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Monomial Ideals with Quasi-Linear Quotients

S. Nazir[1*], I. Anwar[2], and A. Ahmad[3***]**

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[1] *Lahore University of Management Sciences, Lahore, Punjab, 54792 Pakistan*

[2] *Abdus Salam School of Mathematical Sciences, GC University, Lahore, 54000 Pakistan*

[3] *COMSATS Institute of Information Technology, Lahore, Punjab, 54000 Pakistan*

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Abstract—In this paper, we extend the notion of quasi-linear quotients for a pure monomial ideal (not necessarily square-free) of degree d. We introduce the notion of quasi-linear free resolution and show that if a pure monomial ideal $I = (u_1, u_2, \ldots, u_m)$ of degree d in the polynomial ring $S = k[x_1, \ldots, x_n]$ admits quasi-linear quotients then $L_q = (u_1, \ldots, u_{q-1}) : u_q$ admits quasi-linear free resolution for all $q \leq m$. Moreover, we show that if a pure monomial ideal I of degree d admits quasi-linear quotients then $I_{\langle t \rangle}$ will also have quasi-linear quotients for $t \geq d$.

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1. INTRODUCTION

The class of square-free monomial ideals with quasi-linear quotients was introduced in [1]. A monomial ideal is said to have *quasi-linear quotients*, if there exists a minimal monomial system of generators $u_1, u_2, \ldots, u_m$ of I such that $\beta_{01}(L_k) \neq 0$ for all $1 < k \leq m$, where $L_k = (u_1, u_2, \ldots, u_{k-1}) : (u_k)$. The concept of shellability has been the subject of a considerable number of geometric and algebraic investigations for instance see [2–4], and [6]. The facet ideal $I_{\mathcal{F}}(\Delta)$ (introduced in [5]) of a pure shellable simplicial complex Δ admits quasi-linear quotients [2]. Moreover, the class of monomial ideals with linear quotients (introduced by Herzog and Takayama [7]) is contained in the class of monomial ideals with quasi-linear quotients.

The potential aim of this article is to study the monomial ideals with quasi-liner quotients. We introduce the notion of *quasi-linear free resolution* and show that if a pure monomial ideal $I = (u_1, u_2, \ldots, u_m)$ of degree d admits quasi-linear quotients then $L_q = (u_1, \ldots, u_{q-1}) : u_q$ admits *quasi-linear free resolution* for all $q \leq m$. We also show that if a pure monomial ideal I of degree d admits quasi-linear quotients then $I_{\langle t \rangle}$ will also have quasi-linear quotients for $t \geq d$.

*E-mail: shaheen.nazir@lums.edu.pk
**E-mail: iimrananwar@gmail.com
***E-mail: aniqa.mathematics@gmail.com

2. BACKGROUND

In this section, we define some basic notions that we will use in the paper along with some background theory.

Let $S = k[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over an infinite field k with each $\deg(x_i) = 1$ and I a monomial ideal of S. The graded Betti numbers $\beta_{i,j} = \beta_{i,j}(I)$ appear in the graded minimal free resolution

$$\mathfrak{F} : 0 \to \bigoplus_{j \in Z} S(-j)^{\beta_{h,j}} \to \cdots \to \bigoplus_{j \in Z} S(-j)^{\beta_{0,j}} \to I$$

of I in S. Then $\beta_{i,j}(I) = \dim_k \mathrm{Tor}_i(I, k)_j$ for all i and j. There are two important invariants attached to a monomial ideal $I \subset S$ defined in terms of the minimal graded free resolution of I:

$$\text{proj}\dim(I) = \max\{i : \beta_{i,j}(I) \neq 0\},$$

$$\mathrm{reg}(I) = \max\{j : \beta_{i,i+j}(I) \neq 0 \text{ for some } i\}$$

namely *projective dimension* and *regularity* of I respectively. We say that I has a **linear resolution** if there exists an integer $m \geq 1$ such that $\beta_{i,i+j} = 0$ for all i and j with $j \neq m$. Thus, in particular, if I has a linear resolution, then I is generated by homogeneous polynomials of the same degree.

Definition 2.1. *Let I be a monomial ideal in S and u be any monomial in S. Then the* colon ideal *of I with respect to (u) is defined as*

$$I : (u) = \{f \in S \mid fu \in I\}.$$

Here, we recall an important class of monomial ideals namely *ideals with linear quotients* along with some important results.

Definition 2.2. *Let $I \subset S = k[x_1, \ldots, x_n]$ be a monomial ideal, we say that I has* linear quotients*, if there exists a minimal monomial system of generators $u_1, u_2, \ldots, u_m$ of I such that L_k is generated by linear forms for all $1 < k \leq m$, where*

$$L_k = (u_1, u_2, \ldots, u_{k-1}) : (u_k).$$

Moreover for each k, we denote the number of generators of L_k by r_k or in other words $\mu(L_k) = r_k$.

One of the reasons that makes ideals with linear quotient an important subject for considerations is due to the following result [6, Proposition 8.2.1].

Theorem 2.3. *Suppose that $I \subset S$ is a graded ideal generated in degree d and that I has linear quotients. Then I has a d-linear resolution.*

3. PURE MONOMIAL IDEALS WITH QUASI-LINEAR QUOTIENTS

In this section, we will consider only monomial ideals with quasi-linear quotients in the polynomial ring $S = k[x_1, \ldots, x_n]$. We say that a monomial ideal $I \subset S$ is pure of degree d, if and only if $\beta_{0,j}(I) = 0$ for all $j \neq d$.

Definition 3.1. *Let I be a monomial ideal in S. We define the $\mathrm{indeg}(I)$ as follows*

$$\mathrm{indeg}(I) = \min\{j : \beta_{0,j}(I) \neq 0\}.$$

Definition 3.2. *Let $I \subset S$ be a monomial ideal, we say that I has* quasi-linear quotients*, if there exists a minimal monomial system of generators $u_1, u_2, \ldots, u_m$ of I such that* $\mathrm{indeg}(L_i) = 1$ *for all $1 < i \leq m$, where*

$$L_i = (u_1, u_2, \ldots, u_{i-1}) : (u_i).$$

Remark 3.3. *Let $I \subset S$ be a monomial ideal having* linear quotients*, then I has quasi-linear quotients as well. Therefore, the class of ideals with quasi-linear quotients is a generalization of the class of monomial ideals with linear quotients.*

Definition 3.4. *We say that I has a quasi-linear resolution if for each integer i with $0 \leq i <$ proj $\dim(I)$, there exists an integer j such that, $\beta_{i,j}(I) \neq 0$ implies $\beta_{i+1,j+1}(I) \neq 0$. Thus, in* *particular, if I will have a quasi-linear resolution, then there is at least one linear shift in each step of the minimal free resolution of I.*

The following example is given to show the various aspects of quasi-linear resolution of an ideal I.

Example 3.5. *Let us consider the polynomial ring $S = k[x_1, \ldots, x_7]$ and monomial ideal $I = (x_1x_2x_3, x_1x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7)$. Its minimal free resolution is as follows:*

$$0 \to S(-7) \to S(-4)^4 \oplus S(-6) \to S(-3)^5 \to I$$

with Betti diagram:

−	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0
3	5	4	0
4	0	0	0
5	0	1	1

One can see the linear shift in each step clearly.

Theorem 3.6. *Let $I \subset S = k[x_1, \ldots, x_n]$ be a pure monomial ideal of degree d, minimally generated by $\{u_1, u_2, \ldots, u_m\}$. If I has quasi-linear quotients, then $L_q = (u_1, u_2, \ldots, u_{q-1}) : u_q$ has a quasi-linear free resolution for all $1 < q \leq m$.*

Proof. Since I has quasi-linear quotients, so L_q has a variable as a generator, for all $1 < q \leq m$. To show L_q has a quasi-linear free resolution, it is enough to show that a monomial ideal containing a variable has a quasi-linear free resolution.
Let L be an ideal containing a variable. We may assume that $L = (J, x_1)$ with $x_1 \notin \mathrm{supp}(J)$. By using the definition of minimal graded free resolution, one may obtain a minimal graded free resolution of L from a minimal graded free resolution of J, as

$$\mathrm{Tor}_i^S(L,k)_j = \mathrm{Tor}_i^S(J,k)_j \oplus \mathrm{Tor}_{i-1}^S(J(-1),k)_j, \quad \text{for all } i \geq 1 \quad \text{and}$$
$$\mathrm{Tor}_0^S(L,k)_j = \mathrm{Tor}_0^S((x_1,k)_j \oplus \mathrm{Tor}_0^S(J,k)_j.$$

Now suppose that $\beta_{i+1}(L) \neq 0$. Then $\beta_i(J) \neq 0$. (In fact, if $\beta_i(J) = 0$, then $\beta_{i+1}(J) = 0$, which implies $\beta_{i+1}(L) = 0$, a contradiction.) In particular, there exists j such that $\beta_{i,j}(J) \neq 0$. The presence of $\mathrm{Tor}_i^S(J,k)_j \oplus \mathrm{Tor}_{i-1}^S(J(-1),k)_j$ ensures that $\beta_{i,j}(L) \neq 0$ and $\beta_{i+1,j+1}(L) \neq 0$. Therefore, L has a quasi-linear free resolution for all $k \leq m$. □

Here we give an example to demonstrate the step recursively mentioned in the proof of above theorem.

Example 3.7. Let $I = (a, b, cd, ce, de)$ *be a monomial ideal in the polynomial ring* $S = k[a, b, c, d, e]$. *Therefore, we have the following chain of ideals*

$$I_0 = (cd, ce, de) \subset I_1 = (a, I_0) \subset I_2 = I.$$

The Betti diagram of I_0 is given as:

−	0	1
0	0	0
1	0	0
2	3	2

The Betti diagram of I_1 can be obtained from I_0 (changes in shift are shown in bold entries):

−	0	1	2
0	0	0	0
1	**1**	0	0
2	3	$5 = \mathbf{3} + 2$	$2 = \mathbf{2} + 0$

The Betti diagram of $I_2 = I$ can be obtained from I_1 (changes in shift are shown in bold entries):

−	0	1	2	3
0	0	0	0	0
1	$\mathbf{1} + 1$	$1 = 1 + 0$	0	0
2	3	$8 = \mathbf{3} + 5$	$7 = \mathbf{5} + 2$	$2 = \mathbf{2} + 0$

One can notice the changes in the shifts by comparing the above Betti diagrams. It is clear from above that

$$\operatorname{Tor}_i^S(I_1, k)_j = \operatorname{Tor}_i^S(I_0, k)_j \oplus \operatorname{Tor}_{i-1}^S(I_0(-1), k)_j \quad \text{for} \quad i > 0$$

and

$$\operatorname{Tor}_i^S(I, k)_j = Tor_i^S(I_1, k)_j \oplus \operatorname{Tor}_{i-1}^S(I_1(-1), k)_j \quad \text{for} \quad i > 0.$$

Moreover,

$$\operatorname{Tor}_0^S(I_1, k)_j = \operatorname{Tor}_0^S((a), k)_j \oplus \operatorname{Tor}_0^S(I_0, k)_j$$

and

$$\operatorname{Tor}_0^S(I, k)_j = Tor_0^S((b), k)_j \oplus \operatorname{Tor}_0^S(I_1, k)_j.$$

For a pure monomial ideal $I \subset S$ of degree d, we have $I_{\langle t \rangle}$ is a monomial ideal generated by t degree monomials in I. In particular, $I_{\langle t \rangle} = I.(x_1, x_2, \ldots, x_n)^{t-d}$.

Theorem 3.8. *Let $I \subset S$ be a pure monomial ideal of degree d having quasi-linear quotients with respect to the generating set $G(I) = \{u_1, u_2, \ldots, u_m\}$. Then $I_{\langle t \rangle}$ will also have quasi-linear quotients for $t \geq d$.*

Proof. We prove our claim by giving the an ordered generating set of $I_{\langle t \rangle} = I.(x_1, x_2, \ldots, x_n)^{t-d}$ that admits quasi-linear quotients. We claim that $I_{\langle t \rangle}$ have quasi-linear quotients with respect to the following generating system.

$$\begin{gathered} v_1u_1, v_1u_2, \ldots, v_1u_m, \\ v_2u_1, v_2u_2, \ldots, v_2u_m, \\ \vdots \\ v_ru_1, v_ru_2, \ldots, v_ru_m, \end{gathered}$$

such that $r = \binom{n+t-d-1}{n-1}$. We order the above list of generators in lexicographic order with respect to v_i by ignoring the monomial u_j. One may observe that all the monomials in the above list are of type $v_i.u_j$ where v_i is a monomial of degree $t - d - 1$. Firstly, observe that if $u_1, \ldots, u_m$ admits quasi-linear quotients then $x_1^{t-d}u_1, x_1^{t-d}u_2, \ldots, x_1^{t-d}u_m$ also admits the quasi-linear quotients, therefore all the colon $L_{v_1u_k}$ will have a linear term for all $1 \leq k \leq m$. Now take any term from the list of monomials v_iu_k for $2 \leq i$ from the above list with the assumption that $v_i = x_i^p\mathbf{x}$ (with $\deg(\mathbf{x}) = t - d - p$ and $\operatorname{supp}(\mathbf{x}) = \{x_j \mid j > i\}$) then there exists a term $v_j = x_1x_i^{p-1}\mathbf{x}$ such that $v_i <_{lex} v_j$ (or equivalently $i > j$). Therefore, the colon ideal $L_{v_iu_k}$ will have a linear term $\frac{v_ju_k}{GCD(v_iu_k, v_ju_k)} = x_1$, as required. □

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