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On Continuous Multifunctions in Ideal Topological Spaces

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Abstract—The purpose of the present paper is to introduce the concepts of upper and lower $\star$-continuous multifunctions. Several characterizations of upper and lower $\star$-continuous multifunctions are investigated. The relationships between upper and lower $\star$-continuous multifunctions and the other types of continuity are discussed.

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1. INTRODUCTION

Continuity is a basic concept for the study in topological spaces. α-open sets, preopen sets, β-open sets and semi-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets several authors introduced and studied various types of weak forms of continuity for functions and multifunctions. Levine [1] introduced the notion of weakly continuous functions. Popa [2] and Smithson [3] independently introduced the concept of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly quasicontinuous multifunctions, almost weakly continuous multifunctions, weakly α-continuous multifunctions and weakly β-continuous multifunctions. These multifunctions have similar properties. The analogy in their definitions and results suggests the need of formulating a unified theory. Noiri and Popa [4] introduced the concepts of upper and lower M-continuous multifunctions as a multifunction defined between sets satisfying certain minimal conditions and obtained some characterizations of such multifunctions.

In 2006, Noiri and Popa [5] introduced the notions of upper and lower weakly m-continuous multifunctions as a multifunction from a set satisfying certain minimal conditions into a topological space. Singal and Singal [6] introduced the notion of almost continuous functions. Popa [7] introduced the concepts of upper and lower almost continuous multifunctions. The notions of upper and lower almost α-continuous multifunctions are introduced in [8] and the further properties are studied in [9]. Popa et al. [10] introduced and studied the concepts of upper and lower almost precontinuous multifunctions. The notions of upper and lower almost β-continuous multifunctions are introduced by Noiri and Popa [11]. Almost α-continuity, almost precontinuity and almost β-continuity for multifunctions have properties similar to these of almost continuous multifunctions. Further, analogies in their definitions and results suggest the need formulating a unified theory in the setting of multifunctions. In [12], the present authors introduced and studied the notion of almost m-continuous functions. Noiri and Popa [13] introduced the notions of upper and lower almost m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space and obtained several characterizations of such multifunctions. In 2009, Ekici et al. [14] introduced the concept of almost contra-continuous multifunctions and investigated several characterizations of almost contra-continuous multifunctions.

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The concept of ideals in topological spaces has been studied by Kuratowski [15] and Vaidyanathaswamy [16] which is one of the important areas of research in the branch of mathematics. Janković and Hamlett [17] introduced the notion of $\mathcal{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [18] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [19] introduced and investigated the notions of weakly-$\mathcal{I}$-continuous and weak*-$\mathcal{I}$-continuous functions in ideal topological spaces. Donthev [20] introduced the notion of pre-$\mathcal{I}$-open sets and obtained a decomposition of $\mathcal{I}$-continuity. Hatir and Noiri [21] introduced the notions of semi-$\mathcal{I}$-open sets, α-$\mathcal{I}$-open sets and β-$\mathcal{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [22], the present authors investigated further properties of semi-$\mathcal{I}$-open sets and semi-$\mathcal{I}$-continuity. Hatir et al. [23] introduced and investigated the notions of strong β-$\mathcal{I}$-open sets and strongly β-$\mathcal{I}$-continuous functions.

The paper is organized as follows. In Section 3, we introduce the notions of upper and lower $\star$-continuous multifunctions and investigate some characterizations of such multifunctions. Section 4 is devoted to introducing and studying upper and lower almost $\star$-continuous multifunctions. Finally, several interesting characterizations of upper and lower weakly $\star$-continuous multifunctions are discussed.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. For an ideal topological space $(X, \tau, \mathcal{I})$ and a subset A of X, $A^\star(\mathcal{I})$ is defined as follows: $A^\star(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathcal{I})$ is simply written as $A^\star$. In [15], $A^\star$ is called the local function of A with respect to $\mathcal{I}$ and τ and $Cl^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathcal{I})$ finer than τ. A subset A is said to be $\star$-closed [17] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathcal{I}))$ is denoted by $Int^\star(A)$.

Definition 2.1. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be:

(1) pre-$\mathcal{I}$-open [20] if $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(A))$;

(2) semi-$\mathcal{I}$-open [21] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$;

(3) semi*-$\mathcal{I}$-open [24] if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$;

(4) semi*-$\mathcal{I}$-closed [24] if its complement is semi*-$\mathcal{I}$-open;

(5) R–$\mathcal{I}$-open [25] if $A = \mathrm{Int}(\mathrm{Cl}^\star(A))$;

(6) R–$\mathcal{I}$-closed [25] if its complement is R–$\mathcal{I}$-open;

(7) strong β-$\mathcal{I}$-open [26] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$;

(8) $\beta^\star_\mathcal{I}$-open [27] if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A)))$.

Lemma 2.2. [28] *The following properties hold for a subset A of an ideal topological space* $(X, \tau, \mathcal{I})$.

(1) $s^\star_\mathcal{I}\mathrm{Cl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}^\star(A))$.

(2) $s_\mathcal{I}\mathrm{Cl}(A) = A \cup \mathrm{Int}^\star(\mathrm{Cl}(A))$.

(3) $p_\mathcal{I}\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}^\star(A))$.

(4) $s\beta_\mathcal{I}\mathrm{Cl}(A) = A \cup \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(A)))$.

Proposition 2.3. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* V *is open in* X*, then* $s^\star_\mathcal{I}\mathrm{Cl}(V) = \mathrm{Int}(\mathrm{Cl}^\star(V))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [29], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X | F(x) \subseteq B\}$ and $F^-(B) = \{x \in X | F(x) \cap B \neq \emptyset\}$. In particular,

$$F^-(y) = \{x \in X | y \in F(x)\}$$

for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

3. CHARACTERIZATIONS OF UPPER AND LOWER $\star$-CONTINUOUS MULTIFUNCTIONS

We begin this section by introducing the notions of upper and lower $\star$-continuous multifunctions.

Definition 3.1. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be:

(1) *upper $\star$-continuous if for each $x \in X$ and each $\star$-open set V containing $F(x)$, there exists a $\star$-open set U containing x such that $F(U) \subseteq V$;*

(2) *lower $\star$-continuous if for each $x \in X$ and each $\star$-open set V such that $F(x) \cap V \neq \emptyset$, there exists a $\star$-open set U containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

The following theorems give some characterizations of upper and lower $\star$-continuous multifunctions.

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper $\star$-continuous;*

(2) *$F^+(V)$ is $\star$-open in X for every $\star$-open set V in Y;*

(3) *$F^-(H)$ is $\star$-closed in X for every $\star$-closed set H in Y;*

(4) *$\mathrm{Cl}^\star(F^-(B)) \subseteq F^-(\mathrm{Cl}^\star(B))$ for every subset B of Y;*

(5) *$F^+(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^+(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set and $x \in F^+(V)$. Then $F(x) \subseteq V$, there exists a $\star$-open set U containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$. This implies that $x \in \mathrm{Int}^\star(F^+(V))$ and so $F^+(V) \subseteq \mathrm{Int}^\star(F^+(V))$. Consequently, we obtain $F^+(V)$ is $\star$-open.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $\mathrm{Cl}^\star(B)$ is $\star$-closed and by (3), we have $F^-(\mathrm{Cl}^\star(B))$ is $\star$-closed and hence, $\mathrm{Cl}^\star(F^-(B)) \subseteq \mathrm{Cl}^\star(F^-(\mathrm{Cl}^\star(B))) = F^-(\mathrm{Cl}^\star(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \mathrm{Int}^\star(F^+(B)) &= \mathrm{Cl}^\star(X - F^+(B)) = \mathrm{Cl}^\star(F^-(Y - B)) \\ &\subseteq F^-(\mathrm{Cl}^\star(Y - B)) = F^-(Y - \mathrm{Int}^\star(B)) = X - F^+(\mathrm{Int}^\star(B)). \end{aligned}$$

Therefore, we obtain $F^+(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\star$-open set containing $F(x)$. Then $x \in F^+(V) = F^+(\mathrm{Int}^\star(V)) \subseteq \mathrm{Int}^\star(F^+(V))$. There exists a $\star$-open set U containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\star$-continuous. $\square$

Theorem 3.3. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower $\star$-continuous;*

(2) *$F^-(V)$ is $\star$-open in X for every $\star$-open set V in Y;*

(3) *$F^+(H)$ is $\star$-closed in X for every $\star$-closed set H in Y;*

(4) *$\mathrm{Cl}^\star(F^+(B)) \subseteq F^+(\mathrm{Cl}^\star(B))$ for every subset B of Y;*

(5) *$F(\mathrm{Cl}^\star(A)) \subseteq \mathrm{Cl}^\star(F(A))$ for every subset A of X;*

(6) *$F^-(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^-(B))$ for every subset B of Y.*

Proof. We prove only the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) being the proofs of the other similar to those of Theorem 3.2.

(4) $\Rightarrow$ (5): Let A be any subset of X. Then, we have $\mathrm{Cl}^\star(A) \subseteq \mathrm{Cl}^\star(F^+(F(A))) \subseteq F^+(\mathrm{Cl}^\star(F(A)))$ and so $F(\mathrm{Cl}^\star(A)) \subseteq \mathrm{Cl}^\star(F(A))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. Then, we have

$$\begin{aligned} F(X - \mathrm{Int}^\star(F^-(B))) &= F(\mathrm{Cl}^\star(X - F^-(B))) = F(\mathrm{Cl}^\star(F^+(Y - B))) \\ &\subseteq \mathrm{Cl}^\star(F(F^+(Y - B))) \subseteq \mathrm{Cl}^\star(Y - B) = Y - \mathrm{Int}^\star(B) \end{aligned}$$

and hence, $X - \mathrm{Int}^\star(F^-(B)) \subseteq F^+(Y - \mathrm{Int}^\star(B)) = X - F^-(\mathrm{Int}^\star(B))$. Consequently, we obtain $F^-(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^-(B))$. □

Definition 3.4. The *$\star$-$\mathcal{I}$-frontier* of a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\mathrm{fr}^\star_{\mathcal{I}}(A)$, is defined by $\mathrm{fr}^\star_{\mathcal{I}}(A) = \mathrm{Cl}^\star(A) \cap \mathrm{Cl}^\star(X - A) = \mathrm{Cl}^\star(A) - \mathrm{Int}^\star(A)$.

Theorem 3.5. *The set of all points x of X at which a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is not upper $\star$-continuous is identical with the union of the $\star$-$\mathcal{I}$-frontier of the upper inverse images of $\star$-open sets containing $F(x)$.*

Proof. Let $x \in X$ at which F is not upper $\star$-continuous. There exists a $\star$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\star$-open set U containing x. Then, we have $x \in \mathrm{Cl}^\star(X - F^+(V)) = X - \mathrm{Int}^\star(F^+(V))$ and $x \in \mathrm{Cl}^\star(F^+(V))$. Consequently, we obtain $x \in \mathrm{fr}^\star_{\mathcal{I}}(F^+(V))$.

Conversely, suppose that V is $\star$-open set of Y containing $F(x)$ such that $x \in \mathrm{fr}^\star_{\mathcal{I}}(F^+(V))$. If F is upper $\star$-continuous at x, there exists a $\star$-open set U containing x such that $U \subseteq F^+(V)$. This implies that $x \in \mathrm{Int}^\star(F^+(V))$. This is a contradiction and so F is not upper $\star$-continuous. □

Theorem 3.6. *The set of all points x of X at which a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is not lower $\star$-continuous is identical with the union of the $\star$-frontier of the lower inverse images of $\star$-open sets meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 3.5. □

Definition 3.7. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. The set

$$\cap\{G | A \subseteq G \text{ and } G \text{ is } \star\text{-open}\}$$

is called the *$\star$-kernel* of A and is denoted by $\ker^\star(A)$.

Lemma 3.8. *For subsets A, B of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

(1) $A \subseteq \ker^\star(A)$.

(2) *If* $A \subseteq B$, *then* $\ker^\star(A) \subseteq \ker^\star(B)$.

(3) *If* A *is $\star$-open, then* $\ker^\star(A) = A$.

(4) $x \in \ker^\star(A)$ *if and only if* $A \cap K \neq \emptyset$ *for any $\star$-closed set K containing x.*

Theorem 3.9. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a multifunction. If* $F^+(\ker^\star(B)) \subseteq \mathrm{Int}^\star(F^+(B))$ *for every subset B of Y, then F is upper $\star$-continuous.*

Proof. Let V be any $\star$-open set of Y. By Lemma 3.8, $F^+(V) = F^+(\ker^\star(V)) \subseteq \mathrm{Int}^\star(F^+(V))$ and so $\mathrm{Int}^\star(F^+(V)) = F^+(V)$. This shows that $F^+(V)$ is $\star$-open and by Theorem 3.2, F is upper $\star$-continuous. □

Theorem 3.10. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a multifunction. If* $F^-(\ker^\star(B)) \subseteq \mathrm{Int}^\star(F^-(B))$ *for every subset B of Y, then F is lower $\star$-continuous.*

Proof. The proof is similar to that of Theorem 3.9. □

Definition 3.11. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be *$\star$-$\mathcal{I}$-compact* if every cover of X by $\star$-open sets of X has a finite subcover.

A subset K of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be *$\star$-$\mathcal{I}$-compact* if every cover of K by $\star$-open sets has a finite subcover.

Theorem 3.12. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be an upper $\star$-continuous surjective multifunction such that $F(x)$ is $\star$-$\mathcal{J}$-compact for each $x \in X$. If X is $\star$-$\mathcal{I}$-compact, then Y is $\star$-$\mathcal{J}$-compact.*

Proof. Let $\{V_\alpha | \alpha \in \nabla\}$ be a $\star$-open cover of Y. For each $x \in X$, $F(x)$ *is* $\star$-$\mathcal{J}$-compact and there exists a finite subset $\nabla(x)$ of ∇ such that

$$F(x) \subseteq \cup\{V_\alpha | \alpha \in \nabla(x)\}.$$

Put $V(x) = \cup\{V_\alpha | \alpha \in \nabla(x)\}$. Since F is upper $\star$-continuous, there exists a $\star$-open set $U(x)$ containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x)|x \in X\}$ is a $\star$-open cover of X and there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{U(x_i)|1 \leq i \leq n\}$. Therefore, we have $Y = F(X) = F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) = \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \subseteq \overset{n}{\underset{i=1}{\cup}} V(x_i) = \overset{n}{\underset{i=1}{\cup}} \cup_{\alpha \in \nabla(x_i)} V_\alpha$. This shows that Y is $\star$-$\mathcal{J}$-compact. □

Definition 3.13. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\star$-$\mathcal{I}$-*connected* if X cannot be written as the union of two non-empty disjoint $\star$-open sets.

Definition 3.14. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\star$-*clopen* if A is both $\star$-open and $\star$-closed.

Theorem 3.15. *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an upper or lower $\star$-continuous surjective multifunction such that $F(x)$ is $\star$-$\mathcal{J}$-connected for each $x \in X$ and X is $\star$-$\mathcal{I}$-connected, then Y is $\star$-$\mathcal{J}$-connected.*

Proof. Suppose that Y is not $\star$-$\mathcal{J}$-connected. There exist non-empty $\star$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\star$-$\mathcal{J}$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and so $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$, $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$, $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$. Next, we show that $F^+(U)$ and $F^+(V)$ are $\star$-open in X. (i) Let F be upper $\star$-continuous. By Theorem 3.2, we obtain $F^+(U)$ and $F^+(V)$ are $\star$-open in X. (ii) Let F be lower $\star$-continuous. By Theorem 3.3, we have $F^+(U)$ is $\star$-closed in X since U is $\star$-clopen in Y. Therefore, $F^+(V)$ is $\star$-open in X. Similarly, we obtain $F^+(U)$ is $\star$-open in X. Consequently, X is not $\star$-$\mathcal{I}$-connected. This completes the proof. □

4. CHARACTERIZATIONS OF UPPER AND LOWER ALMOST $\star$-CONTINUOUS MULTIFUNCTIONS

In this section, we introduce the notions of upper and lower almost $\star$-continuous multifunctions. Moreover, several interesting characterizations of upper and lower almost $\star$-continuous multifunctions are investigated.

Definition 4.1. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be:

(1) *upper almost $\star$-continuous* if for each $x \in X$ and each $\star$-open set V containing $F(x)$, there exists a $\star$-open set U containing x such that $F(U) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(V))$;

(2) *lower almost $\star$-continuous* if for each $x \in X$ and each $\star$-open set V such that $F(x) \cap V \neq \emptyset$, there exists a $\star$-open set U containing x such that $F(z) \cap \mathrm{Int}(\mathrm{Cl}^\star(V)) \neq \emptyset$ for each $z \in U$.

Theorem 4.2. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper almost $\star$-continuous at $x \in X$;*

(2) *$x \in \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ for every $\star$-open set V of Y containing $F(x)$;*

(3) *$x \in \mathrm{Int}^\star(F^+(V))$ for every R–$\mathcal{J}$-open set V of Y containing $F(x)$;*

(4) *for each R–$\mathcal{J}$-open set V of Y containing $F(x)$, there exists a $\star$-open set U containing x such that $F(U) \subseteq V$.*

Proof. (1) ⇒ (2): Let V be any $\star$-open set of Y containing $F(x)$. There exists a $\star$-open set U containing x such that $F(U) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(V))$. Thus, we have $x \in U \subseteq F^+(\mathrm{Int}(\mathrm{Cl}^\star)(V))$ and so $x \in \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let V be any R–$\mathcal{J}$-open set of Y containing $F(x)$. By (3), we have $x \in \mathrm{Int}^\star(F^+(V))$ and so there exists a $\star$-open set U of X containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(4) ⇒ (1): Let V be any $\star$-open set of Y containing $F(x)$. Since $\mathrm{Int}(\mathrm{Cl}^\star(V))$ is R–$\mathcal{J}$-open, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Int}(\mathrm{Cl}^\star(V))$. This shows that F is upper almost $\star$-continuous at $x \in X$. □

Theorem 4.3. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower almost $\star$-continuous at $x \in X$;*

(2) $x \in \mathrm{Int}^\star(F^-(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ *for every* $\star$-open *set V of Y such that $F(x) \cap V \neq \emptyset$;*

(3) $x \in \mathrm{Int}^\star(F^-(V))$ *for every R–$\mathcal{J}$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(4) *for each R–$\mathcal{J}$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\star$-open set U containing x such that $F(U) \subseteq V$.*

Proof. The proof is similar to that of Theorem 4.2. □

Theorem 4.4. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper almost $\star$-continuous;*

(2) $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ *for every $\star$-open set V of Y;*

(3) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \subseteq F^-(K)$ *for every $\star$-closed set K* of Y;

(4) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(B))))) \subseteq F^-(\mathrm{Cl}(B))$ *for every subset B of Y;*

(5) $F^+(\mathrm{Int}(B)) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))))$ *for every subset B of Y;*

(6) *$F^+(V)$ is $\star$-open in X for every R–$\mathcal{J}$-open set V of Y;*

(7) *$F^-(K)$ is $\star$-closed in X for every R–$\mathcal{J}$-closed set K of Y.*

Proof. (1) ⇒ (2): Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and by Theorem 4.2, we have $x \in \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$. This implies that $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$.

(2) ⇒ (3): Let K be any $\star$-closed set of Y. Then $Y - K$ is $\star$-open in Y and by (2), we have

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(Y - K)))) = \mathrm{Int}^\star(X - F^-(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \\ &= X - \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \end{aligned}$$

and so $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \subseteq F^-(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then $\mathrm{Cl}^\star(B)$ is a $\star$-closed set of Y and by (3), we have $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(B))))) \subseteq F^-(\mathrm{Cl}^\star(B)) \subseteq F^-(\mathrm{Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. By (4), we obtain

$$\begin{aligned} F^+(\mathrm{Int}(B)) = X - F^-(\mathrm{Cl}(Y - B)) &\subseteq X - \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y - B))))) \\ &= X - \mathrm{Cl}^\star(F^-(Y - \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))) = \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))). \end{aligned}$$

(5) ⇒ (6): Let V be any R–$\mathcal{J}$-open set of Y. By (5), we have

$$\begin{aligned} F^+(V) = F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))) = F^+(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}^\star(V)))) &\subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Int}(\mathrm{Cl}^\star(V))))))) \\ \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))))))) &= \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))))) \\ \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V)))) &= \mathrm{Int}^\star(F^+(V)). \end{aligned}$$

Consequently, we obtain $F^+(V)$ is $\star$-open in X.

(6) ⇒ (7): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(7) ⇒ (1): Let $x \in X$ and V be any R–$\mathcal{J}$-open set of Y containing $F(x)$. Then $Y - V$ is R–$\mathcal{J}$-closed and by (7), we have $X - F^+(V) = F^-(Y - V)$ is $\star$-closed and so $F^+(V)$ is $\star$-open in X. Put $U = F^+(V)$, then U is a $\star$-open set containing x such that $F(U) \subseteq V$. It follows from Theorem 4.2 that F is upper almost $\star$-continuous. □

Remark 4.5. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following implication hold:

upper $\star$-continuity ⇒ upper almost $\star$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 4.6. *Let $X = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and ideal $\mathcal{I} = \{\emptyset, \{1\}\}$. Let $Y = \{a, b, c\}$ with topology $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and ideal $\mathcal{J} = \{\emptyset, \{c\}\}$. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is defined as follows: $F(1) = \{c\}$ and $F(2) = F(3) = \{a, b\}$. Then F is upper almost $\star$-continuous but F is not upper $\star$-continuous, since $\{a, b\}$ is $\star$-open in Y but $F^+(\{a, b\})$ is not $\star$-open in X.*

Theorem 4.7. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower almost $\star$-continuous;*

(2) $F^-(V) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ *for every $\star$-open set V of Y;*

(3) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \subseteq F^+(K)$ *for every $\star$-closed set K of Y;*

(4) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(B))))) \subseteq F^+(\mathrm{Cl}(B))$ *for every subset B* of Y;

(5) $F^-(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))))$ *for every subset B of Y;*

(6) *$F^-(V)$ is $\star$-open in X for every R–$\mathcal{J}$-open set V of Y;*

(7) *$F^+(K)$ is $\star$-closed in X for every R–$\mathcal{J}$-closed set K of Y.*

Proof. The proof is similar to that of Theorem 4.4. □

The following theorems give some characterizations of upper and lower almost $\star$-continuous multifunctions.

Theorem 4.8. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper almost $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))))) \subseteq F^-(\mathrm{Cl}^\star(V))$ *for every pre-$\mathcal{J}$-open set V of Y;*

(3) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$ *for every pre-$\mathcal{J}$-open set V of Y;*

(4) $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ *for every pre-$\mathcal{J}$-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any pre-$\mathcal{J}$-open set V of Y. Then $\mathrm{Cl}^\star(V)$ is $\star$-closed in Y and by Theorem 4.4(3), we have $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(2) $\Rightarrow$ (3): Let V be any pre-$\mathcal{J}$-open set V of Y. Then, we have

$$\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(V)))) \subseteq \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))))) \subseteq F^-(\mathrm{Cl}^\star(V)).$$

(3) $\Rightarrow$ (4): Let V be any pre-$\mathcal{J}$-open set V of Y. By (3), we have

$$\begin{aligned} X - \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V)))) &= \mathrm{Cl}^\star(X - F^+(\mathrm{Int}(\mathrm{Cl}^\star(V)))) \\ = \mathrm{Cl}^\star(F^-(Y - \mathrm{Int}(\mathrm{Cl}^\star(V)))) &= \mathrm{Cl}^\star(F^-(\mathrm{Cl}(Y - \mathrm{Cl}^\star(V)))) \\ = \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(Y - \mathrm{Cl}^\star(V))))) &\subseteq F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V))) \\ = F^-(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V))) &\subseteq F^-(Y - V) = X - F^+(V) \end{aligned}$$

and so $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V))))$.

(4) $\Rightarrow$ (1): Since every R–$\mathcal{J}$-open set is pre-$\mathcal{J}$-open. It follows from Theorem 4.4(6) that F is upper almost $\star$-continuous. □

Theorem 4.9. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower almost $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))))) \subseteq F^+(\mathrm{Cl}^\star(V))$ *for every pre-$\mathcal{J}$-open set V of Y;*

(3) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(V)))) \subseteq F^+(\mathrm{Cl}^\star(V))$ *for every pre-$\mathcal{J}$-open set V of Y;*

(4) $F^-(V) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Int}(\mathrm{Cl}^\star(V))))$ *for every pre-$\mathcal{J}$-open set V of Y.*

Proof. The proof is similar to that of Theorem 4.8. □

Definition 4.10. [25] A point x in an ideal topological space $(X, \tau, \mathcal{I})$ is called a δ-$\mathcal{I}$-cluster point of A if $\mathrm{Int}(\mathrm{Cl}^\star(V)) \cap A \neq \emptyset$ for each open neighbourhood V of x. The set of all δ-$\mathcal{I}$-cluster points of A is called the δ-$\mathcal{I}$-closure of A and is denoted by $\delta_{\mathcal{I}}\mathrm{Cl}(A)$.

Definition 4.11. [25] A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called

(1) δ-$\mathcal{I}$-closed if $\delta_{\mathcal{I}}\mathrm{Cl}(A) = A$.

(2) δ-$\mathcal{I}$-open if its complement is $\delta-\mathcal{I}$-closed.

Lemma 4.12. [25] *For subsets A and B of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

(1) $\mathrm{Int}(\mathrm{Cl}^\star(A))$ *is $R-\mathcal{I}$-open.*

(2) If A and B are *$R-\mathcal{I}$-open, then $A \cap B$ is $R-\mathcal{I}$-open.*

(3) *If A is regular open, then it is $R-\mathcal{I}$-open.*

(4) *If A is $R-\mathcal{I}$-open, then it is δ-$\mathcal{I}$-open.*

(5) *Every δ-$\mathcal{I}$-open set is the union of a family of $R-\mathcal{I}$-open sets.*

Lemma 4.13. *For subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

(1) *If A is open in X, then* $\mathrm{Cl}(A) = \delta_{\mathcal{I}}\mathrm{Cl}(A)$.

(2) $\delta_{\mathcal{I}}\mathrm{Cl}(A)$ *is closed in X.*

Proof. (1). Suppose that $x \notin \mathrm{Cl}(A)$. There exists an open set U containing x such that $U \cap A = \emptyset$; hence $\mathrm{Int}(\mathrm{Cl}^\star(U)) \cap A = \emptyset$. This shows that $x \notin \delta_{\mathcal{I}}\mathrm{Cl}(A)$ and so $\delta_{\mathcal{I}}\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A)$. On the other hand, we have $\mathrm{Cl}(A) \subseteq \delta_{\mathcal{I}}\mathrm{Cl}(A)$. Consequently, we obtain $\mathrm{Cl}(A) = \delta_{\mathcal{I}}\mathrm{Cl}(A)$.

(2). Let $x \in X - \delta_{\mathcal{I}}\mathrm{Cl}(A)$. Then $x \notin \delta_{\mathcal{I}}\mathrm{Cl}(A)$ and there exists an open set U_x containing x such that $\mathrm{Int}(\mathrm{Cl}^\star(U_x)) \cap A = \emptyset$. Then, we have $\delta_{\mathcal{I}}\mathrm{Cl}(A) \cap U_x = \emptyset$ and so $x \in U_x \subseteq X - \delta_{\mathcal{I}}\mathrm{Cl}(A)$. Therefore, we obtain $X - \delta_{\mathcal{I}}\mathrm{Cl}(A) = \cup_{x \in X - \delta_{\mathcal{I}}\mathrm{Cl}(A)} U_x$. This shows that $\delta_{\mathcal{I}}\mathrm{Cl}(A)$ is closed. □

Theorem 4.14. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper almost $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\delta_{\mathcal{J}}\mathrm{Cl}(B))))) \subseteq F^-(\delta_{\mathcal{J}}\mathrm{Cl}(B))$ *for every subset B* of Y;

(3) $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(B))))) \subseteq F^-(\delta_{\mathcal{J}}\mathrm{Cl}(B))$ *for every subset B of Y.*

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 4.13, $\delta_{\mathcal{J}}\mathrm{Cl}(B)$ is closed and so $\delta_{\mathcal{J}}\mathrm{Cl}(B)$ is $\star$-closed. By Theorem 4.4, we obtain $\mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\delta_{\mathcal{J}}\mathrm{Cl}(B))))) \subseteq F^-(\delta_{\mathcal{J}}\mathrm{Cl}(B))$.

(2) ⇒ (3): This is obvious since $\mathrm{Cl}(B) \subseteq \delta_{\mathcal{J}}\mathrm{Cl}(B)$ for every subset B of Y.

(3) ⇒ (1): Let K be any $R-\mathcal{J}$-closed set of Y. Then by (3) and Lemma 4.12(4), we have

$$\mathrm{Cl}^\star(F^-(K)) = \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(K)))) \subseteq \mathrm{Cl}^\star(F^-(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(K))))) \subseteq F^-(\delta_{\mathcal{J}}\mathrm{Cl}(K)) = F^-(K)$$

and so $F^-(K)$ is $\star$-closed in X. By Theorem 4.4, F is upper almost $\star$-continuous. □

Theorem 4.15. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower almost $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(\delta_{\mathcal{J}}\mathrm{Cl}(B))))) \subseteq F^+(\delta_{\mathcal{J}}\mathrm{Cl}(B))$ *for every subset B of Y;*

(3) $\mathrm{Cl}^\star(F^+(\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(B))))) \subseteq F^+(\delta_{\mathcal{J}}\mathrm{Cl}(B))$ *for every subset B of Y.*

Proof. The proof is similar to that of Theorem 4.14. □

5. CHARACTERIZATIONS OF UPPER AND LOWER WEAKLY $\star$-CONTINUOUS MULTIFUNCTIONS

In this section, we introduce the notions of upper and lower weakly $\star$-continuous multifunctions and investigate some characterizations of such multifunctions.

Definition 5.1. A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be:

(1) *upper weakly $\star$-continuous* if for each $x \in X$ and each $\star$-open set V containing $F(x)$, there exists a $\star$-open set U containing x such that $F(U) \subseteq \mathrm{Cl}^\star(V)$;

(2) *lower weakly $\star$-continuous* if for each $x \in X$ and each $\star$-open set V such that $F(x) \cap V \neq \emptyset$, there exists a $\star$-open set U containing x such that $F(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$ for each $z \in U$.

Definition 5.2. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called

(1) R–$\mathcal{I}^\star$-open if $A = \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$.

(2) R–$\mathcal{I}^\star$-closed if its complement is R–$\mathcal{I}^\star$-open.

The following theorem give some characterizations of upper weakly $\star$-continuous multifunctions.

Theorem 5.3. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) F is upper weakly $\star$-continuous;

(2) $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$ for every $\star$-open set V of Y;

(3) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;

(4) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(B)))) \subseteq F^-(\mathrm{Cl}^\star(B))$ for every subset B of Y;

(5) $F^+(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))$ for every subset B of Y;

(6) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$ for every $\star$-open set V of Y;

(7) $\mathrm{Cl}^\star(F^-(V)) \subseteq F^-(\mathrm{Cl}^\star(V))$ for every $\star$-open set V of Y;

(8) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$ for every R–$\mathcal{J}^\star$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$, there exists a $\star$-open set U containing x such that $F(U) \subseteq \mathrm{Cl}^\star(V)$. Thus, $x \in U \subseteq F^+(\mathrm{Cl}^\star(V))$. This implies that $x \in \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$ and so $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$.

(2) $\Rightarrow$ (3): Let K be any $\star$-closed set of Y. Then $Y - K$ is $\star$-open in Y and by (2), we have

$$\begin{aligned} X - F^-(K) = F^+(Y-K) &\subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(Y-K))) = \mathrm{Int}^\star(F^+(Y - \mathrm{Int}^\star(K))) \\ &= \mathrm{Int}^\star(X - F^-(\mathrm{Int}^\star(K))) = X - \mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))). \end{aligned}$$

Consequently, we obtain $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $\mathrm{Cl}^\star(B)$ is a $\star$-closed set of Y and by (3), we obtain $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(B)))) \subseteq F^-(\mathrm{Cl}^\star(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then by (4), we have

$$\begin{aligned} X - \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))) &= \mathrm{Cl}^\star(X - F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))) = \mathrm{Cl}^\star(F^-(Y - \mathrm{Cl}^\star(\mathrm{Int}^\star(B)))) \\ &= \mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y-B)))) \subseteq F^-(\mathrm{Cl}^\star(Y-B)) = X - F^+(\mathrm{Int}^\star(B)) \end{aligned}$$

and hence, $F^+(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Then $x \in F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$. Therefore, there exists a $\star$-open set U of X containing x such that $U \subseteq F^+(\mathrm{Cl}^\star(V))$; hence $F(U) \subseteq \mathrm{Cl}^\star(V)$ and so F is upper weakly $\star$-continuous.

(4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (8): Let K be any R–$\mathcal{J}^\star$-closed set of Y. By (7), we have

$$\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))) = F^-(K).$$

(8) $\Rightarrow$ (3): Let K be any $\star$-closed set of Y. Then $\mathrm{Cl}^\star(\mathrm{Int}^\star(K))$ is R–$\mathcal{J}^\star$-closed in Y and by (8), we have

$$\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))) \subseteq \mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))))) \subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))) \subseteq F^-(\mathrm{Cl}^\star(K)) = F^-(K).$$

□

Remark 5.4. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following implication hold:

upper almost $\star$-continuity $\Rightarrow$ upper weakly $\star$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 5.5. *Let $X = \{1, 2, 3\}$ with topology $\tau = \{\emptyset, \{2\}, \{1, 3\}, X\}$ and ideal $\mathcal{I} = \{\emptyset\}$.* Let $Y = \{a, b, c\}$ *with topology* $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ *and ideal* $\mathcal{J} = \{\emptyset, \{c\}\}$. *Define*

$$F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$$

as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. One can deduce that F is upper weakly $\star$-continuous but F is not upper almost $\star$-continuous, since $\{a\}$ is $R-\mathcal{J}$-open in Y but $F^+(\{a\})$ is not $\star$-open in X.

The following theorem give some characterizations of lower weakly $\star$-continuous multifunctions.

Theorem 5.6. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower weakly $\star$-continuous;*

(2) $F^-(V) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Cl}^\star(V)))$ *for every $\star$-open set V of Y;*

(3) $\mathrm{Cl}^\star(F^+(\mathrm{Int}^\star(K))) \subseteq F^+(K)$ *for every $\star$-closed set K of Y;*

(4) $\mathrm{Cl}^\star(F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(B)))) \subseteq F^+(\mathrm{Cl}^\star(B))$ *for every subset B* of Y;

(5) $F^-(\mathrm{Int}^\star(B)) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))$ *for every subset B of Y;*

(6) $\mathrm{Cl}^\star(F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^+(\mathrm{Cl}^\star(V))$ *for every $\star$-open set V of Y;*

(7) $\mathrm{Cl}^\star(F^+(V)) \subseteq F^+(\mathrm{Cl}^\star(V))$ *for every $\star$-open set V of Y;*

(8) $\mathrm{Cl}^\star(F^+(\mathrm{Int}^\star(K))) \subseteq F^+(K)$ *for every $R-\mathcal{J}^\star$-closed set K of Y.*

Proof. The proof is similar to that of Theorem 5.3. □

Definition 5.7. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called

(1) $\mathcal{I}^\star$-preopen if $A \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$.

(2) $\mathcal{I}^\star$-preclosed if its complement is $\mathcal{I}^\star$-preopen.

Theorem 5.8. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is upper weakly $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$ for every $\mathcal{J}^\star$-preopen set V of Y;

(3) $\mathrm{Cl}^\star(F^-(V)) \subseteq F^-(\mathrm{Cl}^\star(V))$ *for every $\mathcal{J}^\star$-preopen set V of Y;*

(4) $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$ *for every $\mathcal{J}^\star$-preopen set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\mathcal{J}^\star$-preopen set V of Y. Since $\mathrm{Int}^\star(\mathrm{Cl}^\star(V))$ is $\star$-open, by Theorem 5.3(7), we have

$$\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V)).$$

(2) $\Rightarrow$ (3): Let V be any $\mathcal{J}^\star$-preopen set V of Y. By (2), we have $\mathrm{Cl}^\star(F^-(V)) \subseteq \mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(3) $\Rightarrow$ (4): Let V be any $\mathcal{J}^\star$-preopen set V of Y. By (3), we have

$$\begin{aligned} X - \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V))) &= \mathrm{Cl}^\star(X - F^+(\mathrm{Cl}^\star(V))) = \mathrm{Cl}^\star(F^-(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V))) = X - F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \subseteq X - F^+(V) \end{aligned}$$

and so $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$.

(4) $\Rightarrow$ (1): Let V be any $\star$-open set V of Y. Then V is $\mathcal{J}^\star$-preopen and by (4), we have $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))$. It follows from Theorem 5.3(2) that F is upper weakly $\star$-continuous. □

Theorem 5.9. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *F is lower weakly $\star$-continuous;*

(2) $\mathrm{Cl}^\star(F^+(\mathrm{Int}(\mathrm{Cl}^\star(V)))) \subseteq F^+(\mathrm{Cl}^\star(V))$ for every *$\mathcal{J}^\star$-preopen set V of Y;*

(3) $\mathrm{Cl}^\star(F^+(V)) \subseteq F^+(\mathrm{Cl}^\star(V))$ *for every $\mathcal{J}^\star$-preopen set V of Y;*

(4) $F^-(V) \subseteq \mathrm{Int}^\star(F^-(\mathrm{Cl}^\star(V)))$ *for every* $\mathcal{J}^\star$*-preopen set* V *of* Y.

Proof. The proof is similar to that of Theorem 5.8. □

Definition 5.10. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\star$-$\mathcal{I}$-*normal* if for every pair of disjoint $\star$-closed sets F and K, there exist disjoint $\star$-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Lemma 5.11. *For an ideal topological space* $(X, \tau, \mathcal{I})$, *the following properties are equivalent:*

(1) X *is* $\star$-$\mathcal{I}$-*normal.*

(2) *For each* $\star$*-closed set* F *and each* $\star$*-open set* V *containing* F, *there exists a* $\star$*-open set* U *such that* $F \subseteq U \subseteq \mathrm{Cl}^\star(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let F be a $\star$-closed set and V be a $\star$-open set containing F. Since F and $X - V$ are disjoint $\star$-closed sets, there exist disjoint $\star$-open sets U and W such that $F \subseteq U$ and $X - V \subseteq W$. Then, we have $X - W \subseteq V$. Since $U \cap W = \emptyset$, $U \subseteq X - W$ and so $\mathrm{Cl}^\star(U) \subseteq \mathrm{Cl}^\star(X - W) = X - W$. Thus, we have $F \subseteq U \subseteq \mathrm{Cl}^\star(U) \subseteq X - W \subseteq V$ which proves (2).

(2) $\Rightarrow$ (1): Let F and K be two disjoint $\star$-closed sets of X. By the hypothesis, there exists a $\star$-open set U such that $F \subseteq U \subseteq \mathrm{Cl}^\star(U) \subseteq X - K$. Put $V = X - \mathrm{Cl}^\star(U)$, then U and V are the required disjoint $\star$-open sets containing F and K, respectively. This shows that X is $\star$-$\mathcal{I}$-normal. □

Theorem 5.12. *For a multifunction* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *such that* $F(x)$ *is* $\star$*-closed in* Y *for each* $x \in X$ *and* $(Y, \sigma, \mathcal{J})$ *is a* $\star$-$\mathcal{J}$-*normal space, the following properties are equivalent:*

(1) F *is upper* $\star$*-continuous*;

(2) F *is upper almost* $\star$*-continuous*;

(3) F *is upper weakly* $\star$*-continuous*.

Proof. We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that F is upper weakly $\star$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\star$-closed in Y and Y is $\star$-$\mathcal{J}$-normal, there exists a $\star$-open set G of Y such that $F(x) \subseteq G \subseteq \mathrm{Cl}^\star(G) \subseteq V$. Since F is upper weakly $\star$-continuous, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Cl}^\star(G)$; hence $F(U) \subseteq V$. This shows that F is upper $\star$-continuous. □

Theorem 5.13. *For a multifunction* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *such that* $F(x)$ *is* $\star$*-open in* Y *for each* $x \in X$, *the following properties are equivalent:*

(1) F *is lower* $\star$*-continuous*;

(2) F *is lower almost* $\star$*-continuous*;

(3) F *is lower weakly* $\star$*-continuous*.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs of these implications are obvious.

(3) $\Rightarrow$ (1): Suppose that F is lower weakly $\star$-continuous. Let $x \in X$ and V be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. There exists a $\star$-open set U of X containing x such that $F(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\star$-open, we have $F(z) \cap V \neq \emptyset$ for each $z \in U$ and so F is lower $\star$-continuous. □

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