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Rates of Power Series Statistical Convergence of Positive Linear Operators and Power Series Statistical Convergence of *q*-Meyer–König and Zeller Operators

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Abstract—In this paper we compute the rates of convergence of power series statistical convergence of sequences of positive linear operators. We also investigate some Korovkin type approximation properties of the q-Meyer–König and Zeller operators and Durrmeyer variant of the q-Meyer–König and Zeller operators via power series statistical convergence. We show that the approximation results obtained in this paper expand some previous approximation results of the corresponding operators.

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1. INTRODUCTION

The classical Korovkin type approximation theory provides a simple criteria whether a sequence of a positive linear operators on $C[0,1]$, the space of all real continuous functions defined on the closed interval $[0,1]$, converges to the identity operator or not [25]. This theory has been studied by many mathematicians with various motivations such as extending the interval to whole real line, relaxing the continuity of the functions, relaxing the positivity of the operators or considering some special sequences of positive linear operators (see e.g., [1, 2]). Another motivation is to introduce the summability theory whenever a sequence of positive linear operator is not convergent in the ordinary sense. Actually, the main aim of the summability theory is to make a non-convergent sequence or series to converge in a more general sense. Therefore, summability theory has many applications in probability limit theorems, approximation theory with positive linear operators, and differential equations, whenever the ordinary limit does not exist (see, [5, 21, 22, 26]). Gadjiev and Orhan [21] proved a Korovkin type theorem by considering statistical convergence instead of ordinary convergence. Following that study many authors have given several approximation results via summability theory (see, e.g., [4, 8, 12, 13, 30, 33, 34, 42]).

Let $x=(x_j)$ be a real sequence and let $A=(a_{nj})$ be a summability matrix. If the sequence $\{(Ax)_n\}$ is convergent to a real number L, then we say that the sequence x is *A-summable* to the real number L, where the series$(Ax)_n := \sum_{j=0}^{\infty} a_{nj}x_j$ is convergent for any $n\in\mathbb{N}_0$ and $\mathbb{N}_0=\{0,1,...\}$. A summability matrix A is said to be *regular* if $\lim(Ax)_n = L$ whenever $\lim x = L$ (see, [9]). In fact, any summability method is said to be regular if it preserves ordinary limit.

Let $A=(a_{nj})$ be a non-negative regular summability matrix and let $E\subset\mathbb{N}$. Then the number

$$\delta_A(E) := \lim \sum_{j\in E} a_{nj}$$

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is said to be the *A-density* of E whenever the limit exists (see, [10, 18, 24]). Regularity of the summability matrix A ensures that $0 \leq \delta_A(E) \leq 1$ whenever $\delta_A(E)$ exists. If we consider $A = C$, the Cesàro matrix, then $\delta(E) := \delta_C(E)$ is called the (*natural* or *asymptotic*) *density* of E (see, [17]), where $C = (c_{nj})$ is the summability matrix defined by

$$c_{nj} = \begin{cases} 1/(n+1), & \text{if} \quad j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

A real sequence $x = (x_j)$ is said to be *A-statistically convergent* (see, [11, 19]) to a real number L if for any $\varepsilon > 0$,

$$\delta_A(\{j \in \mathbb{N}_0 : |x_j - L| \geq \varepsilon\}) = 0.$$

In this case we write $st_A - \lim x = L$. If we consider the Cesàro matrix, then C-statistical convergence is called *statistical convergence* [16, 31, 36]. In general, A-statistical convergence is regular and there exists some sequences which are A-statistically convergent but not ordinary convergent.

Now, we recall the concept of power series (summability) method [9]. Let (p_j) be a real sequence with $p_0 > 0$ and $p_1, p_2, ... \geq 0$, and such that the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L,$$

then we say that $x = (x_j)$ is convergent in the sense of power series method (P_p convergent). This summability method is a general version of Abel and Borel summability methods. Korovkin type theorems related to these methods can be found in [4, 6, 7, 37, 38, 41, 42].

The following theorem characterizes the regularity of a power series method.

Theorem 1 [9]. *A power series method P_p is regular if and only if for any $j \in \mathbb{N}_0$*

$$\lim_{0<t \to R^-} \frac{p_j t^j}{p(t)} = 0.$$

Ünver and Orhan [43] introduced the concept of power series statistical convergence which is stronger than the ordinary convergence. Power series statistical convergence is defined with the help of the concept of density with respect to the power series methods.

Definition 1 [43]. *Let P_p be a regular power series method and let $E \subset \mathbb{N}_0$. If*

$$\delta_{P_p}(E) := \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{j \in E}^{\infty} p_j t^j$$

exists then $\delta_{P_p}(E)$ is called the P_p-density of E. It is easy to see that $0 \leq \delta_{P_p}(E) \leq 1$ whenever $\delta_{P_p}(E)$ exists.

Definition 2 [43]. *Let $x = (x_j)$ be a sequence and let P_p be a regular power series method. Then x is said to be P_p-statistically convergent to L if for any $\varepsilon > 0$*

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{|x_j - L| \geq \varepsilon} p_j t^j = 0,$$

i.e., $\delta_{P_p}(\{j \in \mathbb{N}_0 : |x_j - L| \geq \varepsilon\}) = 0$. In this case we write $st_{P_p} - \lim x = L$.

The main goals of this paper are to give the rates of P_p-statistical convergence and to prove some approximation properties of q-Meyer–König and Zeller operators and Durrmeyer variant of the q-Meyer König and Zeller operators by considering P_p-statistical convergence. Both sequences of operators are constructed with the use of q- calculus.

Now, let us recall some notations from q-calculus [29]: for any fixed real number $q>0$, the q-integer $[j]$ is defined by

$$[j]:=[j]_q=\sum_{k=1}^{j}q^{k-1}=\begin{cases}\frac{1-q^j}{1-q}, & q\neq 1,\\ j, & q=1,\end{cases}$$

where j is a positive integer and $[0]=0$, the q-factorial $[j]!$ of $[j]$ is given with

$$[j]!:=\begin{cases}\prod_{k=1}^{j}[k], & j=1,2,...,\\ 1, & j=0.\end{cases}$$

For integers $j\geq r\geq 0$, the q-binomial coefficient is defined by

$$\begin{bmatrix} j\\ r\end{bmatrix}_q=\frac{[j]_q!}{[r]_q!\,[j-r]_q!}$$

and q-shifted factorial is defined by

$$(t;q)_j:=\begin{cases}1, & j=0,\\ \prod_{j=0}^{j-1}\left(1-tq^j\right), & j=1,2,...\end{cases}$$

Thomae [39] introduced the q-integral of function f defined on the interval $[0,a]$ as follows:

$$\int_0^a f(t)\,d_qt:=a(1-q)\sum_{j=0}^{\infty}f\left(aq^j\right)q^j,\quad 0<q<1.$$

Finally, the q-beta function [39] is defined by

$$B_q(m,j)=\int_0^1 t^{m-1}(qt;q)_{j-1}\,d_qt.$$

In [3], it can be found some different type operators which were constructed via q-calculus. In [40], q-Meyer–König and Zeller operators were defined by

$$M_j^q(f;x)=\begin{cases}\prod_{m=0}^{j}(1-q^mx)\sum_{k=0}^{\infty}f\left(\frac{[k]}{[j+k]}\right)\begin{bmatrix} j+k\\ k\end{bmatrix}x^k, & x\in[0,1),\\ f(1), & x=1\end{cases}\tag{1.1}$$

for $f\in C[0,1]$, $x\in[0,1]$, $j\in\mathbb{N}$ and $q\in(0,1]$. In [13], a different modification of q-Meyer–König and Zeller operators were defined. In [23], Durrmeyer variant of the q-Meyer–König and Zeller operators were introduced for $f\in C[0,1]$, $x\in[0,1]$, $j\in\mathbb{N}$ and $\alpha>0, q\in(0,1]$ as follows:

$$D_j^q(f;x)=\begin{cases}\sum_{k=1}^{\infty}M_j^q(x)\int_0^1\frac{1}{B_q(j,k)}t^{k-1}(qt;q)_{j-1}f(t)\,d_qt+m_{j,0,q}(x)f(0), & x\in[0,1)\\ f(1), & x=1,\end{cases}\tag{1.2}$$

where

$$m_{j,k,q}(x)=(x;q)_{j+1}\begin{bmatrix} j+k\\ k\end{bmatrix}x^k.$$

Statistical convergence of the operators (1.1) and (1.2) and different modifications of these operators were examined in [13, 14, 27, 32]. Using the Abel convergence some properties of the operators (1.1) and (1.2) were studied in [35]. In that work, the authors compare the conditions of Abel convergence results with the below classical conditions (1.3) which are necessary for ordinary convergence.

It is well known that if the classical conditions

$$\lim_{j\to\infty} q_j = 1 \text{ and } \lim_{j\to\infty} \frac{1}{[j]} = 0 \tag{1.3}$$

hold, then for each $f \in C[0,1]$ the sequences $(M_j^q f)$ and $(D_j^q f)$ converge uniformly to f over $[0,1]$ (see [23, 40]). We use the norm of the Banach space $B[0,1]$ defined for any $f \in C[0,1]$ by $||f|| := \sup_{0\le x\le 1} |f(x)|$, where $B[0,1]$ is the space of all bounded real functions defined over $[0,1]$.

We need the following known lemmas in our proofs:

Lemma 1 [40]. *Let $j \geq 3$ be a positive integer. Then the following hold for the operators (1.1):*

$$M_j^q (e_0; x) = 1, \tag{1.4}$$

$$M_j^q (e_1; x) = x, \tag{1.5}$$

$$x^2 \le M_j^q (e_2; x) \le \frac{x}{[j-1]} + x^2, \tag{1.6}$$

where $e_i(x) = x^i$ for $i = 0, 1, 2$.

Lemma 2 [23]. *Let $j \geq 3$ be a positive integer. Then the following hold for the operators (1.2):*

$$D_j^q (e_0; x) = 1, \tag{1.7}$$

$$D_j^q (e_1; x) = x, \tag{1.8}$$

$$D_j^q (e_2; x) = x^2 + \frac{[2]\, x (1-x) \left(1-q^j x\right)}{[j-1]} - E_{j,q}(x), \tag{1.9}$$

where

$$0 \le E_{j,q}(x) \le \frac{x\,[2]\,[3]\,q^{j-1}}{[j-1]\,[j-2]} (1-x)(1-qx)\left(1-q^j x\right).$$

For the operators (1.2) the following lemma was given in [35].

Lemma 3. *Let $j \geq 3$ be a positive integer. Then we have*

$$D_j^q (e_2; x) - x^2 \ge \frac{[2]\, x (1-x) \left(1-q^j x\right)}{[j-1]} - \frac{x\,[2]\,[3]\,q^{j-1}}{[j-1]\,[j-2]} (1-x)(1-qx)\left(1-q^j x\right) \ge 0.$$

Note that in Lemma 1 and Lemma 2 the results were obtained for $j \geq 3$. Like ordinary convergence and statistical convergence, an exclusion of a finite number of terms does not affect the P_p-statistical convergence of a sequence since the P_p-density of a finite set is equal to zero.

The remainder of this paper is organized as follows: In Section 2, we study the rates of P_p-statistical convergence by means of modulus of continuity and class of Lipschitz functions. Furthermore, we give some examples to prove the results obtained in this paper are stronger than some previous ones. In Section 3 and Section 4, we study some Korovkin type approximation properties of q-Meyer–König and Zeller operators and Durrmeyer variant of q-Meyer–König and Zeller operators via P_p-statistical convergence, respectively.

2. RATES OF THE P_p-STATISTICAL CONVERGENCE

In this section, we compute the rates of the P_p-statistical convergence of the sequences of positive linear operators by means of the modulus of continuity and the elements of the Lipschitz class. Some results related to the rate of the convergence may be found in [15, 20, 28].

Let (L_j) be a sequence of positive linear operators from $C[0,1]$ into $B[0,1]$. The modulus of continuity of $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[0,1]}} |f(x)-f(y)|.$$

It is well known that, for any $f\in C[0,1]$,

$$\lim_{\delta\to 0^+}\omega(f,\delta)=0 \tag{2.1}$$

and for any $\delta>0$

$$|f(x)-f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right) \tag{2.2}$$

and for all $c>0$

$$\omega(f,c\delta)\leq(1+\lfloor c\rfloor)\omega(f,\delta),$$

where $\lfloor c\rfloor$ is the greatest integer less than or equal to c. Now we are ready to give the following theorem.

Theorem 2. *Let (L_j) be a sequence of positive linear operators from $C[0,1]$ into $B[0,1]$. If*

(i) $st_{P_p}\text{-}\lim\|L_je_0-1\|=0$,

(ii) $st_{P_p}\text{-}\lim\omega(f,\varphi_j(t))=0$ *for any t*

then, for any $f\in C[0,1]$ we have $st_{P_p}\text{-}\lim\|L_jf-f\|=0$, where

$$\varphi_j(t) := \left(\sup_{0\leq x\leq 1} L_j\left((t-x)^2;x\right)\right)^{\frac{1}{2}}. \tag{2.3}$$

Proof. Using (2.2), for any $f\in C[0,1]$, any $x,t\in[a,b]$ and any $\delta>0$ we can write

$$|L_j(f;x)-f(x)| \leq L_j(|f(t)-f(x)|;x)+|f(x)|\,|L_j(e_0)-1)| \leq L_j\left(\left(1+\left\lfloor\frac{|t-x|}{\delta}\right\rfloor\right)\omega(f,\delta);x\right)$$
$$+|f(x)|\,|L_j(e_0)-1)| \leq \omega(f,\delta)L_j(e_0)+\frac{1}{\delta^2}\omega(f,\delta)L_j\left((t-x)^2;x\right)+|f(x)|\,|L_j(e_0)-1)|.$$

Thus, we obtain

$$|L_j(f;x)-f(x)| \leq \omega(f;\delta)L_j(e_0)+\frac{1}{\delta^2}\omega(f;\delta)L_j\left((t-x)^2;x\right)+|f(x)|\,|L_j(e_0)-1)|.$$

Now, if we take $\delta=\left\{\sup_{0\leq x\leq 1}L_j\left((t-x)^2;x\right)\right\}^{\frac{1}{2}}$, then we get $0\leq\|L_jf-f\|\leq 2\omega(f,\varphi_j(t))+\omega(f,\varphi_j(t))\,|L_j(e_0)-1)|+C\,|L_j(e_0)-1)|$, where $C=\|f\|$. From (ii), we have $st_{P_p}\text{-}\lim\|L_jf-f\|=0$ which completes the proof. □

Now, we give the rate of P_p-statistical convergence with the help of Lipschitz class $Lip_M(\alpha)$ where $0<\alpha\leq 1$, $M>0$. Recall that a function belongs to $Lip_M(\alpha)$ if for any $t,x\in[0,1]$

$$|f(t)-f(x)|\leq M|t-x|^\alpha.$$

We have the following theorem.

Theorem 3. *Let (L_j) be a sequence of positive linear operators from $C[0,1]$ into $B[0,1]$. If*

(i) $st_{P_p}\text{-}\lim\|L_je_0-1\|=0$,

(ii) $st_{P_p}\text{-}\lim\varphi_j(t)=0$ *for any t then, for any $f\in Lip_M(\alpha)$ such that $0<\alpha\leq 1$, $M\in\mathbb{R}^+$, we have* $st_{P_p}\text{-}\lim\|L_jf-f\|=0$.

Proof. Since $f \in Lip_M(\alpha)$, we can write from linearity

$$|L_j(f;x) - f(x)| \leq L_j(|f(t) - f(x)|;x) + |f(x)|\,|L_j(e_0) - 1)|$$
$$\leq ML_j(|t-x|^{\alpha};x) + |f(x)|\,|L_j(e_0) - 1)|.$$

By applying the Hölder inequality, we get

$$||L_jf - f|| \leq M(\varphi_j(t))^{\alpha}\left|(L_j(e_0))^{\frac{2-\alpha}{\alpha}} - 1\right| + M(\varphi_j(t))^{\alpha} + C\,|L_j(e_0) - 1|,$$

from (i) and (ii) we have $st_{P_p} - \lim ||L_jf - f|| = 0$, which ends the proof. □

3. POWER SERIES STATISTICAL CONVERGENCE OF THE SEQUENCE (M_j^q)

In this section, using power series statistical convergence, we obtain Korovkin type approximation of the operators $\left(M_j^q\right)$ defined with (1.1). Throughout this section, we deal with the sequence (q_j) such that $0 < q_j \leq 1$, $q_0 = 0$ and we assume that $M_0^q f = 0$ for any $f \in C[0,1]$.

Now, we recall the following Korovkin type P_p-statistical approximation theorem which is given in [43].

Theorem 4. *Let P_p be regular power series method and let (L_j) be a sequence of linear positive operators on $C[0,1]$ such that for $i = 0,1,2$*

$$st_{P_p}\text{-}\lim ||L_je_i - e_i|| = 0, \tag{3.1}$$

then for any $f \in C[0,1]$ we have $st_{P_p}\text{-}\lim ||(L_jf - f)|| = 0$.

We are ready to prove the following Korovkin type P_p-statistical approximation theorem:

Theorem 5. *Assume that P_p is a regular power series method. If $st_{P_p}\text{-}\lim q_j = 1$ and $st_{P_p}\text{-}\lim \dfrac{1}{[j-1]} = 0$, then for each $f \in C[0,1]$ we have $st_{P_p}\text{-}\lim \left|\left|M_j^q f - f\right|\right| = 0$.*

Proof. From Theorem 4, it is enough to demonstrate that (3.1) holds for (M_j^q). Now, considering Lemma 1, we get for $i = 0,1$ that $st_{P_p}\text{-}\lim \left|\left|M_j^q e_i - e_i\right|\right| = 0$. Moreover, using (1.6), we have for $j \geq 3$ that

$$0 \leq M_j^q(e_2;x) - x^2 \leq \frac{x}{[j-1]}.$$

Here, let us define the following sets for any $\varepsilon > 0$,

$$N := \left\{j \in \mathbb{N} : \left|M_j^q(e_2;x) - x^2\right| \geq \varepsilon\right\}, \quad N_1 := \left\{j \in \mathbb{N} : \frac{1}{[j-1]} \geq \varepsilon\right\}.$$

It is obvious that $N \subset N_1$ which implies with the the hypothesis that

$$0 \leq \delta_{P_p}\left(\left\{j \in \mathbb{N} : \left|\left|M_j^q e_2 - e_2\right|\right|_{C_B} \geq \varepsilon\right\}\right) \leq \delta_{P_p}\left\{j \in \mathbb{N} : \frac{1}{[j-1]} \geq \varepsilon\right\} = 0.$$

Hence, we obtain

$$st_{P_p}\text{-}\lim \left|\left|\left(M_j^q e_2 - e_2\right)\right|\right| = 0.$$

So, the proof is completed. □

The following remark shows that the conditions of Theorem 5 are weaker than the classical conditions:

Remark 1. *It is not difficult to see that the classical conditions (1.3) entail that the sequence $\left(\dfrac{1}{[j-1]}\right)_{j=3}^{\infty}$ is P_p-statistically convergent to zero. Conversely, we assume that P_p is the power series method with (p_j)*

$$p_j := \begin{cases} 0, & j = 2k \\ 1, & j = 2k+1 \end{cases}$$

and we consider the sequence (q_j)

$$q_j := \begin{cases} 0, & j = 2k \\ 1, & j = 2k+1 \end{cases}$$

for some non-negative integer k. Note that (q_j) *does not satisfy the classical conditions. Besides, we have for any* $j \geq 3$ *that*

$$\frac{1}{[j-1]} = \begin{cases} 1, & j = 2k \\ \dfrac{1}{j-1}, & j = 2k+1. \end{cases}$$

Thus, the sequence $\left(\frac{1}{[j-1]}\right)_{j=3}^{\infty}$ *is* P_p*-statistically convergent to zero.*

The rate of P_p-statistical convergence by means of modulus of continuity for the operators $\left(M_j^q\right)$ is given in the following

Theorem 6. *If* $0 < q_j \leq 1$*,* $st_{P_p}\text{-}\lim q_j = 1$ *and* $st_{P_p}\text{-}\lim \frac{1}{[j-1]} = 0$*, then for any* $f \in C[0,1]$ *we have* $\left|\left|M_j^q(f) - f\right|\right| \leq 2\omega(f, \alpha_j)$*, where* $\alpha_j = \frac{1}{[j-1]}$.

4. POWER SERIES STATISTICAL CONVERGENCE OF THE OPERATORS $\left(D_j^q\right)$

In this section, we study the Korovkin type approximation of the operators $\left(D_j^q\right)$ defined with (1.2) by considering the P_p-statistical convergence. Throughout this section, we deal with the sequence (q_j) such that $0 < q_j < 1$, $q_0 = 0$ and we assume that $D_0^q f = 0$ for any $f \in C[0,1]$.

Theorem 7. *If the sequence* $st_{P_p}\text{-}\lim q_j = 1$ *and* $st_{P_p}\text{-}\lim \dfrac{1}{[j-1]} = 0$ *then for each* $f \in C[0,1]$ *we have*

$$st_{P_p}\text{-}\lim \left|\left|D_j^q f - f\right|\right| = 0.$$

Proof. From Theorem 4, it sufficies to show that (3.1) holds for (D_j^q). Using (1.7) and (1.8), we obtain for $i = 0, 1$ that $st_{P_p} - \lim \left|\left|D_j^q e_i - e_i\right|\right| = 0$. On the other hand from (1.9) and Lemma 3 we have for any $j \geq 3$ that

$$\frac{[2]\, x(1-x)\left(1-q^j x\right)}{[j-1]} - \frac{x\,[2]\,[3]\,q^{j-1}}{[j-1]\,[j-2]}(1-x)(1-qx)\left(1-q^j x\right)$$
$$\leq D_j^q(e_2; x) - x^2 \leq \frac{[2]\, x(1-x)\left(1-q^j x\right)}{[j-1]},$$

which implies

$$0 \leq \left|\left|D_j^q e_2 - e_2\right|\right| \leq \sup_{0 \leq x \leq 1}\left(\frac{[2]\, x(1-x)\left(1-q^j x\right)}{[j-1]}\right) \leq \frac{[2]}{[j-1]}.$$

Now, let us define the following sets for any $\varepsilon > 0$,

$$M := \left\{j \in \mathbb{N} : \left|M_j^q(e_2; x) - x^2\right| \geq \varepsilon\right\} \quad M_1 := \left\{j \in \mathbb{N} : \frac{2}{[j-1]} \geq \varepsilon\right\}.$$

It is obvious that $M \subset M_1$ which implies with the hypothesis that

$$0 \leq \delta_{P_p}\left(\left\{j \in \mathbb{N} : \left|\left|M_j^q e_2 - e_2\right|\right|_{C_B} \geq \varepsilon\right\}\right) \leq \delta_{P_p}\left\{j \in \mathbb{N} : \frac{[2]}{[j-1]} \geq \varepsilon\right\} = 0.$$

Therefore, we obtain st_{P_p}- $\lim \left|\left|D_j^q e_2 - e_2\right|\right| = 0$. Hence, the proof ends. □

Following remark shows that the condition of Theorem 7 is weaker than the classical conditions (1.3):

Remark 2. *Note that if the classical conditions (1.3) hold then condition of Theorem 7 holds. In fact, if* $\lim_{j\to\infty} q_j = 1$ *and* $\lim_{j\to\infty} \frac{1}{[j]} = 0$ *then we have*

$$\lim_{j\to\infty} \frac{1}{[j-1]} = \lim_{j\to\infty} \frac{1}{[j] - q_j^{j-1}} = 0.$$

Therefore, it is P_p-statistically convergent to zero. Conversely, we assume that P_p is the power series method with (p_j)

$$p_j := \begin{cases} 0, & j = k^2, \\ 1, & otherwise \end{cases}$$

and we consider the sequence (q_j) given by

$$q_j := \begin{cases} 0, & j = k^2, \\ 1 - \frac{1}{j}, & otherwise \end{cases}$$

for some non-negative integer k. We see that (q_j) does not satisfy the conditions of classical Korovkin theorem. On the other hand, we have for any $j \geq 2$ that

$$\frac{1 - q_j}{1 - q_j^{j-1}} = \begin{cases} 1, & j = k^2, \\ \frac{\frac{1}{j}}{1 - \left(1 - \frac{1}{j}\right)^{j-1}}, & otherwise. \end{cases}$$

Thus, the sequence $\left(\frac{1}{[j-1]}\right)_{j=2}^{\infty}$ is P_p-statistically convergent to zero.

The following theorem which gives the rate of the P_p-statistical convergence by means of modulus of continuity for (D_j^q) can be proved easily.

Theorem 8. *If $0 < q_j \leq 1$, st_{P_p}-$\lim q_j = 1$ and st_{P_p}-$\lim \dfrac{1}{[j-1]} = 0$, then for any $f \in C[0,1]$ we have* $\left|\left|D_j^q(f) - f\right|\right| \leq 2\omega(f, \beta_j)$, *where* $\beta_j = \frac{[2]}{[j-1]}$.

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