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Some Spaces of Harmonic Functions in the Unit Ball of $\mathbb{R}^n$

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Abstract—We introduce the Banach spaces $h_\infty(\varphi)$, $h_0(\varphi)$ and $h^1(\psi)$ functions harmonic in the unit ball $B \subset \mathbb{R}^n$. These spaces depend on weight functions φ, ψ. We prove that if φ and ψ form a normal pair, then $h^1(\psi)^* \sim h_\infty(\varphi)$ and $h_0(\varphi)^* \sim h^1(\psi)$.

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1. INTRODUCTION

A positive continuous decreasing function φ on $[0,1)$ is called weight function if $\lim_{r\to 1}\varphi(r) = 0$. Let $h_\infty(\varphi)$ be the Banach space of complex-valued harmonic functions u on the unit disc with the norm $||u||_\varphi = \sup\{u(z)\varphi(|z|)\colon\ |z| < 1\}$, and $h_0(\varphi)$ be the closed subspace of functions u satisfying $|u(z)| = o\big(1/\varphi(|z|)\big)$ as $|z| \to 1$.

It has been shown by Rubel and Shields [1], that $h_\infty(\varphi)$ is isometrically isomorphic to the second dual of $h_0(\varphi)$. In [2] it was posed and solved the duality problem: for which φ there exists a finite positive Borel measure $d\eta$ on $[0,1)$ such that the space $h^1(\eta) = \big\{v \in L^1(d\eta(r)d\theta)\colon\ v \text{ is harmonic in } |z| < 1\big\}$ represents the intermediate space, the dual of $h_0(\varphi)$ and the predual of $h_\infty(\varphi)$, i.e. $h^1(\eta) \sim h_0(\varphi)^*$ and $h^1(\eta)^* \sim h_\infty(\varphi)$. A positive finite Borel measure η on $[0,1)$ is called weighting measure, if it is not supported on any subinterval $[0,\rho)$, $0 < \rho < 1$.

In the mentioned article [2] only the case $n = 2$ is considered. In the particular case, when φ is a normal function, we can take $d\eta(r) = \psi(r)dr$, where ψ is the paired with respect to weight function φ. This uses the fact that the solvability of the duality problem for the analytic spaces implies also solvability for the harmonic spaces. For the analytic spaces duality problem in the case of normal function was solved in [3].

In the multidimensional case ($n > 2$) a connection between the analytic and the harmonic functions does not exist and it is necessary to solve the problem of duality directly.

Our solving is based on the fact that there are bounded projections from $L^1(B)$ onto the subspace isometric isomorphic to $h^1(\psi)$, and bounded projection from the space $C_0(B)$ to the subspace, isometric isomorphic to $h_0(\varphi)$. Here $C_0(B)$ denotes the Banach space of continuous functions on the closed ball that vanish on the boundary with the supremum norm (for details see [4]).

Below the following notations are used: $B = \{x \in \mathbb{R}^n\colon\ |x| < 1\}$ stands for the open unit ball in $\mathbb{R}^n$; $S = \{x \in \mathbb{R}^n\colon\ |x| = 1\}$ for its boundary, i.e. the unit sphere in $\mathbb{R}^n$; $h(B)$ denotes the vector space of functions harmonic in B; $d\nu$ denotes normalized measure in $\mathbb{R}^n$, i.e. $\nu(B) = 1$; σ denotes normalized area element on S, i.e. $\sigma(S) = 1$.

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2. THE SPACES OF HARMONIC FUNCTIONS

Let the functions $\varphi(r)$ and $\psi(r)$ are positive and continuous on $[0,1)$ and

$$\lim_{r\to 1}\varphi(r)=0 \quad \text{and} \quad \int\limits_0^1 \psi(r)dr<\infty. \tag{1}$$

Among these functions we choose so-called normal functions and normal pairs.

Definition 1. The positive and continuous on $[0,1)$ function φ is called *normal,* if there are three constants $0<a<b$ and $0\le r_0<1$ such that

$$\frac{\varphi(r)}{(1-r)^a} \quad \text{almost decreases in} \quad r_0\le r<1 \quad \text{and} \quad \lim_{r\to 1^-}\frac{\varphi(r)}{(1-r)^a}=0,$$
$$\frac{\varphi(r)}{(1-r)^b} \quad \text{almost increases in} \quad r_0\le r<1 \quad \text{and} \quad \lim_{r\to 1^-}\frac{\varphi(r)}{(1-r)^b}=\infty.$$

Note that a and b are not uniquely determined by φ.

Recall that a real function $g(r)$ is called almost increasing, if there exists a constant $C>0$ such that $g(r_1)\le Cg(r_2)$ for any $r_1<r_2$. Almost decreasing functions are defined similarly.

Definition 2. The functions $\{\varphi,\psi\}$ will be called *a normal pair* if φ is normal and if there is a number α (the pair index) such that

$$\varphi(r)\psi(r)=(1-r^2)^\alpha, \quad 0\le r<1, \quad \text{and} \quad \int\limits_0^1\psi(r)dr<\infty.$$

Lemma 1. *If φ is normal, then there exists a function ψ such that $\{\varphi,\psi\}$ is a normal pair.*

Proof. Choose b such that $\varepsilon(r)=(1-r)^b/\varphi(r)\to 0$. Then the function

$$\psi(r)=(1-r^2)^\alpha/\varphi(r)=\varepsilon(r)(1-r^2)^\alpha/(1-r)^b$$

will be integrable, if we choose $\alpha>b-1$. □

Note, that if $\alpha>b$, then ψ is normal too, with corresponding indices $\alpha-b$ and $\alpha-a$.

Let φ and ψ satisfy (1). We extend these functions to B by defining $\varphi(x)=\varphi(|x|)$ and $\psi(x)=\psi(|x|)$ and define the spaces of harmonic functions:

$$h_\infty(\varphi)=\Big\{u\in h(B)\colon\ ||u||_\varphi=\sup_{x\in B}|u(x)|\varphi(x)<\infty\Big\};$$
$$h_0(\varphi)=\big\{u\in h(B)\colon\ \lim_{|x|\to 1}|u(x)|\varphi(x)=0\big\};$$
$$h^1(\psi)=\Big\{u\in h(B)\colon\ ||u||_\psi=\int\limits_B|u(x)|\psi(x)d\nu(x)<\infty\Big\}.$$

Clearly $h_0(\varphi)\subset h_\infty(\varphi)$ so we can use the norm $||u||_\varphi$ on $h_0(\varphi)$. These three spaces are all normed linear spaces with the indicated norms. In [4] it is also proved their completeness.

3. REPRODUCING KERNEL

Let $\{\varphi,\psi\}$ be a normal pair. Introduce duality (bilinear form) between $h_\infty(\varphi)$ and $h^1(\psi)$:

$$\langle u,v\rangle=\int\limits_B u(x)\overline{v(x)}\left(1-|x|^2\right)^\alpha d\nu(x), \quad u\in h_\infty(\varphi), \quad v\in h^1(\psi). \tag{2}$$

The function $Q_\alpha(x,y)$ is called a reproducing kernel for the bilinear form (2) if $\langle u(\cdot), Q_\alpha(x,\cdot)\rangle = u(x)$ for all $u \in h_\infty(\varphi) \cup h^1(\psi)$. For the first time, the expansion in the zonal harmonics for this kernel were obtained in [5] (for integer α) and [6] (for $\alpha \geq 0$). This expansion is

$$Q_\alpha(x,y) = \frac{2}{nV(B)} \sum_{m=0}^{\infty} \frac{\Gamma(n/2+m+\alpha+1)}{\Gamma(n/2+m)\Gamma(\alpha+1)} Z_m(x,y),$$

where the functions $Z_m(x,y)$ are the zonal harmonics of degree m.

Consider the integral linear operators

$$(Ph)(y) = \varphi(y) \int_B Q_\alpha(x,y)h(x)\psi(x)d\nu(x), \quad h \in L^\infty(B), \quad y \in B,$$

$$(Sf)(y) = \psi(y) \int_B Q_\alpha(x,y)f(x)\varphi(x)d\nu(x), \quad f \in L^1(B), \quad y \in B.$$

Theorem 1. *The following conditions are equivalent:*

(i) $||Q_\alpha(\cdot,y)||_\eta \leq \dfrac{c}{\varphi(y)}, y \in B$;

(ii) *P is a bounded projection of $L^\infty(B)$ onto the subspace $\varphi h_\infty(\varphi)$;*

(iii) *S is a bounded projection of $L^1(B)$ onto the subspace $\psi h^1(\psi)$;*

(iv) $h^1(\psi)^* \sim h_\infty(\varphi)$;

(v) $h_0(\varphi)^* \sim h^1(\psi)$.

This Theorem is proved in [4] for the more general case, when φ is not necessarily a normal function and ψ is the derivative of the weighting measure $d\eta$, i.e. $d\eta = \psi dr$.

4. THE MAIN RESULT

We need the following lemma, which has a technical character.

Lemma 2. *For* $\gamma > -1$ and $m > 1+\gamma$ *we have*

$$\int_0^1 (1-\rho r)^{-m}(1-r)^\gamma dr \leq c(1-\rho)^{1+\gamma-m}, \quad 0 < \rho < 1. \tag{3}$$

Proof. Integrating by parts, we obtain

$$\int_0^1 (1-\rho r)^{-m}(1-r)^\gamma dr = \frac{1}{\gamma+1} + \frac{m}{\gamma+1}\rho \int_0^1 (1-\rho r)^{-m-1}(1-r)^{\gamma+1} dr$$

$$\leq c_1 + c_2\rho \int_0^1 (1-\rho r)^{-m+\gamma} dr,$$

since $(1-r)^{\gamma+1} \leq (1-\rho r)^{\gamma+1}$. The result follows. □

The following lemma is basic.

Lemma 3. *If $\{\varphi,\psi\}$ is a normal pair and if $\alpha+\beta-b > -1$ and $m \geq 1+\alpha+\beta$, then*

$$\int_0^1 (1-\rho r)^{-m}(1-r)^\beta \psi(r) dr \leq c(1-\rho)^{1+\alpha+\beta-m}/\varphi(\rho), \quad 0 \leq \rho < 1.$$

Proof. With the notation in Definitions 1 and 2 we have

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\beta}\psi(r)dr = \int_0^{r_0} + \int_{r_0}^1 = I_1 + I_2.$$

The first integral is bounded for $0 \le \rho < 1$. As for the second integral, for $\rho > r_0$ we have

$$I_2 = \int_{r_0}^1 (1-\rho r)^{-m}(1-r)^{\beta}\left(1-r^2\right)^{\alpha}/\varphi(r)dr = \int_{r_0}^{\rho} + \int_{\rho}^1$$

$$\le c\int_{r_0}^{\rho}(1-\rho r)^{-m}(1-r)^{\alpha+\beta-a}(1-r)^a/\varphi(r)dr + c\int_{\rho}^1 (1-\rho r)^{-m}(1-r)^{\alpha+\beta-b}(1-r)^b/\varphi(r)dr$$

$$\le c\frac{(1-\rho)^a}{\varphi(\rho)}\int_{r_0}^{\rho}(1-\rho r)^{-m}(1-r)^{\alpha+\beta-a}dr + c\frac{(1-\rho)^b}{\varphi(\rho)}\int_{\rho}^1 (1-\rho r)^{-m}(1-r)^{\alpha+\beta-b}dr,$$

and the result follows from Lemma 2, since $\alpha+\beta-a > \alpha+\beta-b > -1$ and $m > 1+\alpha+\beta-a > 1+\alpha+\beta-b$. The constant c is not necessarily the same in different expressions. □

The following Theorem is the main result of this paper.

Theorem 2. *If $\{\varphi,\psi\}$ is a normal pair, then $h^1(\psi)^* \sim h_\infty(\varphi)$ and $h_0(\varphi)^* \sim h^1(\psi)$.*

Proof. As follows from Theorem 1, it is sufficient to prove the inequality

$$\int_B |Q_\alpha(x,y)|\psi(x)d\nu(x) \le c/\varphi(y), \quad y \in B. \tag{4}$$

In the multidimensional case, $Q_\alpha(x,y)$ cannot be written in the explicit form, but for our purposes it is sufficient the estimate

$$|Q_\alpha(x,y)| \le C/|x-y|^{n+\alpha}, \quad |x|<1, \quad |y|=1, \tag{5}$$

obtained in [7] (Lemma 2.7).

Let $x = rx'$ and $y = \rho y'$ be the polar form of points $x, y \in B$ (i.e. $x', y' \in S$, $0 < r, \rho < 1$). Passing to the polar coordinates (taking into account that $d\nu(x) = nr^{n-1}drd\sigma(x')$) and using the estimate (5), we get

$$\int_B |Q_\alpha(x,y)|\psi(x)d\nu(x) = n\int_0^1\int_S |Q_\alpha(x,y)|\psi(r)r^{n-1}drd\sigma(x')$$

$$\le C\int_0^1 \psi(r)r^{n-1}dr\int_S \frac{d\sigma(x')}{|r\rho x'-y'|^{n+\alpha}}. \tag{6}$$

As is known, for any $m > n-1$

$$\int_S \frac{d\sigma(y)}{|x-y|^m} \le \frac{C}{(1-|x|)^{m-n+1}}, \quad |x|<1$$

(see, for instance, [7] (Lemma 2.9)). From this we get

$$\int_S \frac{d\sigma(x')}{|r\rho x'-y'|^m} \le \frac{C}{(1-r\rho)^{m-n+1}}, \tag{7}$$

independently of $y' \in S$ (because the measure $d\sigma(x')$ is invariant with respect to rotations). Therefore, taking into account (7), the inner integral in the right-hand side of (6) can be estimated by

$$\int_S \frac{d\sigma(x')}{|r\rho x' - y'|^{n+\alpha}} \le \frac{C}{(1-r\rho)^{1+\alpha}}. \tag{8}$$

It follows from (6) and (8) that

$$\int_B |Q_\alpha(x,y)|\psi(x)d\nu(x) \le C\int_0^1 (1-r\rho)^{-1-\alpha}\psi(r)dr. \tag{9}$$

Applying Lemma 3 for $m = 1+\alpha$ and $\beta = 0$ to the right side of (9), we obtain the required inequality (4). □

REFERENCES

1. L. A. Rubel and A. L. Shields, "The second duals of certain spaces of analytic functions," J. Austral. Math. Soc. **11**, 276–280 (1970).
2. A. L. Shields and D. L. Williams, "Bounded projections, duality and multipliers in spaces of harmonic functions," J. Reine Angew. Math. **299–300**, 256–279 (1978).
3. A. L. Shields and D. L. Williams, "Bounded projections, duality and multipliers in spaces of analytic functions," Trans. Am. Math. Soc. **162**, 287–302 (1971).
4. A. I. Petrosyan and E. S. Mkrtchyan, "Duality in spaces of functions harmonic in the unit ball," Proc. YSU, Phys. Math. Sci., No. 3, 28–35 (2013).
5. R. Coifman and R. Rochberg, "Representation theorems for holomorphic and harmonic functions in L^p," Asterisque **77**, 11–66 (1980).
6. A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of* A^p_α *Spaces* (Teubner, Leipzig, 1988).
7. M. Jevtić and M. Pavlović, "Harmonic Bergman functions on the unit ball in $\mathbb{R}^n$," Acta Math. Hung. **85**, 81–96 (1999).