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On Generalized Solutions of Problems of Electromagnetic Wave Diffraction by Screens in the Closed Cylindrical Waveguides

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Abstract—In this article we propose to use special classes of the generalized functions in order to state the correct statement of some diffraction problems of electromagnetic waves by thin conducting screens in the cylindrical waveguides with conducting walls. As the generalized solutions, such mappings are considered which assign a linear functional defined on the linear shell of the set of the functions satisfying the corresponding boundary conditions to every value of longitudinal space coordinate. The traces of the solutions on the cross-section of the cylindrical domain are interpreted in the generalized sense. The infinite sets of linear algebraic equations are derived immediately from the generalized boundary conditions. We show that it is advisable to use the boundary conditions for the normal components of the electromagnetic field.

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1. INTRODUCTION

We suppose that the eigen wave propagates in the cylindrical waveguide with conducting walls. If an infinite thin and ideally conducting plate (screen) is placed on the lateral sections of the waveguide, then electromagnetic field arises in the form of the wave outgoing at infinity on two sides of the screen. The mathematical statement of this problem is the following. We seek the solutions of the set of Maxwell equations on two sides of the screen satisfying the boundary conditions on the walls of the waveguide, the radiation conditions and boundary conditions in the plane of the screen.

Let the cylindrical waveguide be placed along the axis z of Cartesian coordinates, and the lateral screen be located in the plane $z = 0$. As it is known (the first publication in this area is the article [1]), the longitudinal components of electric and magnetic vectors of the field can be represent inside the bounded by coordinate z area in the form

$$H_z(x,y,z) = \sum_{m=0}^{+\infty} \lambda_m \varphi_m(x,y) \Big[a_m^- e^{-i\gamma_m z} + a_m^+ e^{i\gamma_m z}\Big],$$
$$E_z(x,y,z) = \sum_{m=0}^{+\infty} \chi_m \psi_m(x,y) \Big[b_m^- e^{-i\delta_m z} + b_m^+ e^{i\delta_m z}\Big], \qquad (1)$$

where $\gamma_m = \sqrt{k^2 - \lambda_m}$, $\delta_m = \sqrt{k^2 - \chi_m}$ and $\varphi_m(x,y)$, $\psi_m(x,y)$ are eigen values and eigen functions of the spectral problems

$$\psi_m(x,y), \quad m = 0,1,\dots \quad | \quad \Delta\psi_m + \chi_m\psi_m = 0, \quad \psi_m\big|_C = 0,$$

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$$\varphi_m(x,y), \quad m=0,1,\dots \quad | \quad \Delta\varphi_m + \lambda_m\varphi_m = 0, \quad \frac{\partial\varphi}{\partial\nu}\Big|_C = 0.$$

We suppose the piecewise smooth bound C of the cross-section is such that these spectral problems have the complete sets of the eigen functions. In addition, let these functions be orthonormal. Either the longitudinal propagate constants γ_m and δ_m are positive, or they have the positive imaginary parts.

The coefficients $a_m^{\pm}$ and $b_m^{\pm}$ determine the electric and magnetic wave in the complete field respectively. The solution of Maxwell equations satisfy the radiation condition in the semi-infinite area $z>0$ if $a_m^- = 0$ and $b_m^- = 0$. Analogously, the solution of Maxwell equations satisfy the radiation condition in the area $z<0$ if $a_m^+ = 0$ and $b_m^+ = 0$.

The tangential components of the field are represented by potential functions E_z and H_z in the following way (see, for example, [2])

$$E_x^{\mp} = i\omega\mu_0\mu\sum_m a_m^{\mp}\frac{\partial\varphi_m}{\partial y}e^{\mp i\gamma_m z} \mp i\sum_m b_m^{\mp}\delta_m\frac{\partial\psi_m}{\partial x}e^{\mp i\delta_m z},$$

$$E_y^{\mp} = -i\omega\mu_0\mu\sum_m a_m^{\mp}\frac{\partial\varphi_m}{\partial x}e^{\mp i\gamma_m z} \mp i\sum_m b_m^{\mp}\delta_m\frac{\partial\psi_m}{\partial y}e^{\mp i\delta_m z},$$

$$H_x^{\mp} = \mp i\sum_m a_m^{\mp}\gamma_m\frac{\partial\varphi_m}{\partial x}e^{\mp i\gamma_m z} - i\omega\varepsilon_0\varepsilon\sum_m b_m^{\mp}\frac{\partial\psi_m}{\partial y}e^{\mp i\delta_m z},$$

$$H_y^{\mp} = \mp i\sum_m a_m^{\mp}\gamma_m\frac{\partial\varphi_m}{\partial y}e^{\mp i\gamma_m z} + i\omega\varepsilon_0\varepsilon\sum_m b_m^{\mp}\frac{\partial\psi_m}{\partial x}e^{\mp i\delta_m z}. \tag{2}$$

If $z \neq 0$ then the series in the expressions of the components of the field converge for any bounded sequences of the coefficients $a_m^{\pm}$ and $b_m^{\pm}$ because the functions $e^{\mp i\gamma_m z}$ and $e^{\mp i\delta_m z}$ decrease exponentially. The limiting process under $z \to 0$ is correct if some additional restrictions on the classes of unknown solutions of Maxwell equations are imposed.

We will show that these restrictions can be removed if we replace the classical statement with the diffraction problem to a generalized statement. We will consider the mappings which assign a linear functional defined on the linear shell of the set of the functions satisfying the corresponding boundary conditions to every value of longitudinal space coordinate.

2. GENERALIZED FUNCTIONS ON THE SEGMENT

We choose the set of all linear combinations of the functions $\varphi_n(x) = \sqrt{2/a}\sin(\pi n x/a)$, $n = 1,2,\dots$, as the space of basic functions. The generalized functions are defined as linear functionals on the space of basic functions $\varphi(\cdot)$. The analogous approach was used in the theory φ-distributions [3].

The linear functional $f[\cdot]$ on the space of basic functions is determined uniquely by set of numbers $f_n = f[\varphi_n(\cdot)]$, $n=1,2,\dots$. Therefore, the generalized functions are identified with sequences of the numbers f_n. The sequence of Fourier coefficients

$$f_n = \int_0^a f(x)\varphi_n(x)\,dx, \quad n=1,2,\dots,$$

corresponds to functions $f(\cdot)$ integrable on the segment $[0,a]$. We call the generalized function regular if usual function expanded into Fourier series corresponds to it.

We will consider the formal series

$$\sum_{n=1}^{+\infty} f_n\varphi_n(\cdot)$$

just as a generalized function. Such record is only the list of the basic functions and of the values of generalized functions on these functions.

The generalized derivatives are defined in the following way (further only the second derivatives will be used).

If $f(\cdot)$ is twice differentiable on $[0,a]$ function and $f(0)=0$, $f(a)=0$, then

$$\int\limits_0^a f''(x)\varphi_n(x)\,dx = -\Big(\frac{\pi n}{a}\Big)^2\int\limits_0^a f(x)\varphi_n(x)\,dx.$$

Therefore, we call the second order derivative of generalized function $f[\cdot]$ such generalized function $g[\cdot]$ that

$$g[\varphi_n(\cdot)] = -\Big(\frac{\pi n}{a}\Big)^2 f[\varphi_n(\cdot)] \quad \forall n=1,2,\ldots$$

We will say that the sequence of generalized functions $f_j[\cdot]$ converges under $j\to+\infty$ to generalized function $f[\cdot]$ if the numerical sequence $f_j[\varphi(\cdot)]$ converges to number $f[\varphi(\cdot)]$ for every basic function $\varphi(\cdot)$. It is sufficient to consider only functions $\varphi_n(\cdot)$ in the set of all basic functions. If the parameter set of generalized functions $f_\alpha[\cdot]$ is given, then the limit pass under $\alpha\to\alpha_0$ is defined analogously.

The generalized functions $f[\cdot]$ and $g[\cdot]$ are equal if $f[\varphi_n(\cdot)] = g[\varphi_n(\cdot)]$ for every $n=1,2,\ldots$ (or $f_n = g_n$, $n=1,2,\ldots$). But as in the classical theory of distributions [4], the values of the generalized function are not defined at the separate points of the segment $[0,a]$.

Let the segment $[0,a]$ consist of two parts $\mathcal{M}$ and $\mathcal{N}$. Further the sets of numbers

$$I_{nm} = \int\limits_{\mathcal{M}} \varphi_n(x)\varphi_m(x)\,dx \quad \text{and} \quad J_{nm} = \int\limits_{\mathcal{N}} \varphi_n(x)\varphi_m(x)\,dx, \quad n,m=1,2,\ldots,$$

will be used. These integrals can be calculated easily, they are connected with each other by the formula $J_{nm} = \delta_{nm} - I_{nm}$.

We will say that the generalized function $f[\cdot]=0$ on $\mathcal{N}$ if

$$\sum_{m=1}^{+\infty} J_{nm} f_m = 0 \quad \forall n=1,2,\ldots$$

(more precisely, if all these series converge and their sums are equal to zero) or in the equivalent form

$$f_n = \sum_{m=1}^{+\infty} I_{nm} f_m \quad \forall n=1,2,\ldots.$$

We illustrate the sense of this definition. If $f[\cdot]$ is a regular generalized function and $f(\cdot)=0$ on $\mathcal{N}$ then

$$f(x) = \sum_{m=1}^{+\infty} f_m\varphi_m(x), \quad \text{where} \quad f_m = \int\limits_{\mathcal{M}} f(x)\varphi_m(x)\,dx.$$

Hence

$$f_n = \int\limits_{\mathcal{M}} \Big(\sum_{m=1}^{+\infty} f_m\varphi_m(x)\Big)\varphi_n(x)\,dx = \sum_{m=1}^{+\infty} f_m I_{mn}.$$

The equality of two generalized functions on $\mathcal{N}$ or on $\mathcal{M}$ can be defined analogously.

It is easy to see that if $f[\cdot]=g[\cdot]$ on $\mathcal{N}$ and on $\mathcal{M}$ then $f_n = g_n$ $\forall n=1,2,\ldots$

Finally, we will say that generalized function $f[\cdot]$ has on $\mathcal{M}$ the same values as the usual function $g(\cdot)$ if

$$f_n = \int\limits_{\mathcal{M}} g(x)\varphi_n(x)\,dx, \quad n=1,2,\ldots,$$

or if $f_n = g_n$, where $g[\cdot]$ is the regular generalized function corresponding to usual function $g(\cdot)$, which is equal to zero on the other part of the segment $[0,a]$.

3. CONJUNCTION PROBLEM FOR HELMHOLTZ EQUATION IN THE STRIP

We consider firstly two-dimensional boundary value problem for Helmholtz equation in the stripe which corresponds to the problem of TM-polarized electromagnetic wave diffraction by the lateral screen in the place waveguide. In this case it is necessary to find the solutions of Helmholtz equation in the strip $0 < x < a$ separatively for $z > 0$ and for $z < 0$. These solutions should satisfy the radiation conditions and boundary conditions on section $z = 0$.

It is known that any function, which is twice continuously differentiable on $(0, a)$, is continuous on the segment $[0, a]$ and is equal to zero at the ends on the segment, by Steklov theorem can be expanded into uniformly convergent series of functions $\varphi_n(\cdot)$. Then any twice continuously differentiable solution of Helmholtz equation in the semi-strip $0 < x < a$, $z > 0$ which satisfies the radiation condition has the form

$$u^1(x, z) = \sum_{n=1}^{+\infty} a_n e^{i\gamma_n z}\varphi_n(x)$$

(the positively oriented solution). Analogously, every twice continuously differentiable and satisfying the radiation condition solution of Helmholtz equation in the semi-strip $0 < x < a$, $z < 0$ has the form

$$u^2(x, z) = \sum_{n=1}^{+\infty} b_n e^{-i\gamma_n z}\varphi_n(x)$$

(negatively oriented solution).

We will consider $u^1(x, z)$ and $u^2(x, z)$ as the functions of argument z. These functions have the generalized function belong to the space constructed above as their values. Their limit values (traces) under $t \to 0$ are denoted as $u^1_0[\cdot]$ and $u^2_0[\cdot]$. The values $u^1[\cdot](z)$ and $u^2[\cdot](z)$ at the basic function $\varphi_n(\cdot)$ for the fixed z are $u^1_n = a_n e^{i\gamma_n z}$ and $u^2_n = b_n e^{-i\gamma_n z}$. We pass to limit under $z \to 0$ and obtain $u^1_{0,n} = a_n$ and $u^2_{0,n} = b_n$. Thus, the traces (generalized limits) of the functions $u^1(x, z)$ and $u^2(x, z)$ on the cross-section $z = 0$ of the strip $0 < x < a$ are generalized functions. The traces of normal derivatives of the functions $u^1(x, z)$ and $u^2(x, z)$ on the cross-section $z = 0$ of the strip are also generalized functions, and their values are $u^1_{1,n} = i\gamma_n b_n$ and $u^2_{1,n} = -i\gamma_n b_n$.

We will state the problem of electromagnetic TM-wave diffraction by the lateral screen in the place waveguide in the following way. Let the part $\mathcal{M}$ of the segment $[0, a]$ correspond to the screen, and $\mathcal{N}$ be the free media interface. Let $u^0(x, z)$ be a potential function of the wave from an external source. If this wave consists of finite set of modes then its traces on the cross-section $z = 0$ are usual functions.

It is necessary to find the generalized solutions of Helmholtz equation $u^1[\cdot](\cdot)$ and $u^2[\cdot](\cdot)$ satisfying the boundary conditions and the conjunction conditions

$$\begin{gathered} u^0_0[\cdot] + u^1_0[\cdot] = 0, \quad u^0_0[\cdot] + u^2_0[\cdot] = 0 \quad \text{on} \quad \mathcal{M}, \\ u^1_0[\cdot] = u^2_0[\cdot], \quad u^1_1[\cdot] = u^2_1[\cdot] \quad \text{on} \quad \mathcal{N}. \end{gathered} \tag{3}$$

Here $u^0_0[\cdot]$ is the generalized function built by the usual functions which is equal to $u^0_0(\cdot)$ on $\mathcal{M}$ and to zero on $\mathcal{N}$.

As $u^1_0[\cdot] = u^2_0[\cdot]$ both on $\mathcal{N}$ and on $\mathcal{M}$, then, firstly, $a_n = b_n$ $\forall n = 1, 2, \ldots$ and, secondly, $u^1_1[\cdot] = -u^2_1[\cdot]$. Therefore, $u^1_1[\cdot] = u^2_1[\cdot] = 0$ on $\mathcal{N}$.

Thus, we have the boundary conditions in terms of generalized functions to determinate the unknown coefficients a_n:

$$u^1_0[\cdot] = -u^0_0[\cdot] \quad \text{on} \quad \mathcal{M}, \quad u^1_1[\cdot] = 0 \quad \text{on} \quad \mathcal{N}. \tag{4}$$

These conditions in the terms of the values of the generalized functions have the form of infinite set of linear algebraic equations (ISLAE):

$$a_k + u^0_{0,k} = \sum_{m=1}^{+\infty} J_{km} a_m, \quad k = 1, 2, \ldots \qquad \gamma_m a_m = \sum_{n=1}^{+\infty} I_{mn}\gamma_n a_n, \quad m = 1, 2, \ldots . \tag{5}$$

There exist two transforms of ISLAE (5) which (as it seems) "twice" decrease the number of the equations. If we insert the expressions a_m from the second set of equations into the right-hand sides of the equations from the first set of equations, then we obtain ISLAE

$$-a_k + \sum_{m=1}^{+\infty} J_{km} \frac{1}{\gamma_m} \sum_{n=1}^{+\infty} I_{mn}\gamma_n a_n = u^0_{0,k}, \quad k = 1, 2, \dots . \tag{6}$$

If we insert the expressions a_n from the first set of equations into the right-hand sides of the equations from the second set of equations, then we obtain

$$-\gamma_m a_m + \sum_{n=1}^{+\infty} I_{mn}\gamma_n \sum_{m=1}^{+\infty} J_{nm} a_m = -\sum_{m=1}^{+\infty} J_{nm} u^0_{n,0}, \quad m = 1, 2, \dots . \tag{7}$$

Numerical experiment shows that the approximate solutions of ISLAE (6) and (7) obtained by truncation method are very similar. But it was stated during the analysis of the numerical result that in the first case the conditions on $\mathcal{M}$ are satisfied more precisely, and in the second case the conditions on $\mathcal{N}$ are satisfied more precisely. Let's explain this effect.

Let the infinite matrixes A and B be composed of elements I_{mn} and J_{mn} respectively. These matrixes define some linear operators in the linear space of numerical sequences.

It is easy to prove that

$$\sum_{m=1}^{+\infty} I_{nm} I_{mk} = I_{nk}, \quad \sum_{m=1}^{+\infty} J_{nm} J_{mk} = J_{nk}, \quad \forall n, k = 1, 2, \dots .$$

Then A and B are complementary to each other orthogonal projectors, i.e. $A^2 = A$, $B^2 = B$ and $A + B = I$ (I is the unit operator), in addition, $AB = 0$, $BA = 0$.

Let $u = (a_1, a_2, \dots)$ and $u^0 = (u^0_{0,1}, u^0_{0,2}, \dots)$ be the infinite vectors, and C be the diagonal matrix consisting of elements $\gamma_1, \gamma_2, \dots$. Then the equations to determinate u have the form

$$u = Bu - Au^0, \quad u = C^{-1}ACu. \tag{8}$$

We note that the operators $C^{-1}AC$ and $C^{-1}BC$ are also complementary to each other orthogonal projectors.

If we insert the right-hand side of the second equation (8) into the right-hand side of the first equation instead u, then we obtain ISLAE (6) in the operator form

$$u = BC^{-1}ACu - Au^0. \tag{9}$$

It is clear that if u is the solution of the equations (8) then u is the solution of (9). Now let u be the solution of the equation (9). We apply the operator A and obtain $Au = -Au^0$. But if we apply the operator B then we obtain only $Bu = BC^{-1}ACu$ instead of $u = C^{-1}ACu$.

Analogously, if we insert the right-hand side of the first equation (8) into the right-hand side of the second equation instead of u, then

$$u = C^{-1}ACBu - C^{-1}ACAu^0$$

(it is ISLAE (7)). In this case we have again no complete equivalence to original equations.

To use all information contained in the original equations, we will minimize the discrepancy of all these equations. We write this condition "in coordinates":

$$\sum_{k=1}^{+\infty} \left| \sum_{m=1}^{+\infty} I_{km} a_m + u^0_k \right|^2 + \sum_{k=1}^{+\infty} \left| \sum_{m=1}^{+\infty} J_{km}\gamma_m a_m \right|^2 \to \min .$$

If we equate the derivatives by a_n to zero then we obtain ISLAE

$$\sum_{m=1}^{+\infty} \left[\sum_{k=1}^{+\infty} I_{km} I_{kn} + \sum_{k=1}^{+\infty} J_{km}\gamma_m J_{kn}\overline{\gamma_n} \right] a_m = -\sum_{k=1}^{+\infty} I_{kn} u^0_k, \quad k = 1, 2, \dots .$$

The approximate solution of this set of equations for the same order of truncation as in the case of ISLAE (6) and (7) gives the most accurate result.

It is possible that the accuracy of the calculations can be increased if we use the different numbers of equations from two groups (5) by analogy with method of solving the branching problem for the place waveguide [5]

4. THE SCREEN IN THE RECTANGULAR WAVEGUIDE

Let the rectangular waveguide have the cross-section $0 < x < a$, $0 < y < b$. In this case we need to determinate four spaces of basic functions and four spaces of the generalized functions.

The spaces on the basic functions consist of the linear combinations of the functions

$$\varphi^{ss}_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b}, \quad \varphi^{cs}_{mn}(x,y) = \frac{2}{\sqrt{ab}}\cos\frac{\pi mx}{a}\sin\frac{\pi ny}{b},$$

$$\varphi^{sc}_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\frac{\pi mx}{a}\cos\frac{\pi ny}{b}, \quad \varphi^{cc}_{mn}(x,y) = \frac{2}{\sqrt{ab}}\cos\frac{\pi mx}{a}\cos\frac{\pi ny}{b}.$$

Here and further the indexes m and n are changed in such way that all basic functions are not constant. The spaces of the generalized functions consist of the linear functionals on the spaces of the basic functions. We denote these spaces as SS, CS, SC, CC.

The values of the functional $f[\cdot] \in SS$ at the function $\varphi^{ss}_{mn}(\cdot)$, for example, is denoted as f^{ss}_{mn}.

We define the operation of the differentiation by x in the following way. This operation transfers the generalized function from one space to another space, and the values of the generalized functions at the basic functions are recalculated by the rule:

$$\left(\frac{\partial f}{\partial x}\right)^{cs}_{mn} = \frac{\pi m}{a} f^{ss}_{mn}; \quad \left(\frac{\partial f}{\partial x}\right)^{ss}_{mn} = -\frac{\pi m}{a} f^{ss}_{mn};$$

$$\left(\frac{\partial f}{\partial x}\right)^{cc}_{mn} = \frac{\pi m}{a} f^{sc}_{mn}; \quad \left(\frac{\partial f}{\partial x}\right)^{sc}_{mn} = -\frac{\pi m}{a} f^{cc}_{mn}.$$

The differentiation by y is defined analogously.

It is known (see, for example, [5]), that any non-oriented TM-wave of the rectangular waveguide has the components

$$E_z = \sum_{m,n}\Big(a_{mn}e^{i\gamma_{mn}z} + b_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{ss}_{mn}, \quad H_z = 0,$$

$$E_x = \sum_{m,n}\frac{i\gamma_{mn}}{\delta^2_{mn}}\frac{\pi m}{a}\Big(a_{mn}e^{i\gamma_{mn}z} - b_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{cs}_{mn},$$

$$E_y = \sum_{m,n}\frac{i\gamma_{mn}}{\delta^2_{mn}}\frac{\pi n}{b}\Big(a_{mn}e^{i\gamma_{mn}z} - b_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{sc}_{mn},$$

$$H_x = \sum_{m,n}\frac{-i\omega\varepsilon_0\varepsilon}{\delta^2_{mn}}\frac{\pi n}{b}\Big(a_{mn}e^{i\gamma_{mn}z} + b_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{sc}_{mn},$$

$$H_y = \sum_{m,n}\frac{i\omega\varepsilon_0\varepsilon}{\delta^2_{mn}}\frac{\pi m}{a}\Big(a_{mn}e^{i\gamma_{mn}z} + b_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{cs}_{mn},$$

where $\delta^2_{mn} = (\pi m/a)^2 + (\pi n/b)^2$.

We have also for the non-oriented TE-wave

$$E_z = 0, \quad H_z = \sum_{m,n}\Big(c_{mn}e^{i\gamma_{mn}z} + d_{mn}e^{-i\gamma_{mn}z}\Big)\varphi^{cc}_{mn},$$

$$E_x = \sum_{m,n} \frac{-i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi n}{b}\left(c_{mn}e^{i\gamma_{mn}z} + d_{mn}e^{-i\gamma_{mn}z}\right)\varphi_{mn}^{cs},$$

$$E_y = \sum_{m,n} \frac{i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi m}{a}\left(c_{mn}e^{i\gamma_{mn}z} + d_{mn}e^{-i\gamma_{mn}z}\right)\varphi_{mn}^{sc},$$

$$H_x = \sum_{m,n} \frac{-i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi m}{a}\left(c_{mn}e^{i\gamma_{mn}z} - d_{mn}e^{-i\gamma_{mn}z}\right)\varphi_{mn}^{sc},$$

$$H_y = \sum_{m,n} \frac{-i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi n}{b}\left(c_{mn}e^{i\gamma_{mn}z} - d_{mn}e^{-i\gamma_{mn}z}\right)\varphi_{mn}^{cs}.$$

It follows that the generalized limits under $z \to 0$ from the region $z > 0$ of the tangential components of the field for the positively oriented wave consist of two addends:

$$(E_x)_{mn}^{cs} = \frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi m}{a}a_{mn} + \frac{-i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi n}{b}c_{mn}, \quad (E_y)_{mn}^{sc} = \frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi n}{b}a_{mn} + \frac{i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi m}{a}c_{mn},$$

$$(H_x)_{mn}^{sc} = \frac{-i\omega\varepsilon_0\varepsilon}{\delta_{mn}^2}\frac{\pi n}{b}a_{mn} + \frac{-i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi m}{a}c_{mn}, \quad (H_y)_{mn}^{cs} = \frac{i\omega\varepsilon_0\varepsilon}{\delta_{mn}^2}\frac{\pi m}{a}a_{mn} + \frac{-i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi n}{b}c_{mn}.$$

We define the conditions when the generalized function is equal to zero on a part of the rectangle $[0, a] \times [0, b]$ analogously to the one-dimensional case. Let

$$I_{mnpq} = \int\limits_{\mathcal{M}} \varphi_{mn}(x, y)\varphi_{pq}(x, y)\, dx\, dy, \quad J_{mnpq} = \int\limits_{\mathcal{N}} \varphi_{mn}(x, y)\varphi_{pq}(x, y)\, dx\, dy$$

for every space of the generalized functions (here we do not write the upper indexes).

By definition, the generalized function $f[\cdot] = 0$ on $\mathcal{N}$ if

$$\sum_{m,n} J_{mnpq}f_{mn} = 0 \quad \forall p, q \quad \text{or} \quad f_{pq} = \sum_{m,n} I_{mnpq}f_{mn} \quad \forall p, q.$$

Consequently, as in the previous case, the boundary conditions of the form (3) for the problem of diffraction of the eigen wave of the rectangular waveguide on the lateral thin conducting screen $\mathcal{M}$ is reduced to the equalities of traces of the form (4)

$$E_{x,0}^0[\cdot] = -E_{x,0}^1[\cdot], \quad E_{y,0}^0[\cdot] = -E_{y,0}^1[\cdot] = 0 \quad \text{on} \quad \mathcal{M},$$

$$H_{x,0}^1[\cdot] = 0, \quad H_{y,0}^1[\cdot] = 0 \quad \text{on} \quad \mathcal{N}$$

or to ISLAE of the form (5)

$$\sum_{m,n} I_{mnpq}^{cs}\left[\frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi m}{a}a_{mn} - \frac{i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi n}{b}c_{mn}\right] = -(E_{x,0}^0)_{pq}^{cs} \quad \forall p, q,$$

$$\sum_{m,n} I_{mnpq}^{sc}\left[\frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi n}{b}a_{mn} + \frac{i\omega\mu_0\mu}{\delta_{mn}^2}\frac{\pi m}{a}c_{mn}\right] = -(E_{y,0}^0)_{pq}^{sc} \quad \forall p, q,$$

$$\sum_{m,n} J_{mnpq}^{sc}\left[\frac{i\omega\varepsilon_0\varepsilon}{\delta_{mn}^2}\frac{\pi n}{b}a_{mn} + \frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi m}{a}c_{mn}\right] = 0 \quad \forall p, q,$$

$$\sum_{m,n} J_{mnpq}^{cs}\left[\frac{i\omega\varepsilon_0\varepsilon}{\delta_{mn}^2}\frac{\pi m}{a}a_{mn} - \frac{i\gamma_{mn}}{\delta_{mn}^2}\frac{\pi n}{b}c_{mn}\right] = 0 \quad \forall p, q.$$

5. THE SCREEN IN THE WAVEGUIDE OF THE ARBITRARY SECTION

In the case of waveguide with metallic walls of the arbitrary section it is advisable to replace the formulations of the boundary conditions and conjunction conditions on the section of the waveguide to other conditions.

We define two spaces Φ and Ψ of the generalized functions as the sets of linear functionals on the linear combinations of the functions $\varphi_m(x,y)$ and $\psi_m(x,y)$. If we introduce further the spaces for the derivatives of these generalized functions, then the components E_x, E_y, H_x, H_y (2) will be composed from two addends which belong to different spaces. The case of rectangular waveguide is exclusive case.

Then we set the non-standard conditions on the cross-section $z=0$: let the components H_z, E_z, $\partial H_z/\partial z$, $\partial E_z/\partial z$ be continuous on the media interface and the components H_z and $\partial E_z/\partial z$ be equal to zero on the screen.

Indeed, if the components E_x, E_y, H_x, H_y satisfy some linear conditions on the media interface, then the sums and differences of their derivatives satisfy analogous conditions. But it follows from Maxwell equations that

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega\mu_0\mu H_z,\quad \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=-\frac{\partial E_z}{\partial z},$$

$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=-i\omega\varepsilon_0\varepsilon E_z,\quad \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=-\frac{\partial H_z}{\partial z}.$$

Therefore it is sufficient further to use only two spaces on generalized functions.

As a result, we set the generalized conditions on the sections $z=0$ for the traces of normal components of the field in the following form:

$$\text{on } \mathcal{M}\ :\ H^+_{z,0}[\cdot]+H^0_{z,0[\cdot]}=0,\quad H^-_{z,0}[\cdot]+H^0_{z,0}[\cdot]=0,$$

$$\Big(\frac{\partial E_z^+}{\partial z}\Big)_0[\cdot]+\Big(\frac{\partial E_z^0}{\partial z}\Big)_0[\cdot]=0,\quad \Big(\frac{\partial E_z^-}{\partial z}\Big)_0[\cdot]+\Big(\frac{\partial E_z^0}{\partial z}\Big)_0[\cdot]=0,$$

$$\text{on } \mathcal{N}\ :\ H^+_{z,0}[\cdot]=H^-_{z,0}[\cdot],\quad \Big(\frac{\partial E_z^+}{\partial z}\Big)_0[\cdot]=\Big(\frac{\partial E_z^-}{\partial z}\Big)_0[\cdot],$$

$$E^+_{z,0}[\cdot]=E^-_{z,0}[\cdot],\quad \Big(\frac{\partial H_z^+}{\partial z}\Big)_0[\cdot]=\Big(\frac{\partial H_z^-}{\partial z}\Big)_0[\cdot].$$

Consequently,

$$H^+_{z,0}[\cdot]=H^-_{z,0}[\cdot],\quad \Big(\frac{\partial E_z^+}{\partial z}\Big)_0[\cdot]=\Big(\frac{\partial E_z^-}{\partial z}\Big)_0[\cdot]$$

both on $\mathcal{M}$ and on $\mathcal{N}$. Then $a^+_m=a^-_m=a_m$ and $b^+_m=-b^-_m=b_m$, besides

$$\Big(\frac{\partial H_z^+}{\partial z}\Big)_0[\cdot]=-\Big(\frac{\partial H_z^-}{\partial z}\Big)_0[\cdot],\quad E^+_{z0}=-E^-_{z0}.$$

Therefore the diffraction problem on the screen is reduced to one-side boundary value problem

$$\text{on } \mathcal{M}\ :\ H^+_{z,0}[\cdot]=-H^0_{z,0}[\cdot],\quad \Big(\frac{\partial E_z^+}{\partial z}\Big)_0[\cdot]=-\Big(\frac{\partial E_z^0}{\partial z}\Big)_0[\cdot],$$

$$\text{on } \mathcal{N}\ :\ \Big(\frac{\partial H_z^+}{\partial z}\Big)_0[\cdot]=0,\quad E^+_{z,0}[\cdot]=0.$$

Now let

$$I^\varphi_{mp}=\int\limits_{\mathcal{M}}\varphi_m(x,y)\varphi_p(x,y)\,dx\,dy,\quad J^\varphi_{mp}=\int\limits_{\mathcal{N}}\varphi_m(x,y)\varphi_p(x,y)\,dx\,dy,$$

$$I^{\psi}_{mp} = \int\limits_{\mathcal{M}} \psi_m(x,y)\psi_p(x,y)\,dx\,dy, \quad J^{\psi}_{mp} = \int\limits_{\mathcal{N}} \psi_m(x,y)\psi_p(x,y)\,dx\,dy$$

(here all integrals are two-dimensional, but the functions φ and ψ have only one index). Then, as before, generalized function $f[\cdot] = 0$ on $\mathcal{N}$ if

$$\sum_m J_{mp} f_m = 0 \quad \forall p \quad \text{or} \quad f_p = \sum_m I_{mp} f_m \quad \forall p.$$

Finally, ISLAE of the diffraction problem has the form (the multipliers λ_m and χ_m are absent in the sums (1))

$$\sum_{m=0}^{+\infty} I_{mp} a_m = -(H^0_{z,0})_p, \quad \sum_{m=0}^{+\infty} I_{mp}\delta_m b_m = -\Big(\frac{\partial E^0_z}{\partial z}\Big)_{0,p},$$

$$\sum_{m=0}^{+\infty} J_{mp}\gamma_m a_m = 0, \quad \sum_{m=0}^{+\infty} J_{mp} b_m = 0, \quad p = 0, 1, \dots .$$

It is easy to see, that we have the independent equations for the unknown a_m and b_m.

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