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Finite Element Approximation of the Minimal Eigenvalue and the Corresponding Positive Eigenfunction of a Nonlinear Sturm–Liouville Problem

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Abstract—The problem of finding the minimal eigenvalue and the corresponding positive eigenfunction of the nonlinear Sturm–Liouville problem for the ordinary differential equation with coefficients nonlinear depending on a spectral parameter is investigated. This problem arises in modeling the plasma of radio-frequency discharge at reduced pressures. A sufficient condition for the existence of a minimal eigenvalue and the corresponding positive eigenfunction of the nonlinear Sturm–Liouville problem is established. The original differential eigenvalue problem is approximated by the finite element method with Lagrangian finite elements of arbitrary order on a uniform grid. The error estimates of the approximate eigenvalue and the approximate positive eigenfunction to exact ones are proved. Investigations of this paper generalize well known results for the Sturm–Liouville problem with linear entrance on the spectral parameter.

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1. INTRODUCTION

The present paper is concerned with investigating the following differential nonlinear Sturm–Liouville problem: find the minimal eigenvalue $\lambda \in \Lambda$, $\Lambda = [0, \infty)$, and the corresponding positive eigenfunction $u = u(x)$, $x \in \Omega$, $\Omega = (0, \pi)$, $\overline{\Omega} = [0, \pi]$, satisfying the following equations

$$-(p(\lambda s(x))u')' = r(\lambda s(x))u, \quad x \in \Omega, \quad u(0) = u(\pi) = 0. \tag{1}$$

We assume that $p(\mu)$, $r(\mu)$, $\mu \in \Lambda$, and $s(x)$, $x \in \overline{\Omega}$, are given infinitely continuously differentiable positive functions. We also assume that the function $p(\mu)$, $\mu \in \Lambda$, and $r(\mu)$, $\mu \in \Lambda$, are nondecreasing functions, $p(\mu) = p_2$, $\mu \in \Lambda_2$, and $r(\mu) = r_2$, $\mu \in \Lambda_2$, $\Lambda_2 = [\alpha, \infty)$, p_2, r_2, and α are given positive numbers.

Nonlinear eigenvalue problems of the form (1) arise in modeling the plasma of radio-frequency discharge at reduced pressures. The existence conditions derived in the present paper defines an existence conditions for maintaining a stationary inductive coupled radio-frequency discharge at reduced presure [1–4].

In the present paper, a sufficient condition for the existence of a minimal eigenvalue and the corresponding positive eigenfunction of the nonlinear Sturm–Liouville problem (1) is established. The original nonlinear differential eigenvalue problem is approximated by the finite element method with

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Lagrangian finite elements of arbitrary order on a uniform grid. The error estimates of the approximate minimal eigenvalue and the approximate positive eigenfunction to exact ones are proved. Investigations of this paper generalize well known results for Sturm–Liouville problems with linear dependence on the spectral parameter and develop the results obtained in the paper [5].

Nonlinear eigenvalue problems also arise in various fields of science and technology [6–26]. Computational methods for solving nonlinear matrix eigenvalue problems were constructed and investigated in the papers [27–41]. Error of the finite difference methods for solving differential eigenvalue problems with nonlinear entrance of the spectral parameter was studied in [42–44]. The finite element method for solving nonlinear eigenvalue problems was investigated in [5, 45], and estimations of the effect of numerical integration in finite element eigenvalue and eigenfunction approximations were established in [46–48] with using the results [49–52]. The investigations of approximate methods for solving eigenvalue problems with nonlinear entrance of the spectral parameter in a Hilbert space were carried out in the paper [54] with help general results for linear spectral problems [55–59]. In the papers [60–66], approximate methods for solving applied nonlinear boundary value problems and variational inequalities have been investigated.

2. VARIATIONAL STATEMENT OF THE PROBLEM

Let $H = L_2(\Omega)$ be the real Lebesgue space with the following norm and scalar product

$$|v|_0 = \left(\int_0^\pi (v(x))^2 dx\right)^{1/2}, \quad (u,v)_0 = \int_0^\pi u(x)v(x)dx, \quad \forall u, v \in H.$$

By $V = \{v : v, v' \in H, v(0) = v(\pi) = 0\}$ we denote the real Sobolev space with the following norm and scalar product

$$|v|_1 = \left(\int_0^\pi (v'(x))^2 dx\right)^{1/2}, \quad (u,v)_1 = \int_0^\pi u'(x)v'(x)dx \quad \forall u, v \in V.$$

Introduce the subset $K = \{v : v \in V, v(x) > 0, x \in \Omega\}$ of the space V.

For fixed $\mu \in \Lambda$, $u, v \in V$, $w \in V \setminus \{0\}$, we define the following bilinear forms and the Rayleigh functional

$$a(\mu,u,v) = \int_0^\pi p(\mu s(x))u'v'dx, \quad b(\mu,u,v) = \int_0^\pi r(\mu s(x))uvdx, \quad R(\mu,w) = \frac{a(\mu,w,w)}{b(\mu,w,w)}.$$

The differential nonlinear eigenvalue problem (1) is equivalent to the following variational nonlinear eigenvalue problem: find the minimal number $\lambda \in \Lambda$ and a function $u \in K$, $b(\lambda,u,u) = 1$, such that

$$a(\lambda,u,v) = b(\lambda,u,v) \quad \forall v \in V. \tag{2}$$

For fixed parameter $\mu \in \Lambda$, we introduce the linear variational parameter eigenvalue problem: find the minimal number $\gamma(\mu) \in \Lambda$ and a function $u = u_\mu \in K$, $b(\mu,u,u) = 1$, such that

$$a(\mu,u,v) = \gamma(\mu)b(\mu,u,v) \quad \forall v \in V. \tag{3}$$

The minimal eigenvalue of problem (3) satisfies the following variational representation

$$\gamma(\mu) = \min_{v \in V \setminus \{0\}} R(\mu,v).$$

Therefore we obtain that the minimal eigenvalue λ of problem (2) is the minimal root of the equation

$$\gamma(\mu) = 1, \quad \mu \in \Lambda. \tag{4}$$

Formulate the auxiliary linear variational eigenvalue problem: find the minimal number $\varkappa \in \Lambda$ and a function $u \in K$, $(u,u)_0 = 1$, such that

$$(u,v)_1 = \varkappa(u,v)_0 \quad \forall v \in V. \tag{5}$$

The eigenvalue and eigenfunction of problem (5) is defined by

$$\varkappa = 1, \quad u(x) = \sqrt{\frac{\pi}{2}}\sin x, \quad x \in \overline{\Omega}, \quad \varkappa = \frac{(u,u)_1}{(u,u)_0} = \min_{v\in V\setminus\{0\}} \frac{(v,v)_1}{(v,v)_0}.$$

For $\mu, \eta \in \Lambda$, we denote

$$\delta_p(\mu,\eta) = \max_{x\in\overline{\Omega}} |p(\mu s(x)) - p(\eta s(x))|, \quad \delta_r(\mu,\eta) = \max_{x\in\overline{\Omega}} |r(\mu s(x)) - r(\eta s(x))|.$$

Theorem 1. *For $\mu, \eta \in \Delta$, the following estimate is valid*

$$|\gamma(\mu) - \gamma(\eta)| \le c(\delta_p(\mu,\eta) + \delta_r(\mu,\eta)),$$

where c is a positive constant independent of $\mu, \eta \in \Delta$, $\Delta = [\alpha, \beta] \subset \Lambda$.

Proof. This result is proved by analogy with the result of the paper [5]. □

Theorem 2. *The convergences $\delta_p(\mu,\eta) \to 0$, $\delta_r(\mu,\eta) \to 0$, as $\eta \to \mu$ hold.*

Proof. This result is proved by analogy with the result of the paper [5]. □

Theorem 3. *Suppose that the following conditions are valid $p_2 < r_2$, $p(\xi s_1) > r(\xi s_2)$ for some number $\xi \in \Lambda$, where numbers s_1 and s_2 are equal, respectively, to the minimum and maximum of the function $s(x)$, $x \in \overline{\Omega}$. Then there exist a minimal simple eigenvalue λ of problem (2) and a corresponding positive eigenfunction u.*

Proof. According to Theorems 1 and 2, $\gamma(\mu)$, $\mu \in \Lambda$, is a continuous function. Using the variational characterizations for the minimal eigenvalues of the problems (3) and (5), we obtain the relations

$$\gamma(\mu) = \min_{v\in V\setminus\{0\}} R(\mu, v) = \min_{v\in V\setminus\{0\}} \frac{\int\limits_0^\pi p(\mu s(x))(v')^2 dx}{\int\limits_0^\pi r(\mu s(x)) v^2 dx} = \frac{p_2}{r_2}\varkappa = \frac{p_2}{r_2} < 1,$$

$$\gamma(\xi) = \min_{v\in V\setminus\{0\}} R(\xi, v) = \min_{v\in V\setminus\{0\}} \frac{\int\limits_0^\pi p(\xi s(x))(v')^2 dx}{\int\limits_0^\pi r(\xi s(x)) v^2 dx} \ge \frac{p(\xi s_1)}{r(\xi s_2)}\varkappa = \frac{p(\xi s_1)}{r(\xi s_2)} > 1,$$

for $\mu s_1 \in \Lambda_2$. Since $\gamma(\mu)$, $\mu \in \Lambda$, is the continuous function, there exists a minimal root of equation (4), which defines the minimal eigenvalue $\lambda \in \Lambda$ of problem (2) corresponding to a positive eigenfunction u. The eigenvalue $\lambda \in \Lambda$ is simple and corresponds to a positive eigenfunction, since $\gamma(\mu)$ is the simple eigenvalue of the parametric problem (3) for $\mu = \lambda$ corresponding to a positive eigenfunction. This completes the proof of the theorem. □

3. FINITE ELEMENT APPROXIMATION OF THE PROBLEM

Let us partition the segment $[0, \pi]$ by equidistant points $x_i = ih$, $i = 0, 1, \ldots, m$, into the elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, $h = \pi/m$. By V_h we denote the subspace of the space V, consisting of continuous functions v^h on $\overline{\Omega}$ that are polynomials of degree at most n on each element e_i, $i = 1, 2, \ldots, m$, $N_h = \dim V_h = mn - 1$. Set

$$K_h = \{v^h : v^h \in V_h, v^h(x) > 0, x \in \Omega\}.$$

The variational nonlinear eigenvalue problem (2) is approximated by the following finite element problem: find the minimal number $\lambda^h \in \Lambda$ and a function $u^h \in K_h$, $b(\lambda^h, u^h, u^h) = 1$, such that

$$a(\lambda^h, u^h, v^h) = b(\lambda^h, u^h, v^h) \quad \forall v^h \in V_h. \tag{6}$$

For fixed $\mu \in \Lambda$, we define linear parameter eigenvalue problem: find the minimal number $\gamma^h(\mu)$ and a function $u^h = u^h_\mu \in K_h$, $b(\mu, u^h, u^h) = 1$, such that

$$a(\mu, u^h, v^h) = \gamma^h(\mu) b(\mu, u^h, v^h) \quad \forall v^h \in V_h. \tag{7}$$

The following variational characterization for the minimal eigenvalue of problem (7) is valid

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R(\mu, v^h).$$

The minimal eigenvalue λ^h of the finite element problem (6) is the minimal root of the equation

$$\gamma^h(\mu) = 1, \quad \mu \in \Lambda, \tag{8}$$

where $\gamma^h(\mu)$ is the minimal eigenvalue of the finite element problem (7).

Theorem 4. *For $\mu, \eta \in \Delta$, the following estimate is valid*

$$|\gamma^h(\mu) - \gamma^h(\eta)| \le c(\delta_p(\mu, \eta) + \delta_r(\mu, \eta)),$$

where c is a positive constant independent of $\mu, \eta \in \Delta$, $\Delta = [\alpha, \beta] \subset \Lambda$.

Proof. The proof of this theorem is similar to that of Theorem 1. □

Theorem 5. *Suppose that the following conditions are valid $p_2 < r_2$, $p(\xi s_1) > r(\xi s_2)$ for some number $\xi \in \Lambda$, where numbers s_1 and s_2 are equal, respectively, to the minimum and maximum of the function $s(x)$, $x \in \overline{\Omega}$. Then for sufficiently small h there exist a minimal simple eigenvalue λ^h of problem (2) and a corresponding positive eigenfunction u^h.*

Proof. By Theorem 4, $\gamma^h(\mu)$, $\mu \in \Lambda$, is a continuous function. Using the variational characterizations for the minimal eigenvalues, we derive the relations

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R(\mu, v^h) = \min_{v^h \in V_h \setminus \{0\}} \frac{\int\limits_0^\pi p(\mu s(x))((v^h)')^2 dx}{\int\limits_0^\pi r(\mu s(x))(v^h)^2 dx} = \frac{p_2}{r_2} \varkappa^h < 1,$$

$$\gamma^h(\xi) = \min_{v^h \in V_h \setminus \{0\}} R(\xi, v^h) = \min_{v^h \in V_h \setminus \{0\}} \frac{\int\limits_0^\pi p(\xi s(x))((v^h)')^2 dx}{\int\limits_0^\pi r(\xi s(x))(v^h)^2 dx} \ge \frac{p(\xi s_1)}{r(\xi s_2)} \varkappa^h \ge \frac{p(\xi s_1)}{r(\xi s_2)} > 1,$$

for $\mu s_1 \in \Lambda_2$ and sufficiently small h. Here we have taken into account that $1 = \varkappa \le \varkappa^h \to 1$ as $h \to 0$, where

$$\varkappa^h = \min_{v^h \in V_h \setminus \{0\}} \frac{(v^h, v^h)_1}{(v^h, v^h)_0}.$$

Since $\gamma^h(\mu)$, $\mu \in \Lambda$, is the continuous function, there exists a minimal root of equation (8), which defines the minimal eigenvalue $\lambda^h \in \Lambda$ of problem (6) corresponding to a positive eigenfunction u^h. The eigenvalue $\lambda^h \in \Lambda$ is simple and corresponds to a positive eigenfunction, since $\gamma^h(\mu)$ is the simple eigenvalue of the parameter eigenvalue problem (7) for $\mu = \lambda^h$ corresponding to a positive eigenfunction for sufficiently small h. This completes the proof of the theorem. □

4. ERROR ANALYSIS OF THE FINITE ELEMENT SCHEME

By c we denote various positive constants independent of h. For fixed $\mu \in \Lambda$, we introduce the operator $P_h(\mu) : V \to V_h$ defined by the formula $a(\mu, u - P_h(\mu)u, v^h) = 0$ for any $v^h \in V_h$, where $u \in V$, $|u - P_h(\mu)u|_1 \leq ch^n$. Put $P_h = P_h(\lambda)$.

By $\gamma_i^h(\mu)$, $u_i^h(\mu) = u_i^h(\mu, x)$, $x \in \overline{\Omega}$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N_h$, we denote eigenvalues and eigenfunctions satisfying equation (7) and such that $\gamma_1^h(\mu) \leq \gamma_2^h(\mu) \leq \ldots \leq \gamma_{N_h}^h(\mu)$, $a(\mu, u_i^h(\mu), u_j^h(\mu)) = \gamma_i^h(\mu)\delta_{ij}$, $b(\mu, u_i^h(\mu), u_j^h(\mu)) = \delta_{ij}$, $i, j = 1, 2, \ldots, N_h$, $u_1^h(\mu) = u_\mu^h \in K_h$, $\gamma_1^h(\mu) = \gamma^h(\mu)$, the functions $u_i^h(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N_h$, form a complete system in the Hilbert space V_h. For fixed $\mu \in \Lambda$ and sufficiently small h, the following error estimates hold: $0 \leq \gamma_i^h(\mu) - \gamma_i(\mu) \leq ch^{2n}$, $i = 1, 2$, $|u_\mu^h - u_\mu|_1 \leq ch^n$.

Theorem 6. *The following convergences hold $\lambda^h \to \lambda$, $u^h \to u$ in V as $h \to 0$.*

Proof. This result follows from the paper [5]. □

Theorem 7. *Suppose that $\gamma'(\lambda) \neq 0$. Then for sufficiently small h the following error estimate holds*

$$0 \leq \lambda^h - \lambda \leq ch^{2n}.$$

Proof. We have $(\gamma^h(\mu))' \to \gamma'(\mu)$ as $h \to 0$, since

$$(\gamma^h(\mu))' = a'(\mu, v^h, v^h) - \gamma^h(\mu)b'(\mu, v^h, v^h) \to a'(\mu, v, v) - \gamma(\mu)b'(\mu, v, v) = \gamma'(\mu)$$

as $h \to 0$, $\mu \in \Lambda$, $v^h = u_\mu^h$, $v = u_\mu$, where $\gamma^h(\mu) \to \gamma(\mu)$, $u_\mu^h \to u_\mu$ in V as $h \to 0$.

For fixed $\varepsilon > 0$, we conclude $\lambda^h \in [\lambda - \varepsilon, \lambda + \varepsilon]$ for sufficiently small h and there exists $\xi^h \in [\lambda - \varepsilon, \lambda + \varepsilon]$ such that

$$c_1(\lambda^h - \lambda) \leq -(\gamma^h(\xi^h))'(\lambda^h - \lambda) = \gamma^h(\lambda) - \gamma^h(\lambda^h) = \gamma^h(\lambda) - \gamma(\lambda) \leq c_2 h^{2n}$$

for sufficiently small h, since $-(\gamma^h(\mu))' \geq c_1$ for $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon]$ and sufficiently small h, $\gamma^h(\lambda^h) = \gamma(\lambda) = 1$. This proves the theorem. □

Theorem 8. *Assume that $\gamma'(\lambda) \neq 0$. Let u be the positive eigenfunction of problem (2), and let u^h be approximate eigenfunction of problem (6). Then for sufficiently small h the following error estimate holds*

$$|u^h - u|_1 \leq ch^n.$$

Proof. Denote $\beta_i^h = b(\lambda^h, P_h u, y_i^h)$, $i = 1, 2, \ldots, N_h$, where $y_i^h = u_i^h(\lambda^h)$, $i = 1, 2, \ldots, N_h$. Since elements y_i^h, $i = 1, 2, \ldots, N_h$, form orthonormal basis in the Hilbert space V_h, it follows that the element $P_h u \in V_h$ can be represented in the form $P_h u = \beta_1^h y_1^h + w_1^h$, where $w_1^h = \beta_2^h y_2^h + \ldots + \beta_N^h y_N^h$.

The inequality $\gamma_2(\lambda) - \gamma_1(\lambda) > 0$ implies that

$$\gamma_2^h(\lambda^h) - 1 = \gamma_2^h(\lambda^h) - \gamma_1(\lambda) = (\gamma_2(\lambda) - \gamma_1(\lambda)) + (\gamma_2^h(\lambda^h) - \gamma_2^h(\lambda))$$
$$+ (\gamma_2^h(\lambda) - \gamma_2(\lambda)) \geq (\gamma_2(\lambda) - \gamma_1(\lambda)) - ch^{2n} \geq c$$

for sufficiently small h. Set

$$\zeta_h(u) = \sup_{v^h \in V_h \setminus \{0\}} \frac{|a(\lambda^h, P_h u, v^h) - \lambda b(\lambda^h, P_h u, v^h)|}{|v^h|_1}.$$

Then we have the estimate $\zeta_h(u) \leq ch^n$.

Let us prove the estimate $|w_1^h|_1 \leq c\zeta_h(u)$. It is readily seen that $a(\lambda^h, P_h u, w_1^h) = a(\lambda^h, w_1^h, w_1^h)$, $b(\lambda^h, P_h u, w_1^h) = b(\lambda^h, w_1^h, w_1^h)$, $a(\lambda^h, w_1^h, w_1^h) \geq \gamma_2^h(\lambda^h) b(\lambda^h, w_1^h, w_1^h)$. These relations imply the inequalities

$$|w_1^h|\zeta_h(u) \geq a(\lambda^h, P_h u, w_1^h) - b(\lambda^h, P_h u, w_1^h)$$
$$= a(\lambda^h, w_1^h, w_1^h) - b(\lambda^h, w_1^h, w_1^h) \geq \frac{\gamma_2^h(\lambda^h) - 1}{\gamma_2^h(\lambda^h)} a(\lambda^h, w_1^h, w_1^h) \geq c^{-1}|w_k^h|_1^2,$$

which in turn imply the required estimate.

Now, using the estimates we have proved, we arrive at the inequalities

$$|P_h u - \beta_1^h y_1^h|_1 = |w_1^h|_1 \le c\zeta_h(u) \le ch^n$$

for sufficiently small h.

Denote $||v||_{b(\mu)} = b(\mu, v, v)$. Then we derive

$$\beta_1^h = ||\beta_1^h y_1^h||_{b(\lambda^h)} \le ||u||_{b(\lambda)} + \Big| ||u||_{b(\lambda)} - ||u||_{b(\lambda^h)} \Big| + ||u - \beta_1^h y_1^h||_{b(\lambda^h)} \le 1 + ch^n,$$

$$\beta_1^h = ||\beta_1^h y_1^h||_{b(\lambda^h)} \ge ||u||_{b(\lambda)} - \Big| ||u||_{b(\lambda)} - ||u||_{b(\lambda^h)} \Big| - ||u - \beta_1^h y_1^h||_{b(\lambda^h)} \ge 1 - ch^n,$$

where we have taken into account that

$$\Big| ||u||_{b(\lambda)} - ||u||_{b(\lambda^h)} \Big| \le \frac{\Big| ||u||_{b(\lambda)}^2 - ||u||_{b(\lambda^h)}^2 \Big|}{||u||_{b(\lambda)} + ||u||_{b(\lambda^h)}} \le c(\lambda^h - \lambda) \le ch^{2n},$$

$$||u - \beta_1^h y_1^h||_{b(\lambda^h)} \le c|u - \beta_1^h y_1^h|_1 \le c(|u - P_h u|_1 + |P_h u - \beta_1^h y_1^h|_1) \le ch^n.$$

Consequently, we obtain $|1 - \beta_1^h| \le ch^n$. As a result we conclude

$$\begin{aligned} c|u^h - u|_1 &\le ||u - y_1^h||_{a(\lambda^h)} \le ||u - \beta_1^h y_1^h||_{a(\lambda^h)} + ||y_1^h - \beta_1^h y_1^h||_{a(\lambda^h)} \\ &\le ||u - \beta_1^h y_1^h||_{a(\lambda^h)} + c|1 - \beta_1^h| \le ch^n. \end{aligned}$$

This proves the theorem. □

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