pi - 1}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta} \left|F\left(1-2\beta, 1-2\beta, 2-2\beta; \frac{t\,(1-2x)}{x+t-2xt}\right)\right|. \tag{41}$$

Since $c-a-b = 2-2\beta-2+4\beta = 2\beta < 0$, then by the aid of the formula:

$$F\,(a,b,c,z) = (1-z)^{c-a-b}\, F\,(c-a, c-b, c; z)\,, \quad |\arg(1-z)| < \pi$$

and by the estimate

$$|F\,(a,b,c,z)| \le \begin{cases} \text{const}, \quad c-a-b>0, \quad 0 \le z \le 1, \\ \text{const}\cdot(1-z)^{c-a-b}, \quad c-a-b<0, \quad 0<z<1, \\ \text{const}\cdot[1+\ln(1-z)]\,, \quad c-a-b=0, \quad 0<z<1 \end{cases} \tag{42}$$

from (41) for $0 \le t \le 1$ we come to an estimate

$$|K(x,t)| \le C_1 C_2\frac{\gamma_3}{\sin\beta\pi - 1}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}\left(\frac{x}{x+t-2xt}\right)^{2\beta} \le \frac{C_3 x}{x+t-2xt}.$$

Now we estimate the right-hand side of equation (36). Differentiating (33) with respect to y, then putting $y = 0$, we obtain

$$\frac{\partial q_2\,(\xi,\eta;t,0)}{\partial y} = k_2\eta\left[(\xi-t)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta-1}. \tag{43}$$

Substituting (43) into (38), we have

$$F(x)=\frac{2\cos\pi\beta}{(\sin\pi\beta-1)\,\Gamma(1-\beta)}x^{1-\beta}D_{0\eta}^{1-\beta}\psi(\eta)$$

$$-\frac{\gamma_3 x^{1-2\beta}}{1-\sin\pi\beta}\int\limits_0^l\left[(\xi(s)-t)^2+\frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{\beta-1}\eta(s)\chi(s)ds. \tag{44}$$

By virtue of properties of the functions $\psi\,(x)$ and $\varphi\,(s)$, it follows from (44) that the function $F\,(x)$ has derivatives of any order in the interval $(0,1)$. Let us find out the behavior of the function $F\,(x)$ and its derivative as $x\to 0$ and as $x\to 1$. Consider the expression

$$F_1(x)=\int\limits_0^l\left[(\xi(s)-t)^2+\frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{\beta-1}\eta(s)\chi(s)ds. \tag{45}$$

We estimate the expression (45). By virtue of $\delta(s),\rho(s),\varphi(s)\in C\,[0,l]$, for enough small $x>0$ there hold inequalities

$$|F_1\,(x)|\le\int\limits_{l-\varepsilon}^l|\chi(s)|\left[(\xi-x)^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta-1}\eta ds+O(1)$$

$$<C_4\int\limits_{l-\varepsilon}^l\left[(\xi-x)^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta-1}\eta ds+O(1).$$

Hence, by virtue of (26), for sufficiently small $\varepsilon>0$ we obtain

$$|F_1\,(x)|<C_5\int\limits_{l-\varepsilon}^l\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|\left[x^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{-\frac{1}{2}-\beta}ds+O(1)$$

$$<C_6\int\limits_0^\delta\left[x^2+\tilde\eta^2\right]^{-\frac{1}{2}-\beta}d\tilde\eta+O(1). \tag{46}$$

By the change $\mu^2=\omega$ in (46), using the integral representation of the hypergeometric function [37, § 2, form. 2.10] and taking into account (42) we obtain

$$|F_1\,(x)|<\frac{\delta^2}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|\left(\frac{\delta^2}{x^2}\right)^{-\frac{1}{2}}$$

$$+\frac{\delta}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta}x^{2\beta+1}F\left(\beta,\frac{1}{2},1+\beta;\frac{x^2}{x^2+\delta^2}\right)$$

$$<\frac{\delta}{x^{2\beta}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|+\left|\frac{\delta\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta}<C_7x^{-2\beta}. \tag{47}$$

If $1-x$ is sufficiently small, by the similarly way we obtain

$$|F_1\,(x)|=C_8\,(1-x)^{-2\beta}\,. \tag{48}$$

Carrying out the same reasoning, we obtain

$$\left|F_1'\,(x)\right|<C_9x^{-2\beta-1},\quad\left|F_1'\,(x)\right|=C_{10}\,(1-x)^{-2\beta-1}\,. \tag{49}$$

By virtue of (47)–(49), from (44) we deduce that $F(x)\in C(\overline{J})\cap C^2(J)$ and the function $F'(x)$ turns to infinity order less then $2\beta+1$ as $x\to 1$, and as $x\to 0$ this derivative $F'(x)$ is bounded.

By the change of variables $\zeta = \frac{t^2}{1-2t+2t^2}$, $z = \frac{x^2}{1-2x+2x^2}$, we represent the equation (36) as follows

$$\omega(z) + \gamma_4 \int_0^1 \frac{\omega(\zeta)d\zeta}{\zeta - z} - \int_0^1 \overline{K}(z,\zeta)\omega(\zeta)d\zeta = \tilde{F}(z), \tag{50}$$

where

$$\omega(z) = (1-2x+2x^2)\tilde{T}(x), \quad \tilde{F}(z) = (1-2x+2x^2)F(x),$$

$$\overline{K}(z,\zeta) = \frac{1-2t+2t^2}{2t(1-t)(1-2x+2x^2)}K(x,t) + \gamma_4 \frac{(1-2x+2x^2)(1-2t+2t^2)}{(1-t)(t+x-2xt)},$$

$$x = \frac{\sqrt{z}}{\sqrt{z}+\sqrt{1-z}}, \quad t = \frac{\sqrt{\zeta}}{\sqrt{\zeta}+\sqrt{1-\zeta}}.$$

Since $1-\gamma_4^2 \neq 0$, then equation (50) is a normal type equation. Its index is zero in the class $h(0)$, i.e. in the class of functions, which bounded as $z \to 0$ and unbounded as $z \to 1$.

Thus, we studied the solution $\omega(z)$ of singular integral equation (50) in the class $h(0)$.

We reduce the singular integral equation (50) by the well-known Carleman–Vekua regularization method [38] to an equivalent Fredholm equation of the second kind, the solvability of which implies from the uniqueness of the solution of problem PT.

From the equality $\tilde{T}(x) = x^{1-2\beta}T(x)$ and $\omega(z) = (1-2x+2x^2)\tilde{T}(x)$ we find the function $T(x)$, which is continuous in $(0,1)$ and integrable on $[0,1]$.

Substituting $T(x)$ into (24), we find $\tau(x)$. Then from (29) and (35) we find $\nu^{\pm}(x)$. When $\tau(x)$ is known function, then the solution of the problem PT for differential equation (3) in the domain D_1 we restore as a solution of the problem PT^+ for differential equation (3) with conditions (5) and (19). The solution of the problem PT for differential equation (3) in the domain D_2 we restore as a generalized solution of the Cauchy problem with conditions (19) and (20) for differential equation (3).

Thus, in the domain D a solution of the problem PT exists. Theorem 2 is proved. □

REFERENCES

1. F. I. Frankl, *Selected Works on Gas Dynamics* (Nauka, Moscow, 1973) [in Russian].
2. I. N. Vekua, *Generalized Analytic Functions* (Fizmatgiz, Moscow, 1959) [in Russian].
3. E. I. Moiseev and M. Mogimi, "On the completeness of eigenfunctions of the Neumann–Tricomi problem for a degenerate equation of mixed type," Differ. Equat. **41**, 1789–1791 (2005).
4. E. I. Moiseev and M. Mogimi, "On the completeness of eigenfunctions of the Tricomi problem for a degenerate equation of mixed type," Differ. Equat. **41**, 1462–1466 (2005).
5. E. I. Moiseev, T. E. Moiseev, and A. A. Kholomeeva, "On the non-uniqueness of the solution of the inner Neumann-Hellerstedt problem for the Lavrent'ev–Bitsadze equation," Sib. Zh. Chist. Prikl. Mat. **17** (3), 52–57 (2017).
6. A. B. Okboeb, "Tricomy problem for second kind parabolic-hyperbolic type equation," Lobachevskii J. Math. **41**, 58–70 (2020).
7. O. A. Repin and S. K. Kumykova, "A nonlocal problem for degenerate hyperbolic equation," Russ. Math. (Iz. VUZ) **61** (7), 43–48 (2017).
8. K. B. Sabitov, "On the theory of the Frankl problem for equations of mixed type," Russ. Math. (Iz. VUZ) **81** (1), 99–136 (2017).
9. K. B. Sabitov and V. A. Novikova, "Nonlocal A. A. Dezin's problem for Lavrent'ev–Bitsadze equation," Russ. Math. (Iz. VUZ) **60** (6), 52–62 (2016).
10. K. B. Sabitov and I. A. Khadzhi, "The boundary-value problem for the Lavrent'ev-Bitsadze equation with unknown right-hand side," Russ. Math. (Iz. VUZ) **55** (5), 35–42 (2011).
11. M. S. Salakhitdinov and B. Islomov, "Boundary value problems for an equation of mixed type with two inner lines of degeneracy," Dokl. Math. **43**, 235–238 (1991).
12. M. S. Salakhitdinov and M. Mirsaburov, "The Bitsadze-Samarskii problem for a class of degenerate hyperbolic equations," Differ. Equat. **38**), 288–293 (2002).
13. M. S. Salakhitdinov and A. K. Urinov, "Eigenvalue problems for a mixed-type equation with two singular coefficients," Sib. Math. J. **48**, 707–717 (2007).

14. M. S. Salakhitdinov and M. Mirsaburov, "A problem with a nonlocal boundary condition on the characteristic for a class of equations of mixed type," Math. Notes **86**, 704–715 (2009).
15. M. S. Salakhitdinov and B. I. Islomov, "A nonlocal boundary-value problem with conormal derivative for a mixed-type equation with two inner degeneration lines and various orders of degeneracy," Russ. Math. (Iz. VUZ) **55** (1), 42–49 (2011).
16. M. S. Salakhitdinov and N. B. Islamov, "Nonlocal boundary-value problem with Bitsadze–Samarskii condition for equation of parabolic-hyperbolic type of the second kind," Russ. Math. (Iz. VUZ) **59** (6), 34–42 (2015).
17. F. G. Tricomi, *Lezioni sulle equazioni a derivate parziali* (Gheroni, Torino, 1954).
18. A. V. Bitsadze, *Some Classes of Partial Differential Equations* (Nauka, Moscow, 1981) [in Russian].
19. M. M. Smirnov, *Equations of a Mixed Type* (Nauka, Moscow, 1970) [in Russian].
20. I. L. Karol, "On a boundary value problem for an equation of mixed elliptic-hyperbolic type," Dokl. Akad. Nauk SSSR **88**, 197–200 (1953).
21. N. K. Mamadaliev, "On representation of solution of the modified Cauchy problem," Sib. Math. J. **41**, 889–899 (2000).
22. S. A. Tersenov, "On the theory of hyperbolic equations with data on lines of degeneration type," Sib. Mat. Zh. **2**, 931–935 (1961).
23. S. S. Isamuhamedov and Zh. Oromov, "On boundary value problems for a mixed type equation of the second kind with a nonsmooth line of degeneration," Differ. Uravn. **18**, 324–334 (1982).
24. N. K. Mamadaliev, "Tricomi problem for strongly degenerate equations of parabolic-hyperbolic type," Math Notes **66**, 310–315 (1999).
25. R. S. Khairullin, *The Tricomi Problem for an Equation of the Second Kind with Strong Degeneracy* (Kazan. Gos. Univ., Kazan, 2015) [in Russian].
26. M. S. Salakhitdinov and N. B. Islamov, "A nonlocal boundary-value problem with the Bitsadze–Samarskii conditon for a parabolic-hyperbolic equation of the second kind," Russ. Math. (Iz. VUZ) **59** (6), 34–42 (2015).
27. K. B. Sabitov and I. P. Egorova, "On the correctness of boundary value problems with periodicity conditions for an equation of mixed type of the second kind," Vestn. Samar. Tekh. Univ., Ser. Fiz.-Mat. Nauki **23**, 430–451 (2019).
28. A. M. Nakhushev, "Loaded equations and their applications," Differ. Uravn. **19**, 86–94 (1983).
29. A. A. Samarsky, "On some problems of the theory of differential equations," Differ. Uravn. **16**, 1925–1935 (1980).
30. M. S. Salakhitdinov and T. G. Ergashev, "Integral representation of a generalized solution of the Cauchy problem in the class for one equation of hyperbolic type of the second kind," Uzb. Mat. Zh., No. 1, 67–75 (1995).
31. K. B. Sabitov and A. K. Suleimanova, "The Dirichlet problem for a mixed-type equation of the second kind in a rectangular domain," Russ. Math. (Iz. VUZ) **51** (4), 42–50 (2007).
32. K. B. Sabitov and A. K. Suleimanova, "The Dirichlet problem for a mixed-type equation with characteristic degeneration in a rectangular domain," Russ. Math. (Iz. VUZ) **53** (11), 37–45 (2009).
33. T. K. Yuldashev, "Nonlocal inverse problem for a pseudohyperbolic-pseudoelliptic type integro-differential equations," Axioms **9** (2), 45-1–21 (2020).
34. T. K. Yuldashev and B. J. Kadirkulov, "Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator," Axioms **9** (2), 68-1–19 (2020).
35. B. I. Islomov and A. A. Abdullayev, "On a problem for an elliptic type equation of the second kind with a conormal and integral condition," J. Nanosyst.: Phys. Chem. Math. **9**, 307–318 (2018).
36. M. S. Salakhitdinov and A. A. Abdullaev, "Nonlocal boundary value problem for a mixed type equation of the second kind," Dokl. Akad. Nauk Uzb., No. 1, 3–5 (2013).
37. M. M. Smirnov, *Equations of a Mixed Type* (Vysshaya Shkola, Moscow, 1985) [in Russian].
38. N. I. Muskhelishvili, *Singular Integral Equations* (Nauka, Moscow, 1968).