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Refined Analysis of Piezoelectric Microcantilevers in Gradient Electroelasticity Theory

Y. Solyaev[1,2*], A. Ustenko[1], and E. Lykosova[1***]**

(Submitted by A. V. Lapin)

[1] *Institute of Applied Mechanics of Russian Academy of Sciences, Moscow, 125040 Russia*

[2] *Federal Research Center "Computer Sciences and Control" of Russian Academy of Sciences, Moscow, 119333 Russia*

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Abstract—In this paper we present analytical solutions for the cantilever beam bending problems obtained in the non-classical electroelasticity theory with strain and electric field gradient effects. We show that considered model allows to provide the refined analysis for the electric field distribution around the supproted end of the cantilever taking into account the extended number of boundary conditions, which cannot be captured in classical models.

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1. INTRODUCTION

Piezoelectric microcantilevers are the typical elements of sensor devices that are widely used in calorimetry, chemistry, medicine etc. [1–4]. Large amount of analytical and numerical models of such structures have been developed, which review can be found in [5–7]. In the present paper we consider the non-classical model of piezoelectric cantilevers in the framework of second gradient electroelasticity theory, that is the extended variant of theory of dielectrics with spatial dispersion [6, 8]. Several 2D and 3D problems in this theory were considered recently in [9–13], while its variational formulation and some basic theorems were established in [14].

Beam theory utilized in the present study for the analysis of piezoelectric cantilevers was developed in [15]. This theory is related to the class of gradient Euler–Bernoulli models with so-called "uniaxial stress state" [16]. This theory was derived based on common Euler–Bernoulli hypothesis introduced in strain-electric field gradient theory and providing the corrected variational formulation. Area of application of this theory is related to the description of the small sized electroelastic beams, which dimensions are comparable to the materials characteristic length. This range of dimensions in ideal crystalline materials may have the order of interatomic distances [17, 18], however in composite structures and in architectured metamaterials characteristic length parameters may have larger values "— of several unit cells and more [19, 20]. From phenomenological point of view, gradient beam models can be also used for the refined analysis of macro-sized structures in the local zones around supports and concentrated loads [21]. In this work we show an example of such analysis of piezoelectric cantilever beam under end point load with different type of non-classical elastic and electric boundary conditions, which physical meaning is discussed.

*E-mail: yos@iam.ras.ru
**E-mail: audlord@mail.ru
***E-mail: elykosova@mail.ru

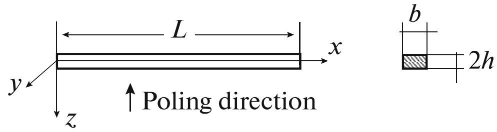

Fig. 1. Electroelastic beam.

2. ELECTROELASTIC GRADIENT BEAM MODEL

Let us consider an electroelastic beam with rectangular cross-section (Fig. 1). The beam is poled along the thickness direction, so that it exhibits the material symmetry of a hexagonal crystal in class 6 mm—transversely isotropic about z-axis. Electric enthalpy density of the beam depends on strain, strain gradients, electric field and electric field gradients as follows [9, 14, 15]:

$$\bar{H}(\varepsilon_{ij}, \varepsilon_{ij,k}, E_i, E_{i,j}) = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}\varepsilon_{ij}E_k - \frac{1}{2}\epsilon_{ij}E_iE_j + \frac{1}{2}A_{ijklmn}\varepsilon_{ij,k}\varepsilon_{lm,n} - \frac{1}{2}\alpha_{ijkl}E_{i,j}E_{k,l}, \tag{1}$$

where C_{ijkl} and A_{ijklmn} are the fourth- and sixth-order tensors of the elastic moduli; $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ is an infinitesimal strain tensor; u_i are the displacements; e_{ijk} is the third-order tensor of piezoelectric moduli; ϵ_{ij} and α_{ijkl} are the second- and the fourth-order tensors of dielectric permittivity constants; $E_i = -\phi_{,i}$ is the electric field vector, ϕ is the electric potential; the comma denotes the differentiation with respect to spatial variables and repeated indices imply summation.

Note, that in (1) we neglect the flexoelectric and high-order coupling effects that generally can persist in considered model. We adopt these simplifications to demonstrate the principal aspects of analytical solutions which arise in piezoelectric gradient models.

Constitutive equations for stress, double stress, electric displacement and electric quadrupole are the following:

$$\tau_{ij} = \frac{\partial \bar{H}}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad \mu_{ijk} = \frac{\partial \bar{U}}{\partial \varepsilon_{ij,k}} = A_{ijklmn}\varepsilon_{lm,n}, \tag{2}$$

$$D_i = -\frac{\partial \bar{H}}{\partial E_i} = e_{ijk}\varepsilon_{jk} + \epsilon_{ij}E_j, \quad Q_{ij} = -\frac{\partial \bar{H}}{\partial E_{i,j}} = \alpha_{ijkl}E_{k,l}. \tag{3}$$

Derivation of the beam model will be shortly presented here and one can find the additional discussion in [15]. Thus, we consider the Bernoulli–Euler hypotheses for the displacements together with parabolic low for the electric potential that allow to take into account electromechanical coupling in the case of elementary flexure of slender beam without shear deformations [5, 6]:

$$u_1 = -zw'(x), \quad u_2 = 0, \quad u_3 = w(x), \quad \phi = z^2\varphi(x).$$

Strain, electric field and their gradients are the following:

$$\varepsilon_{11} = -zw'', \quad \varepsilon_{11,1} = -zw''', \quad \varepsilon_{11,3} = -w'',$$
$$E_1 = -z^2\varphi', \quad E_3 = -2z\varphi, \quad E_{1,1} = -z^2\varphi'', \quad E_{1,3} = E_{3,1} = -2z\varphi', \quad E_{3,3} = -2\varphi. \tag{4}$$

Cauchy stress, double stress, electric displacement and electric quadrupole follow from (4), (2), (3):

$$\tau_{11} = C_{1111}\varepsilon_{11} - e_{311}E_3 = -Ezw'' + 2e_{31}z\varphi,$$
$$\mu_{111} = A_{111111}\varepsilon_{11,1} = -\ell^2 Ezw''', \quad \mu_{113} = A_{113113}\varepsilon_{11,3} = -\ell^2 Ew'', \tag{5}$$

$$D_1 = \epsilon_{11}E_1 = -\epsilon_1 z^2\varphi', \quad D_3 = e_{311}\varepsilon_{11} + \epsilon_{33}E_3 = -e_{31}zw'' - 2\epsilon_3 z\varphi,$$
$$Q_{11} = \alpha_{1111}E_{1,1} = -\ell^2\epsilon_1 z^2\varphi'', \quad Q_{13} = Q_{31} = \alpha_{1313}E_{1,3} = -2\ell^2\epsilon_1 z\varphi',$$
$$Q_{33} = \alpha_{3333}E_{3,3} = -2\ell^2\epsilon_3\varphi, \tag{6}$$

where we use Voigt notations $e_{311} = e_{31}$, $\epsilon_{11} = \epsilon_1$, $\epsilon_{33} = \epsilon_3$, take into account that $C_{1111} = E$, $A_{111113} = A_{113111} = 0$, $\alpha_{1113} = \alpha_{1131} = \alpha_{1311} = \alpha_{3111} = 0$ in the theory of orthotropic gradient beam and assume for simplicity that $A_{111111} = A_{113113} = \ell^2 E$, $\alpha_{1111} = \alpha_{1313} = \alpha_{3131} = \ell^2\epsilon_1$ and $\alpha_{3333} = \ell^2\epsilon_3$; ℓ is the materials characterstic length, which is assumed to be the same in the elastic and electric parts of the problem.

Based on (1), (4)–(6) the total electric enthalpy of the beam can be expressed in the form

$$H = \frac{b}{2}\int_0^L\int_{-h}^{h}(\tau_{11}\varepsilon_{11} + \mu_{111}\varepsilon_{11,1} + \mu_{113}\varepsilon_{11,3} + D_1E_1 + D_3E_3 + Q_{11}E_{1,1} + 2Q_{13}E_{1,3} + Q_{33}E_{3,3})dzdx. \tag{7}$$

Using appropriate simplifications and taking into account boundary conditions on the top and bottom surfaces of the beam [15], variation of (7) can be presented as

$$\delta H = b\int_0^L\int_{-h}^{h}(\tau_{11}\delta\varepsilon_{11} + \mu_{111}\delta\varepsilon_{11,1} + (D_1 - 2Q_{13,3})\delta E_1 + D_3\delta E_3 + Q_{11}\delta E_{1,1})dzdx. \tag{8}$$

Substitution of (4), (5), (6) into (8) and definition of mechanical and electric resultants provide us the representation

$$\delta H = \int_0^L(-M\delta w'' - M_h\delta w''' - P_1\delta\varphi' - P_3\delta\varphi - P_h\delta\varphi'')dx,$$

where we introduce the bending moment M, gradient bending moment M_h, axial polarization resultant P_1, through-thickness polarization resultant P_3 and resultant for gradient of axial polarization P_h (resultant for quadrupole) as follows:

$$M = b\int_{-h}^{h} z\tau_{11}dz = -EIw'' + 2e_{31}I\varphi, \quad M_h = b\int_{-h}^{h} z\mu_{111}dz = -\ell^2 EIw''',$$

$$P_1 = b\int_{-h}^{h} z^2\left(D_1 - 2Q_{13,3}\right)dz = -\epsilon_1(J + 4\ell^2 I)\varphi',$$

$$P_3 = 2b\int_{-h}^{h} zD_3dz = -2e_{31}Iw'' - 4\epsilon_3 I\varphi, \quad P_h = b\int_{-h}^{h} z^2Q_{11}dz = -\ell^2\epsilon_1 J\varphi'', \tag{9}$$

where $I = 2bh^3/3$ and $J = 2bh^5/5$ are the classical and high order cross section moments of inertia.

In the case of mechanical loading, the work done by the external distributed transversal load $q(x)$ is given by

$$W = b\int_0^L q(x)w(x)dx.$$

Boundary value problem statement can be derived from the following variational equation:

$$\delta L = \delta W - \delta H = 0, \quad L = W - H.$$

As result, one can obtain the following system of governing equations

$$\begin{cases} M'' - M_h''' + q = 0, \\ P_1' - P_h'' - P_3 = 0, \end{cases} \qquad x \in (0, L), \tag{10}$$

and natural and essential boundary conditions that should be defined at the beam ends $x=0,L$ with respect to

$$\begin{cases} M_h \quad \text{or} \quad w'', \quad M-M_h' \quad \text{or} \quad w', \quad M'-M_h'' \quad \text{or} \quad w, \\ P_h \quad \text{or} \quad \varphi', \quad P_1-P_h' \quad \text{or} \quad \varphi. \end{cases} \tag{11}$$

One can see that in presented high-order theory there exist an extended set of boundary conditions, that allow to prescribe, e.g. the curvature of the beam (w'') at its end, or the electric field ($-\varphi'$). These additional boundary conditions should not be treated as some artifact of gradient model, but can be used for the refined analysis of the beams state around supports and loaded areas [21].

Governing equations in terms of deflections and electric potential can be derived by substituting (9) into (10):

$$\begin{cases} \ell^2 EIw^{VI} - EIw^{IV} + 2e_{31}I\varphi'' = 0, \\ \ell^2\epsilon_1 J\varphi^{IV} - \epsilon_1(J+4\ell^2 I)\varphi'' + 4\epsilon_3 I\varphi + 2e_{31}Iw'' = 0. \end{cases} \tag{12}$$

From the second equation in (12) we can find the representation for the second gradient of deflections

$$w'' = -\ell^2\frac{\epsilon_1 J}{2e_{31}I}\varphi^{IV} + \frac{\epsilon_1(J+4\ell^2 I)}{2e_{31}I}\varphi'' - \frac{2\epsilon_3}{e_{31}}\varphi \tag{13}$$

and from the first equation in (12) we by obtain then

$$\ell^4\varphi^{VIII} - 2\ell^2\left(1+\frac{10}{3}\frac{\ell^2}{h^2}\right)\varphi^{VI} + \left(1+\frac{20}{3}\frac{\ell^2}{h^2}(1+\bar{\epsilon})\right)\varphi^{IV} - \frac{20}{3h^2}(\bar{\epsilon}+K^2)\varphi'' = 0,$$

where $K = e_{31}^2/(\epsilon_1 E)$ is the electromechanical coupling factor of the beam and $\bar{\epsilon} = \epsilon_3/\epsilon_1$. As result, general solution for the electric potential can be found in the following form

$$\varphi = \sum_{i=1}^{3} C_i e^{x\sqrt{\lambda_i}} + \sum_{i=4}^{6} C_i e^{-x\sqrt{\lambda_i}} + C_7 + C_8 x, \tag{14}$$

where C_i $(i=1...8)$ are the unknown constants to be determined from boundary conditions and λ_i are the roots of the following third-order polynomial equation

$$\frac{20}{h^2\ell^4}(\bar{\epsilon}+K^2) - \left(\frac{3}{\ell^4} + \frac{20}{h^2\ell^2}(\bar{\epsilon}+1)\right)\lambda + \left(\frac{10}{h^2}+\frac{3}{\ell^2}\right)\lambda^2 - 3\lambda^3 = 0.$$

General solution for the displacements can be found then by using (13), taking into account that integration results in additional two constants, such that total number of needed boundary conditions will be 10.

3. BENDING OF PIEZOELECTRIC CANTILEVER BEAM

Let us consider now the problem of cantilever beam bending by introducing the appropriate boundary conditions for the clamped end at $x=0$ and vertical end point force end at $x=L$. In this case the following sets of boundary conditions (11) can be used:

1. Left end—clamped with zero curvature and grounded with zero axial electric field; right end—unelectroded with prescribed end point force conditions:

$$\begin{cases} x=0: \quad w''=0, \quad w'=0, \quad w=0, \quad \varphi'=0, \quad \varphi=0; \\ x=L: \quad M_h=0, \quad M-M_h'=0, \quad M'-M_h''=F, \quad P_h=0, \quad P_1-P_h'=0. \end{cases} \tag{15}$$

This type of boundary conditions can be specified if the left end of the beam is fixed in the rigid metallic grips, which are grounded and provide the constraints for the beams deflections, rotations and change of curvature. Note that conditions for the curvature and electric field are non-classical and cannot be specified in classical models of electroelastic beams, while the large enough fixed zone in real supportes of the cantilevers may provide such a state described by (15).

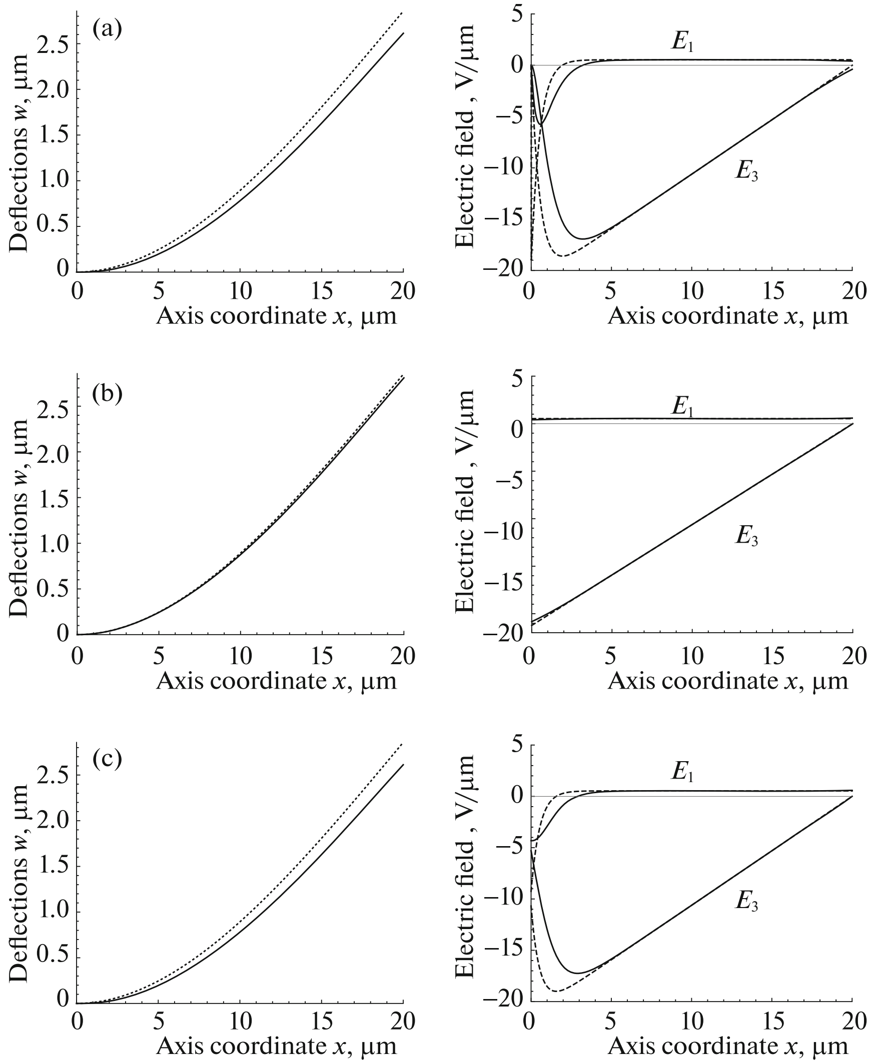

Fig. 2. Deflections and electric field components in the beams with different boundary conditions: (a) Eq. (15), (b) Eq. (16), (c) Eq. (17). Solid lines—gradient solution ($\ell/h = 0.5$), dotted lines—classical solution ($\ell = 0$).

2. Left end—clamped without curvature constraints and unelectroded, right end—grounded with prescribed mechanical force:

$$\begin{cases} x = 0: \quad M_h = 0, \quad w' = 0, \quad w = 0, \quad P_h = 0, \quad P_1 - P_h' = 0, \\ x = L: \quad M_h = 0, \quad M - M_h' = 0, \quad M' - M_h'' = F, \quad P_h = 0, \quad \varphi = 0. \end{cases} \tag{16}$$

This type of boundary conditions can be specified if the left end of the beam is fixed in the unelectroded grips, which provide the constraints for the beams deflections and rotations only. The difference with previous conditions here is that the constrained zone is rather small and curvature of the beam can change under loading even inside the grips. On the right end, the beam is contacted, e.g., with grounded substrate.

3. Left end—clamped with zero curvature and unelectroded, right end—grounded with prescribed force:

$$\begin{cases} x = 0: \quad w'' = 0, \quad w' = 0, \quad w = 0, \quad P_h = 0, \quad P_1 - P_h' = 0; \\ x = L: \quad M_h = 0, \quad M - M_h' = 0, \quad M' - M_h'' = F, \quad P_h = 0, \quad \varphi = 0. \end{cases} \tag{17}$$

This type of boundary conditions is intermediate between previous two. Here we have large fixed zone at the left end of the beam (thought it is modeled as a point condition) and grounded and mechanically loaded right end.

Each of three presented cases of boundary conditions can be used to determine the unknown ten constants in general solutions (14) and (13) by their substitution into (15), (16) or (17) and solution of resulting systems of ten algebraic equations. Resulting solutions can be found analytically, however, they have rather complicated form and do not presented here.

4. NUMERICAL EXAMPLES

Examples are given for the beams made of piezoelectric ceramics PZT-7A with material constants $E = 140$ GPa, $e_{31} = -2.1$ C/m^2, $\epsilon_1 = 460\epsilon_0$, $\epsilon_3 = 235\epsilon_0$, where $\epsilon_0 = 8.885 \times 10^{-12}$ F/m. Point force equals to $F = 0.1$ mN. Beams dimensions are $L = 20\mu$m, $b = h = 1\mu$m. Numerical solutions were found by using Wolfram Mathematica package and presented in Fig. 2.

One can see that the variants of boundary conditions (15) and (17) (Fig. 2a,c) with restricted electric field and curvature conditions results in rather complicated state of the beam polarization near its supported left end. In these cases gradient effects play significant role and change the length of zone affected by the boundary constraints in electric field and even slightly change the deflections of the beam. Solution with "soft" elastic and unelectroded electric support (16) results in almost classical behavior of the beam without strong gradient effects evenfor rather large ratio between characteristic length and thickness of the beam $\ell/h = 0.5$ (see Fig. 2b).

5. CONCLUSIONS

Presented solutions shown the main features that arise in the static problems in the theory of gradient electroelastic beams. This theory can describe the influence of non-classical effects that arise near the constrained areas of the beam, and, generally say, in the areas of strain and electric field concentration. These effects may be important in the case of precise measurements, when even the small noises from supports may affected the quality of the obtained results and should be avoided in the analysis.

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