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Boundary-value Problems with Data on Characteristics for Hyperbolic Systems of Equations

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Abstract—The main subjects of the present paper are the Goursat and Darboux boundary-value problems for hyperbolic systems with two independent variables. We show that obtained by T.V. Chekmarev in terms of successive approximations formulas for solution of the Goursat problem can be built also by the Riemann method, work out an analog of the Riemann–Hadamard method for the system, and introduce its Riemann–Hadamard matrix. We solve the Darboux problem in terms of the introduced matrix.

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1. INTRODUCTION

System of equations of the first order

$$\frac{\partial u_i}{\partial x_i} = \sum_{k=1}^{n} a_{ik}(x_1, \dots, x_n)u_k + f_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \tag{1}$$

was studied by a number of authors (see, for instance, [1–6] and references in these papers). It is of interest, in particular, in connection with differential equations of mixed type. The most number of publications concerns the case $n = 2$.

The author prposed [7] a version of the ariemann method for systems of differential equations both with simple and multiple characteristics.

The Darboux problem for hyperbolic equations and systems is of great interest. This problem for equations of second order with two independent variables is considered by a number of authors. We mention here the works [8, p. 228–233], [9–14]. There exist also investigations of the Darboux problem for hyperbolic equations of the third order [15, 16].

2. THE GOURSAT PROBLEM

We consider the following system of equations

$$\begin{cases} u_{1x} = a_{11}(x,y)u_1 + a_{12}(x,y)u_2 + f_1(x,y), \\ u_{2y} = a_{21}(x,y)u_1 + a_{22}(x,y)u_2 + f_2(x,y), \end{cases} \tag{2}$$

in rectangle $D = \{x_0 < x < x_1,\, y_0 < y < y_1\}$. Here a_{11}, a_{12}, a_{21}, a_{22}, f_1, $f_2 \in C(\overline{D})$. A solution u is called regular in D if u_1, u_2, u_{1x}, $u_{2y} \in C(D)$.

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The Goursat problem: *Find regular solution of system (2) in domain D which is continuously extendable on the boundary of domain D and satisfies boundary conditions*

$$u_1(x_0,y)=\varphi(y),\quad u_2(x,y_0)=\psi(x),$$
$$\varphi(y)\in C([y_0,y_1]),\quad \psi(x)\in C([x_0,x_1]). \tag{3}$$

The Goursat problem has unique solution [3, p. 15–23].

A linear transformation of the desired functions reduces (2) to the case

$$a_{11}\equiv a_{22}\equiv 0. \tag{4}$$

In what follows we conider these identities as fulfilled.

Chekmarev [2, 3, p. 15–22] by means of the successive approximations method obtained solution of the Goursat problem (2), (3) in the form

$$u_1(x,y)=\varphi(y)+\int\limits_{y_0}^{y}\varphi(\tau)L(x,y,x_0,\tau)d\tau+\int\limits_{x_0}^{x}\psi(t)K(x,y,t,y_0)dt+\int\limits_{x_0}^{x}f_1(t,y)dt$$
$$+\int\limits_{x_0}^{x}\int\limits_{y_0}^{y}(L(x,y,t,\tau)f_1(t,\tau)+K(x,y,t,\tau)f_2(t,\tau))d\tau dt, \tag{5}$$

$$u_2(x,y)=\psi(x)+\int\limits_{y_0}^{y}\varphi(\tau)M(x,y,x_0,\tau)d\tau+\int\limits_{x_0}^{x}\psi(t)N(x,y,t,y_0)dt+\int\limits_{y_0}^{y}f_2(x,\tau)d\tau$$
$$+\int\limits_{x_0}^{x}\int\limits_{y_0}^{y}(M(x,y,t,\tau)f_1(t,\tau)+N(x,y,t,\tau)f_2(t,\tau))d\tau dt, \tag{6}$$

where L, K, M, N are determined by coefficients a_{12}, a_{21} as uniformly convergent series, and satiffy relations

$$K(x,y,t,\tau)=a_{12}(t,y)+\int\limits_{t}^{x}a_{12}(\xi,y)N(\xi,y,t,\tau)d\xi,$$
$$L(x,y,t,\tau)=\int\limits_{t}^{x}a_{12}(\xi,y)M(\xi,y,t,\tau)d\xi,$$
$$M(x,y,t,\tau)=a_{21}(x,\tau)+\int\limits_{t}^{y}a_{21}(x,\eta)L(x,\eta,t,\tau)d\eta,$$
$$N(x,y,t,\tau)=\int\limits_{\tau}^{y}a_{21}(x,\eta)K(x,\eta,t,\tau)d\eta. \tag{7}$$

There exists other approach to solving of described aboveproblems. We can apply to system (2) the Riemann method. Rewrite (2) in vector-matrix form

$$L(\mathbf{U})=\mathbf{F},\quad L(\mathbf{U})\equiv\mathbf{A}\mathbf{U}_x+\mathbf{B}\mathbf{U}_y-\mathbf{C}\mathbf{U},\quad \mathbf{U}=\text{colon}(u_1,u_2),$$
$$\mathbf{A}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}0&a_{12}\\a_{21}&0\end{pmatrix},$$
$$\mathbf{F}=\text{colon}(f_1,f_2). \tag{8}$$

Introduce Riemann matrix $\mathbf{R} = \mathrm{colon}(\mathbf{R}_1, \mathbf{R}_2)$, where vectors $\mathbf{R}_i(x, y, \xi, \eta) = (r_{i1}, r_{i2})$, $i = \overline{1, 2}$, are solutions of systems

$$\begin{cases} r_{i1}(x, y) = \delta_{i1} - \int\limits_{\xi}^{x} a_{21}(\alpha, y) r_{i2}(\alpha, y) d\alpha, \\ r_{i2}(x, y) = \delta_{i2} - \int\limits_{\eta}^{y} a_{12}(x, \beta) r_{i1}(x, \beta) d\beta, \end{cases} \tag{9}$$

δ_{ij} is Kronecker symbol. The solutions of systems (9) exist and are unique in the class of continuous functions. We differentiate (9), and see that system (9) is equivalent to two Goursat problems

$$\begin{cases} r_{i1x} = -a_{21}(x, y) r_{12}, \quad r_{i2y}(x, y) = -a_{12}(x, y) r_{11}, \\ r_{i1}|_{x=\xi} = \delta_{i1}, \quad r_{i2}|_{y=\eta} = \delta_{i2}, \quad i = 1, 2. \end{cases} \tag{10}$$

Clearly, function R with respect to the first pair of variables satisfies the conjugated to (2) system

$$L^*(\mathbf{V}) = 0, \quad L^*(\mathbf{V}) \equiv -(\mathbf{VA})_x - (\mathbf{VB})_y - \mathbf{VC}.$$

We integrate identity

$$\mathbf{R}L(\mathbf{U}) = (\mathbf{RAU})_x + (\mathbf{RBU})_y \tag{11}$$

over rectangle $D_1 = \{x_0 < x < \xi, y_0 < y < \eta\}$, $(\xi, \eta) \in D$, and obtain

$$\int\limits_{y_0}^{\eta} r_{11}(\xi, y, \xi, \eta) u_1(\xi, y) dy + \int\limits_{x_0}^{\xi} r_{12}(x, \eta, \xi, \eta) u_2(x, \eta) dx = \int\limits_{y_0}^{\eta} r_{11}(x_0, y, \xi, \eta) u_1(x_0, y) dy$$
$$+ \int\limits_{x_0}^{\xi} r_{12}(x, y_0, \xi, \eta) u_2(x, y_0) dx + \iint\limits_{D_1} (r_{11}(x, y, \xi, \eta) f_1(x, y) + r_{12}(x, y, \xi, \eta) f_2(x, y)) dx dy, \tag{12}$$

$$\int\limits_{y_0}^{\eta} r_{21}(\xi, y, \xi, \eta) u_1(\xi, y) dy + \int\limits_{x_0}^{\xi} r_{22}(x, \eta, \xi, \eta) u_2(x, \eta) dx = \int\limits_{y_0}^{\eta} r_{21}(x_0, y, \xi, \eta) u_1(x_0, y) dy$$
$$+ \int\limits_{x_0}^{\xi} r_{22}(x, y_0, \xi, \eta) u_2(x, y_0) dx + \iint\limits_{D_1} (r_{21}(x, y, \xi, \eta) f_1(x, y) + r_{22}(x, y, \xi, \eta) f_2(x, y)) dx dy. \tag{13}$$

We use the properties of the Riemann matrix, differentiate (12) with respect to variable η and (13) with respect to ξ, and obtain solutions of the Goursat problem in the following form:

$$u_1(\xi, \eta) = \varphi(\eta) + \int\limits_{y_0}^{\eta} r_{11\eta}(x_0, y, \xi, \eta) \varphi(y) dy + \int\limits_{x_0}^{\xi} r_{12\eta}(x, y_0, \xi, \eta) \psi(x) dx + \int\limits_{x_0}^{\xi} f_1(x, \eta) dx$$
$$+ \int\limits_{y_0}^{\eta} \int\limits_{x_0}^{\xi} (r_{11\eta}(x, y, \xi, \eta) f_1(x, y) + r_{12\eta}(x, y, \xi, \eta) f_2(x, y)) dx dy, \tag{14}$$

$$u_2(\xi, \eta) = \psi(\xi) + \int\limits_{y_0}^{\eta} r_{21\xi}(x_0, y, \xi, \eta) \varphi(y) dy + \int\limits_{x_0}^{\xi} r_{22\xi}(x, y_0, \xi, \eta) \psi(x) dx + \int\limits_{y_0}^{\eta} f_2(\xi, y) dy$$

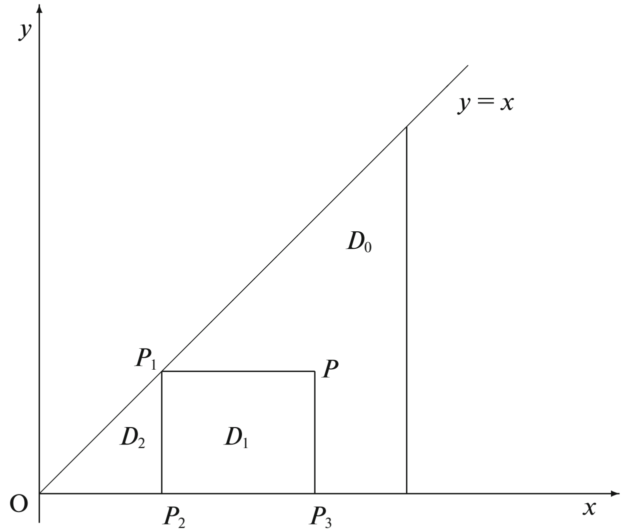

Fig. 1.

$$+\int_{y_0}^{\eta}\int_{x_0}^{\xi}(r_{21\xi}(x,y,\xi,\eta)f_1(x,y)+r_{22\xi}(x,y,\xi,\eta)f_2(x,y))dxdy. \tag{15}$$

The formulas (5), (6), (7), (9), (14), (15) imply relations

$$L(\xi,\eta,x,y)=r_{11\eta}(x,y,\xi,\eta),\quad K(\xi,\eta,x,y)=r_{12\eta}(x,y,\xi,\eta),$$
$$M(\xi,\eta,x,y)=r_{21\xi}(x,y,\xi,\eta),\quad N(\xi,\eta,x,y)=r_{22\xi}(x,y,\xi,\eta).$$

3. THE DARBOUX PROBLEM

We consider the **Darboux problem:** *Find regular solution of system (2) in domain* $D_0=\{(x,y):0<y<x<T\}$*, which is continuable on the boundary of domain D and satisfies boundary conditions*

$$u_1(y,y)=\lambda(y), u_2(x,0)=\mu(x),$$
$$\lambda(y)\in C([0,T]),\quad \mu(x)\in C([0,T]). \tag{16}$$

Solution of the Darboux problem exists, and it is unique [3, p. 26–29].

We fix a point $P(\xi,\eta)\in D_0$, and consider corresponding quadrangle $D_p=\{(x,y):y<x<\xi, 0<y<\eta\}\subset D_0$ with vertices at points $O(0,0)$, $P_1(\eta,\eta)$, $P(\xi,\eta)$, $P_3(\xi,0)$, rectangle $D_1=\{(x,y):\eta<x<\xi, 0<y<\eta\}$ with vertices at points P, P_1, $P_2(\eta,0)$, P_3, and triangle $D_2=\{(x,y):y<x<\eta, 0<y<\eta\}$ with vertices at points O, P_1, P_2 (see Fig. 1).

Define Riemann–Hadamard matrix for the Darboux problem $H(x,y,\xi,\eta)$. Let

$$H(x,y,\xi,\eta)=\begin{cases}R(x,y,\xi,\eta), & (x,y)\in D_1,\\ V(x,y,\xi,\eta), & (x,y)\in D_2,\end{cases} \tag{17}$$

where R is the defined above Riemann matrix, and matrix $\mathbf{V}=\mathrm{colon}(\mathbf{V}_1,\mathbf{V}_2)$, with vectors $\mathbf{V}_i(x,y,\xi,\eta)=(\bar r_{i1},\bar r_{i2})$, $i=\overline{1,2}$, needs to be specified in domain D_2.

We require that V_1, V_2 be solutions the Darboux problems in domain D_2 for systems

$$\bar r_{i1x}=-a_{21}(x,y)\bar r_{12},\quad \bar r_{i2y}(x,y)=-a_{12}(x,y)\bar r_{11},\quad i=1,2, \tag{18}$$

with conditions relatively

$$\bar r_{11}(\eta,y,\xi,\eta)=r_{11}(\eta,y,\xi,\eta),\quad \bar r_{12}(y,y,\xi,\eta)=0 \tag{19}$$

for vector V_1, and

$$\bar{r}_{21}(\eta, y, \xi, \eta) = r_{21}(\eta, y, \xi, \eta), \quad \bar{r}_{22}(y, y, \xi, \eta) = 0 \tag{20}$$

for vector V_2.

We integrate identity (11) over rectangle D_1:

$$\int_0^\eta r_{11}(\xi, y, \xi, \eta)u_1(\xi, y)dy + \int_\eta^\xi r_{12}(x, \eta, \xi, \eta)u_2(x, \eta)dx = \int_0^\eta r_{11}(\eta, y, \xi, \eta)u_1(\eta, y)dy$$

$$+ \int_\eta^\xi r_{12}(x, 0, \xi, \eta)u_2(x, 0))dx + \iint_{D_1} (r_{11}(x, y, \xi, \eta)f_1(x, y) + r_{12}(x, y, \xi, \eta)f_2(x, y))dxdy, \tag{21}$$

$$\int_0^\eta r_{21}(\xi, y, \xi, \eta)u_1(\xi, y)dy + \int_\eta^\xi r_{22}(x, \eta, \xi, \eta)u_2(x, \eta)dx = \int_0^\eta r_{21}(\eta, y, \xi, \eta)u_1(\eta, y)dy$$

$$+ \int_\eta^\xi r_{22}(x, 0, \xi, \eta)u_2(x, 0))dx + \iint_{D_1} (r_{21}(x, y, \xi, \eta)f_1(x, y) + r_{22}(x, y, \xi, \eta)f_2(x, y))dxdy. \tag{22}$$

Integration of identity (11) over triangle D_2 gives

$$- \int_0^\eta \bar{r}_{12}(x, 0, \xi, \eta)u_2(x, 0)dx + \int_0^\eta \bar{r}_{11}(\eta, y, \xi, \eta)u_1(\eta, y)dy + \int_\eta^0 (\bar{r}_{11}(y, y, \xi, \eta)u_1(y, y)$$

$$- \bar{r}_{12}(y, y, \xi, \eta)u_2(y, y))dy + \iint_{D_2} (\bar{r}_{11}(x, y, \xi, \eta)f_1(x, y) + \bar{r}_{12}(x, y, \xi, \eta)f_2(x, y))dxdy, \tag{23}$$

$$- \int_0^\eta \bar{r}_{22}(x, 0, \xi, \eta)u_2(x, 0)dx + \int_0^\eta \bar{r}_{21}(\eta, y, \xi, \eta)u_1(\eta, y)dy + \int_\eta^0 (\bar{r}_{21}(y, y, \xi, \eta)u_1(y, y)$$

$$- \bar{r}_{22}(y, y, \xi, \eta)u_2(y, y))dy + \iint_{D_2} (\bar{r}_{21}(x, y, \xi, \eta)f_1(x, y) + \bar{r}_{22}(x, y, \xi, \eta)f_2(x, y))dxdy. \tag{24}$$

We add (21) with (23) and (22) with (24), and obtain

$$\int_0^\eta r_{11}(\xi, y, \xi, \eta)u_1(\xi, y)dy + \int_\eta^\xi r_{12}(x, \eta, \xi, \eta)u_2(x, \eta)dx$$

$$+ \int_0^\eta \bar{r}_{11}(\eta, y, \xi, \eta)u_1(\eta, y)dy + \int_\eta^0 (\bar{r}_{11}(y, y, \xi, \eta)u_1(y, y) - \bar{r}_{12}(y, y, \xi, \eta)u_2(y, y))dy$$

$$= \int_0^\eta r_{11}(\eta, y, \xi, \eta)u_1(\eta, y)dy + \int_\eta^\xi r_{12}(x, 0, \xi, \eta)u_2(x, 0))dx$$

$$+ \int_0^\eta \bar{r}_{12}(x, 0, \xi, \eta)u_2(x, 0)dx + \iint_{D_1} (r_{11}(x, y, \xi, \eta)f_1(x, y) + r_{12}(x, y, \xi, \eta)f_2(x, y))dxdy$$

$$+\iint\limits_{D_2}(\bar r_{11}(x,y,\xi,\eta)f_1(x,y)+\bar r_{12}(x,y,\xi,\eta)f_2(x,y))dxdy, \tag{25}$$

$$\int\limits_0^\eta r_{21}(\xi,y,\xi,\eta)u_1(\xi,y)dy+\int\limits_\eta^\xi r_{22}(x,\eta,\xi,\eta)u_2(x,\eta)dx$$

$$+\int\limits_0^\eta \bar r_{21}(\eta,y,\xi,\eta)u_1(\eta,y)dy+\int\limits_\eta^0(\bar r_{21}(y,y,\xi,\eta)u_1(y,y)-\bar r_{22}(y,y,\xi,\eta)u_2(y,y))dy$$

$$=\int\limits_0^\eta r_{21}(\eta,y,\xi,\eta)u_1(\eta,y)dy+\int\limits_\eta^\xi r_{22}(x,0,\xi,\eta)u_2(x,0))dx$$

$$+\int\limits_0^\eta \bar r_{12}(x,0,\xi,\eta)u_2(x,0)dx+\iint\limits_{D_1}(r_{21}(x,y,\xi,\eta)f_1(x,y)+r_{22}(x,y,\xi,\eta)f_2(x,y))dxdy$$

$$+\iint\limits_{D_2}(\bar r_{21}(x,y,\xi,\eta)f_1(x,y)+\bar r_{22}(x,y,\xi,\eta)f_2(x,y))dxdy. \tag{26}$$

It follows from (9), (19), (20) that $r_{11}(\xi,y,\xi,\eta)=1$, $r_{12}(x,\eta,\xi,\eta)=0$, $r_{21}(\xi,y,\xi,\eta)=0$, $r_{22}(x,\eta,\xi,\eta)=1$, $\bar r_{12}(y,y,\xi,\eta)=0$, $\bar r_{22}(y,y,\xi,\eta)=0$, and

$$\bar r_{11}(\eta,y,\xi,\eta)-r_{11}(\eta,y,\xi,\eta)=0,\quad \bar r_{21}(\eta,y,\xi,\eta)-r_{21}(\eta,y,\xi,\eta)=0.$$

Consequently, (25) and (26) are representable as

$$\int\limits_0^\eta u_1(\xi,y)dy=F_1(\xi,\eta),\quad \int\limits_\eta^\xi u_2(x,\eta)dx=F_2(\xi,\eta), \tag{27}$$

where $F_1(\xi,\eta)$, $F_2(\xi,\eta)$ are completely determined by the Riemann–Hadamard matrix H and boundary data of the Darboux problem.

We differentiate the first equality (27) in η, and second in ξ, and obtain explicit solution of the Darboux problem in terms of the Riemann–Hadamard matrix.

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