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An Approach to Experimental Computation of an Anisotropic Viscoelastic Plate Stiffnesses

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Abstract—The viscoelastic behavior of anisotropic composite is studied in this paper. Constitutive relations and equilibrium equations are derived for a Kirchhoff plate using general linear viscoelasticity constitutive relations for the anisotropic case. The derived model parameters—plate stiffnesses—are experimental functions. An approach to these parameters identification is given for certain cases of material properties.

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1. INTRODUCTION

Viscoelastic properties study is one of the most dynamically developing branches of material science, as well as composite mechanics. For applications, some simplified models like rods, plates, and shells are significant.

This paper comes to continue papers [1–6], dedicated to the construction of composite mechanics models of nonhomogeneous anisotropic rods and plates. Two results are presented there: following the approach stated in [5] a plate model is derived for time-dependent material properties, as a generalization of the elastic case; and an approach to the said model identification method.

The method of model construction is based on classical for engineering mechanics approach of reducing three-dimensional continuum equilibrium equations into two-dimensional field equations. The considered reduction is based on kinematical and statical hypotheses, but there are also different methods like the power series expansion, asymptotic methods, etc, some of them are illustrated in [7–12]. Such approaches, first arising in linear elasticity tasks, were successfully generalized for an anisotropic, nonhomogeneous case, as well as for isotropic viscoelastic case. This paper presents a version of anisotropic viscoelastic plates theory—an isotropic one is enough for some thin structures like foams [16–18], but an increase of thickness leads to the significant influence of anisotropy. A noticeable difficulty of anisotropic viscoelasticity constitutive relations usage lies in asymmetric modulus tensor [19], which does not occur in the isotropic case, but still allows using Kirchhoff hypothesis [13]. In accordance with an approach [20] of reducing identification problem into a least-squares one a set of experimental tests, as well as a data processing method, were supplied by one of the authors for an elastic plate, this paper is conscripted to suggest a solution for a linear viscoelastic plate.

A classical coordination of a plate is used; small Latin indexes i, j are supposed to $\in [1..3]$, big Latin indexes $I, J \in [1..2]$, a comma represents partial spatial derivative $a_{,j} = \frac{\partial a}{\partial x_j}$, a dot is derivative with respect to time.

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2. MECHANICAL MODEL

First of all, the mechanical models for kinematics and statics and constitutive relations for this plate are introduced.

2.1. Kinematical and Statical Relations of Thin Plate

The thickness (h) of a plate (Π) is assumed to be significantly less than linear sizes of Π (we also suppose Π is rectangular in a plane, so $\Pi = a * b * h$), $h/\min\{a, b\} < 0.1$. This assumption allows to name plate a thin one, and also to use the technique of homogenization concerning the basic plane. Usually, the median plane, equidistant for facial surfaces, is used as a basic one; this paper also follows this approach.

The basic idea of homogenization is tied with the kinematical assumptions method. Some kind of physical assumption must be introduced to establish a relation between deformations (as well as displacements) of the whole plate and its median plane, the said relation allows to compute deformations an arbitrary point of a plate using its coordinates and information about the deformed state of the median plane. There are various assumptions, as well as a generalization of notion "kinematical assumption" in a formal functional way, but this paper is based on a Kirchhoff assumption. So, a deformation of a plate in respect to neutral fiber (straight and orthogonal to the undeformed median plane)

- keeps it straight,
- may not modify its length,
- keeps it orthogonal to the deformed median plane.

As per deformation tensor, the Kirchhoff assumption may be written as $\varepsilon_{IJ} = \varepsilon_{IJ}(x_1, x_2)$, $\varepsilon_{I3} = 0$, $\varepsilon_{33} = 0$. For further computations a relation in terms of displacements is preferred: conditions for deformations may be solved as a system of partial differential equations. The solution

$$u_i = w_i - \delta_{iK} x_3 w_{3,K} \tag{1}$$

is usually called *Kirchhoff plate displacement*. For further computations it is necessary to define kinematical factors: plane curvatures $\varkappa_{IJ}$ and plane deformations e_{IJ},

$$\varepsilon_{IJ} = e_{IJ} + x_3 \varkappa_{IJ} = \frac{1}{2}(w_{I,J} + w_{J,I}) - x_3 w_{3,IJ}.$$

Mechanics of rods and plates appeals to the internal force factors (instead of the stress tensor) as a main force parameter. Force factors of a plate are defined as a homogenization of stress tensor components with respect to thickness:

$$T_{IJ} = \int\limits_{-h/2}^{h/2} \sigma_{IJ} dx_3,\ M_{IJ} = \int\limits_{-h/2}^{h/2} x_3 \sigma_{IJ} dx_3,\ Q_I = \int\limits_{-h/2}^{h/2} \sigma_{I3} dx_3. \tag{2}$$

Here T_{IJ} are in-plane forces, Q_I are transverse shear forces, and M_{IJ} are bending moments. Equilibrium equations for force factors are a generalization of classical plate equations

$$T_{IJ,J} + q_I = 0, \quad M_{IJ,IJ} = q, \tag{3}$$

q_I and q are the surface loads—mass and surface forces and moments homogenization [5].

2.2. Constitutive Equations

The material is supposed to be linear viscoelastic and orthotropic with respect to median plane. The constitutive relations are a geometrical generalization of classical linear viscoelasticity constitutive relations [21]

$$\sigma_{ij} = \int\limits_0^t R_{ijkl}(t-\tau)\dot{\varepsilon}_{kl}(\tau)d\tau + R_{ijkl}(t)\varepsilon_{kl}(0), \quad \varepsilon_{ij} = \int\limits_0^t \Pi_{ijkl}(t-\tau)\dot{\sigma}_{kl}(\tau)d\tau + \Pi_{ijkl}(t)\sigma_{kl}(0). \quad (4)$$

The relaxation and compliance functions are mutually inverse, $\int\limits_0^t R_{ijkl}(t-\tau)d\Pi_{klmn}(\tau) = \Delta_{ijmn}$, Δ is a unit tensor.

Relations (4) are formulated for a general anisotropic linear viscoelastic body. It is essential to transform these relations into special relation(-s), which should establish a correspondence between force factors and kinematical factors of a plate. This transformation is based on force factors definition (2) and first constitutive relation (4). It may be simplified into $\sigma_{ij} = \int\limits_0^t R_{ijKL}(t-\tau)\dot{\varepsilon}_{KL}(\tau)d\tau + R_{ijKL}(t)\varepsilon_{KL}(0)$ taking into account the kinematical assumption. So,

$$T_{IJ} = \int\limits_{-h/2}^{h/2}\int\limits_0^t R_{IJKL}(t-\tau)\dot{\varepsilon}_{KL}(\tau)d\tau + R_{IJKL}(t)\varepsilon_{KL}(0)dx_3,$$

$$M_{IJ} = \int\limits_{-h/2}^{h/2}\int\limits_0^t R_{IJKL}(t-\tau)\dot{\varepsilon}_{KL}(\tau)x_3d\tau + R_{IJKL}(t)\varepsilon_{KL}(0)x_3dx_3.$$

Substitution of (1) leads to

$$T_{IJ} = \int\limits_0^t A_{IJKL}(t-\tau)\dot{e}_{KL}(\tau)d\tau + A_{IJKL}(t)e_{KL}(0) + \int\limits_0^t B_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau)d\tau + B_{IJKL}(t)\varkappa_{KL}(0),$$

$$M_{IJ} = \int\limits_0^t B_{IJKL}(t-\tau)\dot{e}_{KL}(\tau)d\tau + B_{IJKL}(t)e_{KL}(0) + \int\limits_0^t D_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau)d\tau + D_{IJKL}(t)\varkappa_{KL}(0). \quad (5)$$

A generalization of nonhomogeneous plate stiffnesses is defined there: $A_{IJKL}(t) = \int\limits_{-h/2}^{h/2} R_{IJKL}(t)dx_3$ are compressional stiffnesses, $B_{IJKL}(t) = \int\limits_{-h/2}^{h/2} R_{IJKL}(t)x_3dx_3$ are transverse shear stiffness, $D_{IJKL}(t) = \int\limits_{-h/2}^{h/2} R_{IJKL}(t)x_3^2dx_3$ are bending stiffnesses. For a viscoelastic tensor R_{IJKL} of odd with respect to x_3 components multilayer influence stiffnesses are absent: $B_{IJKL} = 0$, bending deformations and plane deformations of this plate are independent. Different components of $R_{IJKL}(t)$ may be different time-functions.

In light of practical importance, only bending is considered.

2.3. Equilibrium Equations of a Plate

Direct substitution of (5) into (3) results into equilibrium equations of a plate

$$\Bigg(\int_0^t A_{IJKL}(t-\tau)\dot{e}_{KL}(\tau)d\tau + A_{IJKL}(t)e_{KL}(0) + \int_0^t B_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau)d\tau + B_{IJKL}(t)\varkappa_{KL}(0)_{,J}\Bigg) = q_I,$$

$$\Bigg(\int_0^t B_{IJKL}(t-\tau)\dot{e}_{KL}(\tau)d\tau + B_{IJKL}(t)e_{KL}(0) + \int_0^t D_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau)d\tau + D_{IJKL}(t)\varkappa_{KL}(0)_{,IJ}\Bigg) = q. \quad (6)$$

This representation may be shorted and transformed by introducing notations $\langle f\rangle = \frac{1}{h}\int\limits_{-h/2}^{h/2} f dx_3$, $R[\varepsilon]_{IJ} = \int\limits_0^t R_{IJKL}(t-\tau)\dot{\varepsilon}_{KL}(\tau)d\tau$. The result may be called a shortened form of equilibrium equations:

$$(\langle R[\varepsilon]_{IJ}\rangle)_{,J} = q_I; \quad (\langle R[\varepsilon x_3]_{IJ}\rangle)_{,IJ} = q;$$

R is an operator in time-domain, $\langle\,\rangle$ is a spatial one with respect to thickness, $_{,I}$ and $_{,IJ}$ are spatial with respect to plane.

Similar to the elastic case, one needs four boundary conditions for every point of the boundary contour to close the system. Also due to a dynamical effect, an initial condition for curvatures and deformation is needed. Two types of boundary conditions may occur:

- kinematical conditions, $w_i\big|_{\partial\Pi} = f_i(x_1, x_2)$, $\frac{dw_i}{dn}\big|_{\partial\Pi} = g_i(x_1, x_2)$,
- statical conditions, $T_{IJ}n_J\Big|_\Gamma = T_I^0$, $(Q_I n_I + \frac{d}{ds}(\epsilon_{KI}n_K M_{IJ}n_J))\Big|_\Gamma = Q^0 + (\epsilon_{JI}M_I^0 n_J)$, $M_{IJ}n_I n_J\Big|_\Gamma = M_I^0 n_I$ [5].

3. MODEL IDENTIFICATION

One of the important focuses of any mechanical model is the identification problem. For a static loading of an orthotropic Kirchhoff plate, it becomes the problem of experimental stiffness computation. This problem has some well-known solutions for an elastic isotropic plate as well as some solutions for the non-isotropic case using different computation methods. It is necessary to note an approach of minimization of the error on constitutive law [15]. Viscoelastic and even linear viscoelastic cases differ significantly: the identification object is a set of time-functions instead of a set of constant values; as it was mentioned above, one of moduli symmetry conditions is absent: $C_{IJKL} \neq C_{KLIJ}$ [19], different components of C are different time-functions. However, we are also going to follow the approach, stated in [15], by rewriting the constitutive equations into a state-space form [22] and solving a linear system.

It is necessary to propose a set of experiments and a data processing algorithm, which allows computing the relaxation function of the investigated material. One of the favorable features of the identification problem is the possibility to choose the most convenient for further computations loading program. A loading, which makes possible an analytical solution of the equation of moments (3), will be called a simple one. The identification problem is stated and studied there for any simple loading for a plate. The particular character of loading is supposed to be predetermined, as well as any necessary loading parameters, that allow concluding $M_{IJ} = M_{IJ}(x_1, x_2)$ are known.

Some restriction on material properties is introduced to simplify the task. Let us assume R to have in-plane areas of constant material properties with a surface, sufficient for at least 5 curvatures measurements. The practical boundaries of these areas are equipment-dependent. Otherwise, a facial plane Σ_0 may be covered by a set of $U_i, i \in (1..n)$, $\Sigma_0 = \cup U_i$, $U_i \cap U_j \subset \partial U_i \cap \partial U_j$, $R_{IJKL}\big|_{U_i} = R_{IJKL}(t)$. Further analysis is processed on a fixed U_i.

The momentum equilibrium equation (6) for a fixed U_i transforms into

$$\Bigg(\int\limits_0^t B_{IJKL}(t-\tau)\dot{e}_{KL}(\tau)d\tau + B_{IJKL}(t)e_{KL}(0) + \int\limits_0^t D_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau)d\tau + D_{IJKL}(t)\varkappa_{KL}(0)\Bigg)_{,IJ} = q(t).$$

For a simple loading identification problem may be solved using the constitutive relations

$$\int\limits_0^t B_{IJKL}(t-\tau)\dot{e}_{KL}(\tau,x_1,x_2)d\tau + B_{IJKL}(t)e_{KL}(0,x_1,x_2) + \int\limits_0^t D_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau,x_1,x_2)d\tau + D_{IJKL}(t)\varkappa_{KL}(0,x_1,x_2) = M_{IJ}(t,x_1,x_2),$$

$$\int\limits_0^t A_{IJKL}(t-\tau)\dot{e}_{KL}(\tau,x_1,x_2)d\tau + A_{IJKL}(t)e_{KL}(0,x_1,x_2) + \int\limits_0^t B_{IJKL}(t-\tau)\dot{\varkappa}_{KL}(\tau,x_1,x_2)d\tau + B_{IJKL}(t)\varkappa_{KL}(0,x_1,x_2) = T_{IJ}(t,x_1,x_2),$$

where M and $\varkappa$ are measured. Continuous-time representation should be replaced by a discrete-time one using a unit-step discretization:

$$\sum_0^l A_{IJKL}(l-k)(e_{KL}(k+1,x_1,x_2) - e_{KL}(k,x_1,x_2)) + A_{IJKL}(l)e_{KL}(0,x_1,x_2) + \sum_0^l B_{IJKL}(l-k)(\varkappa_{KL}(k+1,x_1,x_2) - \varkappa_{KL}(k,x_1,x_2)) + B_{IJKL}(l)\varkappa_{KL}(0,x_1,x_2) = T_{IJ}(l,x_1,x_2). \quad (7)$$

$$\sum_0^l B_{IJKL}(l-k)(e_{KL}(k+1,x_1,x_2) - e_{KL}(k,x_1,x_2)) + B_{IJKL}(l)e_{KL}(0,x_1,x_2) + \sum_0^l D_{IJKL}(l-k)(\varkappa_{KL}(k+1,x_1,x_2) - \varkappa_{KL}(k,x_1,x_2)) + D_{IJKL}(l)\varkappa_{KL}(0,x_1,x_2) = M_{IJ}(l,x_1,x_2). \quad (8)$$

The goal is to identify stiffnesses, using e_{IJ} and $\varkappa_{IJ}$ measurements, as well as information about loading (M_{IJ} and T_{IJ} are supposed to be known). For a fixed l there are two approaches to compute $A, B, D(l)$: one may use previous values $A, B, D(l-1) \ldots A, B, D(0)$ and interpret (7) and (8) as equations on variables $A, B, D(l-1)$ or to recalculate all the stifness values for every new measurement and post-process the array of results.

3.1. State-Space Form of Equilibrium Equations

For a full asymmetry of the plate analysis of the R_{IJKL} identification problem is more efficient in comparison with A, B, D identification. The identification procedure, based on plate equations, may be more efficient only for symmetry, which results into $B = 0$, at least $\forall U_i$ (this condition does not affect $D_{1122} \neq D_{2211}$). So (7) and (8) simplify into the uncoupled equations

$$\sum_0^l A_{IJKL}(l-k)(e_{KL}(k+1, x_1, x_2) - e_{KL}(k, x_1, x_2)) + A_{IJKL}(l)e_{KL}(0, x_1, x_2) = T_{IJ}(l, x_1, x_2),$$

$$\sum_0^l D_{IJKL}(l-k)(\varkappa_{KL}(k+1, x_1, x_2) - \varkappa_{KL}(k, x_1, x_2)) + D_{IJKL}(l)\varkappa_{KL}(0, x_1, x_2) = M_{IJ}(l, x_1, x_2). \quad (9)$$

Let

$$\overrightarrow{a} = \begin{pmatrix} A_{1111} \\ A_{1122} \\ A_{2211} \\ A_{2222} \end{pmatrix}; \quad \overrightarrow{d} = \begin{pmatrix} D_{1111} \\ D_{1122} \\ D_{2211} \\ D_{2222} \end{pmatrix}; \quad \overrightarrow{T} = \begin{pmatrix} T^1_{11} \\ T^1_{22} \\ \vdots \\ T^n_{11} \\ T^n_{22} \end{pmatrix}; \quad \overrightarrow{M} = \begin{pmatrix} M^1_{11} \\ M^1_{22} \\ \vdots \\ M^n_{11} \\ M^n_{22} \end{pmatrix};$$

here $1 \ldots n$ are dots of measurements, located in U_i. The said notation allows rewriting (9) into a system of ordinary linear equations

$$E(1)\overrightarrow{a}(l) + \sum_{k=1}^{l}(E(k+1) - E(k))\overrightarrow{a}(l-k) = \overrightarrow{T};$$

$$K(1)\overrightarrow{d}(l) + \sum_{k=1}^{l}(K(k+1) - K(k))\overrightarrow{d}(l-k) = \overrightarrow{M}, \quad (10)$$

$E(t)$ and $K(t)$ are matrixes of plane deformation and curvatures:

$$E(t) = \begin{pmatrix} e^1_{11}(t) & e^1_{22}(t) & 0 & 0 \\ 0 & 0 & e^1_{11}(t) & e^1_{22}(t) \\ \vdots & \vdots & \vdots & \vdots \\ e^n_{11}(t) & e^n_{22}(t) & 0 & 0 \\ 0 & 0 & e^n_{11}(t) & e^n_{22}(t) \end{pmatrix}; \quad K(t) = \begin{pmatrix} \varkappa^1_{11}(t) & \varkappa^1_{22}(t) & 0 & 0 \\ 0 & 0 & \varkappa^1_{11}(t) & \varkappa^1_{22}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \varkappa^n_{11}(t) & \varkappa^n_{22}(t) & 0 & 0 \\ 0 & 0 & \varkappa^n_{11}(t) & \varkappa^n_{22}(t) \end{pmatrix}.$$

So (10) becomes an overdetermined system of linear algebraic equations on $\overrightarrow{a}$ and $\overrightarrow{d}$.

3.2. Solution

The said system (10) is overdetermined in general case of dependance $M_{IJ} = M_{IJ}(x_1, x_2)$, $T_{IJ} = T_{IJ}(x_1, x_2)$. For example, the case of pure bending corresponds to $M_{IJ} = const$, and system (10) becomes underdetermined due to a linear dependance between strings of E and K. However, components of these matrixes are a result of noisy measurements and numerical spatial differentiation, so in practical tasks the system (10) may be considered as overdetermined. As usual, an overdetermined system may be solved by a least-squares method, but model data testing shows the necessity of a large number of measurements (over 100 spatial measurements) for acceptable accuracy of the method (5% error).

This phenomenon occurs due to the erroneous calculation of deformations and curvatures caused by numerical differentiation of measured displacements.

A solution of an overdetermined system with additive noise in the matrix may be found using the total least squares [23] approach, so $\tilde{d}(l)$, $\tilde{a}(l)$ may be calculated using a singular value decomposition procedure [25].

Practical testing show low accuracy of the method (for some combinations of model properties, estimated stiffnesses are even negative). This error may be corrected by including a set-type restriction [27] on $\tilde{d}$: $\tilde{d}$ components are a result of relaxation function integration, so $\tilde{d}(t)$ must be positive $\forall t$ and must not increase in time: $\tilde{d}(l) \leqslant \tilde{d}(k) \forall k < l$, the same for a. Finally

$$\tilde{a} = argmin \frac{||E(1)x(l) + \sum_{k=1}^{l}(E(k+1) - E(k))\tilde{a}(l-k) - \overrightarrow{T}||^2}{1 + ||x||^2}, \quad 0 < \tilde{a}(l) \leqslant \tilde{a}(l-k),$$

$$\tilde{d} = argmin \frac{||K(1)x(l) + \sum_{k=1}^{l}(K(k+1) - K(k))\tilde{d}(l-k) - \overrightarrow{M}||^2}{1 + ||x||^2}, \quad 0 < \tilde{d}(l) \leqslant \tilde{d}(l-k),$$

here $||a||^2$ is a standard Euclidean vector norm and Frobenius matrix norm.

4. CONCLUSION

To sum up, this paper presents an approach to the identification of an anisotropic viscoelastic plate model. A simple version of model derivation, based on Kirchhoff's kinematical assumption was considered there. To supply an identification procedure we have also introduced some additional assumptions on material properties symmetries. However, even this case became nontrivial to consider: the state-space form of identification problem is an overdetermined system of linear equations with noisy matrix, for particular cases of loading columns of this matrix are close to linear dependency. That does not allow us to apply the classical least-squares approach: one needs to use some modified total least-squares approach. The suggested approach is step-by-step in time, so the further goal is to derive a simultaneous one (to reduce error accumulation). Also, an analysis of estimation quality is planned.

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