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Quasiconformal Mappings in the Theory of Semi-linear Equations

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Abstract—We study the Dirichlet problem with continuous boundary data in simply connected domains D of the complex plane for the semi-linear partial differential equations whose linear part has the divergent form. We prove that if a Jordan domain D satisfies the so-called quasi-hyperbolic boundary condition, then the problem has regular (continuous) weak solutions whose first generalized derivatives by Sobolev are integrable in the second degree. We give a suitable example of a Jordan domain with the quasihyperbolic boundary condition that fails to satisfy both the well-known (A)-condition and the outer cone condition. We also extend these results to some non-Jordan domains in terms of the prime ends by Caratheodory. The proofs are based on our factorization theorem established earlier. This theorem allows us to represent solutions of the semi-linear equations in the form of composition of solutions of the corresponding quasilinear Poisson equation in the unit disk and quasiconformal mapping of D onto the unit disk generated by the measurable matrix function of coefficients. In the end we give applications to relevant problems of mathematical physics in anisotropic inhomogeneous media.

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1. INTRODUCTION

Given a domain D in $\mathbb{C}$, denote by $M_K^{2\times 2}(D)$ the class of all 2×2 symmetric matrix functions $A(z)=\{a_{jk}(z)\}$ with measurable entries and $\det A(z)=1$, satisfying the uniform ellipticity condition

$$\frac{1}{K}\,|\xi|^2 \;\le\; \langle A(z)\xi,\xi\rangle \;\le\; K|\xi|^2 \quad \text{a.e. in} \quad D \tag{1}$$

for every $\xi\in\mathbb{C}$, where $1\le K<\infty$. Further we study the semi-linear equations

$$\operatorname{div}\,[A(z)\nabla u(z)] \;=\; f(u(z)), \quad z\in D, \tag{2}$$

with continuous functions $f:\mathbb{R}\to\mathbb{R}$ either bounded or such that $f(t)/t\to 0$ as $t\to\infty$. Simi-linear equations with such f describe a number of physical phenomena in anisotropic inhomogeneous media. The equations (2) are closely relevant to the so-called Beltrami equations. Let $\mu: D\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. The equation

$$\omega_{\bar z}=\mu(z)\omega_z, \tag{3}$$

where $\omega_{\bar z}=(\omega_x+i\omega_y)/2,\ \omega_z=(\omega_x-i\omega_y)/2,\ z=x+iy,\ \omega_x$ and ω_y are partial derivatives of the function ω in x and y, respectively, is said to be the **Beltrami equation**. The equation (3) is said to be

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nondegenerate if $||\mu||_\infty < 1$. The homeomorphic solutions of nondegenerate Beltrami's equations (3) in $W^{1,2}_{\rm loc}$ are called **quasiconformal mappings**, see e.g. [1, 2] and [27].

We say that a quasiconformal mapping ω satisfying (3) is **agreed with** $A \in M^{2\times2}_K(D)$ if

$$\mu(z) = \frac{a_{22}(z) - a_{11}(z) - 2ia_{12}(z)}{\det\,(I + A(z))}, \tag{4}$$

where I is the unit 2×2 matrix. Condition (1) is now written as

$$|\mu(z)| \ \le\ (K-1)/(K+1) \quad \text{a.e. in } D. \tag{5}$$

Vice versa, given a measurable function $\mu : D \to \mathbb{C}$, satisfying (5), one can invert the algebraic system (4) to obtain the matrix function $A \in M^{2\times2}_K(D)$:

$$A(z) = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} \\ \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{6}$$

By the existence theorem for (3), see e.g. Theorem V.B.3 in [1] and Theorem V.1.3 in [27], any $A \in M^{2\times2}_K(D)$ generates a quasiconfomal mapping $\omega : D \to \mathbb{D}$.

We also would like to pay attention to a strong interaction between linear and non-linear elliptic systems in the plane and quasiconformal mappings. The most general first order linear homogeneous elliptic system with real coefficients can be written in the form $f_{\bar z} + \mu(z)f_z + \nu(z)\overline{f_z} = 0$, with measurable coefficients μ and ν such that $|\mu| + |\nu| \le (K-1)/(K+1) < 1$. This equation is a particular case of a non-linear first order system $f_{\bar z} = H(z, f_z)$ where $H : G \times \mathbb{C} \to \mathbb{C}$ is Lipschitz in the second variable,

$$|H(z, w_1) - H(z, w_2)| \le \frac{K-1}{K+1}|w_1 - w_2|, \quad H(z, 0) \equiv 0.$$

The principal feature of the above equation is that the difference of two solutions need not solve the same equation but each solution can be represented as *a composition of a quasiconformal homeomorphism and an analytic function*. Thus quasiconformal mappings become the central tool for the study of these non-linear systems. A rather comprehensive treatment of the present state of the theory is given in the excellent book of Astala, Iwaniec and Martin [2]. This book contains also an exhaustive bibliography on the topic. In particular, the following fundamental Harmonic Factorization Theorem for the uniformly elliptic divergence equations

$$\text{div}\, A(z, \nabla u) = 0, \quad z \in \Omega, \tag{7}$$

holds, see [2], Theorem 16.2.1: Every solution $u \in W^{1,2}_{\rm loc}(\Omega)$ of the equation (7) can be expressed as *the composition $u(z) = h(f(z))$ of a quasiconformal homeomorphism $f : \Omega \to G$ and a suitable harmonic function h on G.*

The main goal of this paper is to point out another application of quasiconformal mappings to the study of some semi-linear partial differential equations, linear part of which contains the elliptic operator in the divergence form div $[A(z)\nabla u(z)]$.

A fundamental role in the study of the posed problem will play Theorem 4.1 in [20], that can be considered as a suitable counterpart to the mentioned above Factorization theorem: a function $u : D \to \mathbb{R}$ is a weak solution of (2) in the class $C \cap W^{1,2}_{\rm loc}(D)$ if and only if $u = U \circ \omega$ where $\omega : D \to \mathbb{D}$ is a quasiconformal mapping agreed with A and U is a weak solution in the class $C \cap W^{1,2}_{\rm loc}(\mathbb{D})$ of the quasilinear Poisson equation

$$\triangle U(w) = J(w)\, f(U(w)), \quad w \in \mathbb{D}, \tag{8}$$

where J denotes the Jacobian of the inverse quasiconformal mapping $\omega^{-1} : \mathbb{D} \to D$. Here a **weak solution** to (2) is a function $u \in C \cap W^{1,2}_{\rm loc}(\Omega)$ such that

$$\int\limits_D \langle A(z)\nabla u(z), \nabla \eta(z)\rangle\, dm(z) + \int\limits_D f(u(z))\eta(z)\, dm(z) = 0 \quad \forall \eta \in C \cap W^{1,2}_0(D),$$

where $m(z)$ stands for the Lebesgue measure in the plane.

2. DEFINITIONS AND PRELIMINARY REMARKS

In the paper [21], Theorem 3, we have established the following statement on the existence of regular weak solutions of the Dirichlet problem for a quasilinear Poisson equation.

Proposition 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function, $h : \mathbb{D} \to \mathbb{R}$ be a function in the class $L^p(\mathbb{D})$, $p > 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that is either bounded or with the nondecreasing function $|f|$ of $|t|$ such that*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 0. \tag{9}$$

Then there exist weak solutions U of the quasilinear Poisson equation

$$\triangle U(z) = h(z)\, f(U(z)) \tag{10}$$

such that $U \in C(\overline{\mathbb{D}})$, $U|_{\partial\mathbb{D}} \equiv \varphi$. More precisely, $U|_{\mathbb{D}} \in W^{2,p}_{loc}(\mathbb{D})$ and (10) holds a.e. in $\mathbb{D}$. Moreover, $U \in W^{1,q}_{loc}(\mathbb{D})$ for some $q > 2$ and U is locally Hölder continuous in $\mathbb{D}$. If in addition φ is Hölder continuous, then U is Hölder continuous in $\overline{\mathbb{D}}$. If $p > 2$, then $U \in C^{1,\alpha}_{loc}(\mathbb{D})$ where $\alpha = (p-2)/p$. In particular, $U \in C^{1,\alpha}_{loc}(\mathbb{D})$ for all $\alpha \in (0,1)$ if $h \in L^\infty(\mathbb{D})$.

Thus, the degree of regularity of the weak solutions of the Dirichlet problem to (10) essentially depends on the degree of integrability of the multiplier h. Furthermore, by an example in [21] the equation (10) can have no continuous solutions if h is only in the class $L^1(\mathbb{D})$.

Making use of fundamental results on boundary correspondence under conformal and quasiconformal mappings, one can extend the above statement replacing the unit disk $\mathbb{D}$ by a smooth Jordan's domain.

Corollary 1. *Let D be a smooth (C^1) Jordan domain in $\mathbb{C}$, $\Phi : \partial D \to \mathbb{R}$ be a continuous function, $H : D \to \mathbb{R}$ be a function in the class $L^p(D)$, $p > 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is either bounded or satisfying (9) with nondecreasing $|f|$ of $|t|$. Then there exist weak solutions u of the quasilinear Poisson equation*

$$\triangle u(\zeta) = H(\zeta)\, f(u(\zeta)) \quad \text{for a.e.} \quad \zeta \in D \tag{11}$$

such that $u \in C(\overline{D})$, $u|_{\partial D} \equiv \varphi$. More precisely, $u|_{\mathbb{D}} \in W^{2,p}_{loc}(\mathbb{D})$ and (10) holds a.e. in $\mathbb{D}$. Moreover, $u \in W^{1,q}_{loc}(D)$ for some $q > 2$ and u is locally Hölder continuous in D. If in addition Φ is Hölder continuous, then u is Hölder continuous in $\overline{D}$. Furthermore, if $p > 2$, then $u \in C^{1,\alpha}_{loc}(D)$, where $\alpha = (p-2)/p$. In particular, $u \in C^{1,\alpha}_{loc}(D)$ for all $\alpha \in (0,1)$ if $h \in L^\infty(D)$. In the latter case, if in addition Φ is Hölder continuous on ∂D with some order $\beta \in (0,1)$, then u is Hölder continuous in $\overline{D}$ with the same order.

Proof. Let ω be a conformal mapping of D onto $\mathbb{D}$. By the Caratheodory–Osgood–Taylor theorem, ω is extended to a homeomorphism $\tilde{\omega}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$, see [12] and [13], see also [5] and Theorem 3.3.2 in the monograph [14]. Then, setting $\varphi = \Phi \circ \tilde{\omega}^{-1}|_{\partial\mathbb{D}}$, we see that the function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ is continuous. Let $h = J \cdot H \circ \Omega$, where Ω is the inverse mapping $\omega^{-1} : \mathbb{D} \to D$ and J is its Jacobian $J = |\Omega'|^2$. By the known Warschawski result, see Theorem 2 in [34], its derivative Ω' is extended by continuity onto $\overline{\mathbb{D}}$. Consequently, J is bounded and the function h is of the same class in $\mathbb{D}$ as H in D. Let U be a solution of the Dirichlet problem from Proposition 1 for the equation (10) with the given φ and h. Note that $\omega' = 1/\Omega' \circ \omega$ is also extended by continuity onto $\overline{D}$ because $\Omega' \neq 0$ on $\partial\mathbb{D}$ by Theorem 1 in [24]. Thus, $u = U \circ \omega$ is the desired solution of the Dirichlet problem for the equation (11). □

By our factorization theorem, mentioned in the Introduction, the degree of the regularity of the weak solutions of the semi-linear equation (2) will depend on the degree of integrability of the Jacobian J of the quasiconformal mapping $\Omega : \mathbb{D} \to D$ associated with the matrix function A, see equation (8). In turn, the latter depends on geometry of the domain D.

Remark 1. By Theorem 4.7 in [3], the Jacobian of a quasiconformal mapping $\Omega : \mathbb{D} \to D$ is in $L^p(\mathbb{D})$, $p > 1$, if and only if the domain D satisfies the **quasihyperbolic boundary condition**, i.e.

$$k_D(z, z_0) \leq a \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} + b \quad \forall z \in D \tag{12}$$

for some constants a and b and a fixed point $z_0 \in D$, where $k_D(z, z_0)$ is the **quasihyperbolic distance** between the points z and z_0 in the domain D,

$$k_D(z, z_0) := \inf_\gamma \int\limits_\gamma \frac{ds}{d(\zeta, \partial D)}. \tag{13}$$

Here $d(\zeta, \partial D)$ denotes the Euclidean distance from a point $\zeta \in D$ to the boundary of D and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D.

In this connection, recall more definitions. The image of the unit disk $\mathbb{D}$ under a quasiconformal mapping of $\mathbb{C}$ onto itself is called a **quasidisk** and its boundary is called a **quasicircle** or a **quasiconformal curve**. Recall also that a **Jordan curve** is a continuous one-to-one image of the unit circle in $\mathbb{C}$. As known, such a smooth (C^1) or Lipschitz curve is a quasiconformal curve and, at the same time, quasiconformal curves can be even locally non–rectifiable as it follows from the well-known Van Koch snowflake example, see e.g. the point II.8.10 in [27]. The recent book [17] contains a comprehensive discussion and numerous characterizations of quasidisks, see also [1, 16] and [27].

Remark 2. Quasidisks satisfy the quasihyperbolic boundary condition. Indeed, as known, the conformal mapping $\Omega : \mathbb{D} \to D$ is extended to a quasiconformal mapping of $\mathbb{C}$ onto itself if ∂D is a quasicircle, see e.g. Theorem II.8.3 in [27]. By one of the main Bojarski results, see [8], the generalized derivatives of quasiconformal mappings in the plane are locally integrable with some power $q > 2$. Note also that its Jacobian $J(w) = |\Omega_w|^2 - |\Omega_{\bar{w}}|^2$, see e.g. I.A(9) in [1]. Consequently, in this case $J \in L^p(\mathbb{D})$ for some $p > 1$.

A domain D in $\mathbb{R}^n$, $n \geq 2$, is called satisfying **(A)-condition** if

$$\text{mes}\, D \cap B(\zeta, \rho) \leq \Theta_0 \text{mes}\, B(\zeta, \rho) \quad \forall \zeta \in \partial D, \quad \rho \leq \rho_0 \tag{14}$$

for some Θ_0 and $\rho_0 \in (0, 1)$, see 1.1.3 in [26]. Recall also that a domain D in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every boundary point of D from the completion of D after its suitable rotations and shifts. It is clear that the outer cone condition implies (A)-condition.

Remark 3. Note that quasidisks D satisfy (A)-condition. Indeed, the quasidisks are the so-called QED-domains by Gehring–Martio, see Theorem 2.22 in [18], and the latter satisfy the condition

$$\text{mes}\, D \cap B(\zeta, \rho) \geq \Theta_* \text{mes}\, B(\zeta, \rho) \quad \forall \zeta \in \partial D, \quad \rho \leq \text{diam}\, D \tag{15}$$

for some $\Theta_* \in (0, 1)$, see Lemma 2.13 in [18], and quasidisks (as domains with quasihyperbolic boundary) have boundaries of the Lebesgue measure zero, see e.g. Theorem 2.4 in [3]. Thus, it remains to note that, by definition, the completions of quasidisks D in the the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ are also quasidisks up to the inversion with respect to a circle in D.

Probably the first example of a simply connected plane domain D with the quasihyperbolic boundary condition which is not a quasidisk was constructed in [7], Theorem 2. However, this domain satisfies (A)-condition. In the next section, we construct an example of a domain D with the quasihyperbolic boundary condition but without (A)-condition and, consequently, without the outer cone condition, see Lemma 1.

3. DIRICHLET PROBLEM FOR SEMI–LINEAR EQUATIONS

By the mentioned above factorization theorem from [20], the study of semi–linear equations (2) *in Jordan domains D* is reduced, by means of a suitable quasiconformal change of variables, to the study of the corresponding quasilinear Poisson equations (8) *in the unit disk* $\mathbb{D}$.

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition. Suppose that $A \in M^{2\times 2}_K(D)$, $\varphi : \partial D \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which is either bounded or with nondecreasing $|f|$ of $|t|$ such that*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = 0. \tag{16}$$

Then there is a weak solution $u : D \to \mathbb{R}$ of the equation (2) which is locally Hölder continuous in D and continuous in $\overline{D}$ with $u|_{\partial D} = \varphi$. If in addition φ is Hölder continuous, then u is Hölder continuous in $\overline{D}$.

Proof. By Theorem 4.1 in [20], if u is a week solution of (2), then $u = U \circ \omega$, where ω is a quasiconformal mapping of D onto the unit disk $\mathbb{D}$ agreed with A and U is a week solution of the equation (10) with $h = J$, where J stands for the Jacobian of ω^{-1}. It is also easy to see that if U is a week solution of (10) with $h = J$, then $u = U \circ \omega$ is a week solution of (2). It allows us to reduce the Dirichlet problem for equation (2) with a continuous boundary function φ in the simply connected Jordan domain D to the Dirichlet problem for the equation (10) in the unit disk $\mathbb{D}$ with the continuous boundary function $\psi = \varphi \circ \omega^{-1}$. Indeed, ω is extended to a homeomorphism of $\overline{D}$ onto $\overline{\mathbb{D}}$, see e.g. Theorem I.8.2 in [27]. Thus, the function ψ is well defined and really is continuous on the unit circle.

It is well-known that the quasiconformal mapping ω is locally Hölder continuous in D, see Theorem 3.5 in [9]. Taking into account the fact that D is a Jordan domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition, we can show that both mappings ω and ω^{-1} are Hölder continuous in $\overline{D}$ and $\overline{\mathbb{D}}$, correspondingly. Indeed, $\omega = H \circ \Omega$ where Ω is a conformal (Riemann) mapping of D onto $\mathbb{D}$ and H is a quasiconformal mapping of $\mathbb{D}$ onto itself. The mappings Ω and Ω^{-1} are Hölder continuous in $\overline{D}$ and in $\overline{\mathbb{D}}$, correspondingly, by Theorem 1 and its corollary in [7]. Next, by the reflection principle H can be extended to a quasiconformal mapping of $\mathbb{C}$ onto itself, see e.g. I.8.4 in [27], and, consequently, H and H^{-1} are also Hölder continuous in $\overline{\mathbb{D}}$, see again Theorem 3.5 in [9]. The Hölder continuity of ω and ω^{-1} in closed domains follows immediately.

Now it is easy to see that if φ is Hölder continuous, then ψ is also so, and all the conclusions of Theorem 1 follow from Proposition 1. □

Lemma 1. *There exists a Jordan domain D in $\mathbb{C}$ with the quasihyperbolic boundary condition, that does not satisfy either condition* (A) *or the condition of the outer cone.*

Proof. Let C_1 be the cube $\{z = x + iy : \ |x| < 1, \ |y| < 1\}$, R_n are the rectangles $\{z = x + iy : \ 1 \leq x < a_n, \ \varepsilon_n \leq y < \varepsilon_{n-1}\}$ with $a_n = 1/n$ and $\varepsilon_n = 2^{-n}$ and R_n^* are the reflections of R_n with respect to the real axis, $n = 1, 2, \ldots$. Let D be the domain consisting of the cube C_1 and the remainder $R := \bigcup_{n=1}^{\infty}(R_n \cup R_n^*)$. First of all, it is clear that D is a Jordan domain whose boundary consists of a countable collection of segments of horizontal and vertical straight lines and the point $z_0 = 1$.

Let us show that D satisfies the quasihyperbolic boundary condition. Note firstly that the quasihyperbolic distance from 0 to any point in its central closed cube $C_{3/4} := \{z = x + iy : \ |x| \leq 3/4, \ |y| \leq 3/4\}$ is not grater than 3. Now, let C be the continuum consisting of the cube $C_{3/4}$ and the segments $3/4 \leq x \leq 7/4$ on the straight lines $y = 3/4$ and $y = -3/4$. It is clear by the triangle inequality that the quasihyperbolic distance from 0 to any point of C is not grater than 7.

Next, note that all points in the triangle $\triangle$ with the vertices $7/4 + 3i/4$, $2 + i$ and $2 + i/2$ lie more closely to the vertical line $x = 2$ than to the horizontal lines $y = 1$ and $y = 1/2$ because its sides $(7/4 + 3i/4, 2 + i)$ and $(7/4 + 3i/4, 2 + i/2)$ are bisectrices of the right angles at vertices $2 + i$ and $2 + i/2$ of ∂D. Each point in $\triangle$ lies on a segment of a straight line starting from the corresponding point on the side $(2 + i, 2 + i/2) \subset \partial D$ and ending at the point $7/4 + 3i/4 \in C$ and the slope of the line to the side is varied in the limits $\pi/4$ and $\pi/2$. Let s be the natural parameter on one of such segments S with $s = 0$ at the corresponding point of ∂D and $\zeta(s)$ be the natural parametrization of points on γ. Then

$$d(\zeta(s), \partial D) \geq s/\sqrt{2}. \tag{17}$$

By the symmetry of D, the similar statement is true for the triangle $\triangle^*$ that is symmetric for $\triangle$ with respect to the real axes.

Note also that every point in $D \setminus C$, except the points of the triangles $\triangle$ and $\triangle^*$, lies on a segment of a straight line going under the angle $\pi/4$ with respect to horizontal and vertical straight lines, starting from the corresponding point on ∂D and ending at the nearest point on the continuum C. It is clear that (17) holds on such segments, too. The lengths of all segments mentioned above are bounded by the diameter δ of D, $\delta = \sqrt{13} \leq 4$ and, consequently, $k_D(\zeta(s_0), \zeta(s_*)) \leq \sqrt{2}(\ln s_* - \ln s_0) \leq \sqrt{2}(\ln \delta - \ln s_0)$, where s_* and s_0 correspond to points in C and in $D \setminus C$. Thus, by the triangle inequality

$$k_D(z, 0) \leq \sqrt{2} \ln \frac{d(0, \partial D)}{d(z, \partial D)} + 7 + \sqrt{2} \ln \frac{\delta}{\sqrt{2}} < \sqrt{2} \ln \frac{d(0, \partial D)}{d(z, \partial D)} + 10 \quad \forall z \in D,$$

i.e. the domain D really satisfies the quasihyperbolic boundary condition.

Finally, let us show that D does not satisfy $A-$condition at the point $(1,0)$. Indeed, let us consider the sequence of disks D_n centered at the given point with the radii $\rho_n^2 = a_n^2 + \varepsilon_n^2 = n^{-2} + 2^{-2n}$. Note that $D_n \cap D$ contains 2 caps of the disk D_n that are cut off by the horizontal straight lines $y = \varepsilon_n = 2^{-n}$ and $y = -\varepsilon_n = -2^{-n}$. Consequently,

$$\mathrm{I}_n := \frac{\text{mes}\,(D_n \cap D)}{\text{mes}\,D_n} \geq \frac{\alpha_n - \sin\alpha_n}{\pi},$$

where α_n is the angular size of each of these caps. Since $\sin(\alpha_n/2) = a_n/\rho_n$ converges to 1 as $n \to \infty$, we have that $\alpha_n \to \pi$, i.e. I_n converges to 1 as $n \to \infty$. $\Box$

Scaling, rotating and shifting the remainder R from the proof, it is possible to construct Jordan domains D that are similar to the Van Koch snowflake with the quasihyperbolic boundary condition and, simultaneously, without (A)-condition at the everywhere dense set of boundary points.

4. THE DIRICHLET PROBLEM IN TERMS OF PRIME ENDS

It is much more simpler than in Lemma 1 to construct similar examples of domains with the quasihyperbolic boundary condition that are not Jordan.

Lemma 2. *There exist bounded simply connected domains D_* in $\mathbb{C}$ that are not Jordan, satisfy the quasihyperbolic boundary condition, however, without* (A)*-condition and, consequently, without the outer cone condition.*

Proof. Let P be the rectangular $\{z = x + iy : \ -1 < x < 2, \ |y| < 1\}$ and let the domain D be obtained from P through cut along $1 \leq x < 2$ in the real axis. Denote by C the union of the cube $C_{1/2} := \{z = x + iy : \ |x| \leq 1/2, \ |y| \leq 1/2\}$ and 2 segments $1/2 \leq x \leq 3/2$ on the straight lines $y = 1/2$ and $y = -1/2$. Let $\triangle$ be the triangle with the vertices $3/2 + i/2$, $2 + i$ and 2 and let $\triangle^*$ be the triangle which is symmetric for $\triangle$ with respect to the real axis. Arguing as in the proof of Lemma 2, it is easy to get the following estimate

$$k_D(z,0) < \sqrt{2}\ln\frac{d(0,\partial D)}{d(z,\partial D)} + 5 \quad \forall z \in D$$

i.e. the domain D is really with the quasihyperbolic boundary condition, but it is clear that (A)-condition does not hold at the end point of the cut in P. $\Box$

Before to formulate the corresponding results for non-Jordan domains, let us recall the necessary definitions of the relevant notions and notations. Namely, we follow Caratheodory [13] under the definition of the prime ends of domains in $\mathbb{C}$, see also Chapter 9 in [14]. First of all, recall that a continuous mapping $\sigma : \mathbb{I} \to \mathbb{C}$, $\mathbb{I} = (0,1)$, is called a **Jordan arc** in $\mathbb{C}$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations σ, $\overline{\sigma}$ and $\partial\sigma$ for $\sigma(\mathbb{I})$, $\overline{\sigma(\mathbb{I})}$ and $\overline{\sigma(\mathbb{I})} \setminus \sigma(\mathbb{I})$, correspondingly. A **cross–cut** of a simply connected domain $D \subset \mathbb{C}$ is a Jordan arc σ in the domain D with both ends on ∂D splitting D.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of D is called a **chain** in D if:

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \varnothing$ for every $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) σ_m splits D into 2 domains one of which contains σ_{m+1} and another one σ_{m-1} for every $m > 1$;

(iii) $\delta(\sigma_m) \to 0$ as $m \to \infty$ where $\delta(\sigma_m)$ is the diameter of σ_m with respect to the Euclidean metric in $\mathbb{C}$.

Correspondingly to the definition, a chain of cross-cuts σ_m generates a sequence of domains $d_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial d_m = \sigma_m$. Chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called **equivalent** if, for every $m = 1, 2, \ldots$, the domain d_m contains all domains d'_k except a finite number and, for every $k = 1, 2, \ldots$, the domain d'_k contains all domains d_m except a finite number, too. A **prime end** P of the domain D is an equivalence class of chains of cross-cuts of D. Later on, E_D denote the collection of all prime ends of a domain D and $\overline{D}_P = D \cup E_D$ is its completion by its prime ends.

Next, we say that a sequence of points $p_l \in D$ is **convergent to a prime end** P of D if, for a chain of cross-cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains all points p_l except their finite collection. Further, we say that a sequence of prime ends P_l converges to a prime end P if, for a chain of cross−cuts $\{\sigma_m\}$ in P, for every $m = 1, 2, \ldots$, the domain d_m contains chains of cross−cuts $\{\sigma'_k\}$ in all prime ends P_l except their finite collection.

A basis of neighborhoods of a prime end P of D can be defined in the following way. Let d be an arbitrary domain from a chain in P. Denote by d^* the union of d and all prime ends of D having some chains in d. Just all such d^* form a basis of open neighborhoods of the prime end P. The corresponding topology on E_D and, respectively, on $\overline{D}_P$ is called the **topology of prime ends**. The continuity of functions on E_D and $\overline{D}_P$ will be understood with respect to this topology or, the same, with respect to the above convergence.

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition. Suppose $A \in M_K^{2\times 2}(D)$, $\varphi : E_D \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which is either bounded or with nondecreasing $|f|$ of $|t|$ such that*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 0. \tag{18}$$

Then there is a weak solution $u : D \to \mathbb{R}$ of the equation (2) which is locally Hölder continuous in D and continuous in $\overline{D}_P$ with $u|_{E_D} = \varphi$.

Proof. By Theorem 4.1 in [20], if u is a week solution of (2), then $u = U \circ \omega$, where ω is a quasiconformal map of D onto the unit disk $\mathbb{D}$ agreed with A and U is a week solution of the equation (10) with $h = J$, here J stands for the Jacobian of the mapping ω^{-1}. And vice versa, if U is a week solution of (10) with $h = J$, then $u = U \circ \omega$ is a week solution of (2).

Hence the Dirichlet problem for (2) in the domain D can be reduced to the so for the equation (10) in $\mathbb{D}$ with the corresponding boundary function $\psi = \varphi \circ \omega^{-1}$. The existence and continuity of the boundary function ψ in the case of an arbitrary bounded simply connected domain D is a fundamental result of the theory of the boundary behavior of conformal and quasiconformal mappings. Namely, $\omega^{-1} = H \circ \Omega$, where Ω stands for a quasiconformal automorphism of the unit disk $\mathbb{D}$ and H is a conformal mapping of $\mathbb{D}$ onto D. It is known that Ω can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto itself, see e.g. Theorem I.8.2 in [27]. Moreover, by the well-known Caratheodory theorem on the boundary correspondence under conformal mappings, see e.g. Theorems 9.4 and 9.6 in [14], the mapping H is extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}_P$. Thus, the function ψ is well defined and really continuous on the unit circle.

Moreover, ω is locally Hölder continuous in $\mathbb{D}$, see e.g. Theorem 3.5 in [9]. Thus, by Theorem 4.1 in [20], Theorem 2 follows from Proposition 1. □

5. SOME MODEL EQUATIONS

The interest to the study of some model semi-linear equations considered below is well known both from a purely theoretical point of view, due to its deep relations to linear and nonlinear harmonic analysis, and because of numerous applications of equations of this type in various areas of physics, differential geometry, logistic problems etc., see e.g. [2, 11, 15, 19, 24, 25, 28, 30] and the references therein.

In particular, the mathematical modelling of reaction-diffusion problems leads to the study of the corresponding Dirichlet problem for the equation (2) with specified right hand side. Following [4], a nonlinear system can be obtained for the density u and the temperature T of the reactant. Upon eliminating T the system can be reduced to a scalar problem for the concentration

$$\triangle u = \lambda\, f(u), \tag{19}$$

where λ stands for a positive constant. It turns out that the density of the reactant u may be zero in a closed interior region D_0 called *a dead core*. If, for instance, $f(u) = u^q$, $q > 0$, a particularization of the results in Chapter 1 of [15] shows that a dead core may only exist if and only if $0 < q < 1$ and λ is large enough. See also the corresponding examples of cores in [20]. In connection with the above, the following statement may have of independent interest.

Theorem 3. *Let D be a Jordan domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition. Suppose that $A \in M_K^{2\times 2}(D)$ and $\varphi : \partial D \to \mathbb{R}$ is a continuous function. Then there exists a weak solution $u : D \to \mathbb{R}$ of the semi-linear equation*

$$\mathrm{div}\,[A(z)\nabla u(z)] = u^q(z), \quad 0 < q < 1, \tag{20}$$

which is locally Hölder continuous in D, continuous in $\overline{D}$ and satisfies the boundary condition $u|_{\partial D} = \varphi$. If in addition φ is Hölder continuous, then u is Hölder continuous in $\overline{D}$.

Applying Corollary 1, we also arrive at the following consequence.

Corollary 2. *Let D be a smooth Jordan domain in $\mathbb{C}$ and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there exists a weak solution U of the quasilinear Poisson equation*

$$\triangle U(z) = U^q(z), \quad 0 < q < 1, \tag{21}$$

which is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$ and such that $U \in C^{1,\alpha}_{loc}(D)$ for all $\alpha \in (0,1)$. If in addition φ is Hölder continuous with some order $\beta \in (0,1)$, then U is also Hölder continuous in $\overline{D}$ with the same order.

Recall also that certain mathematical models of a heated plasma lead to nonlinear equations of the type (19). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = +\infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [15]. With the replacement of the function $U = \psi(u) = |u|^q \text{sign}\, u$, we have that $u = |U|^Q \text{sign}\, U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \text{sign}\, u$, we come to the equation $\triangle U = |U|^q \text{sign}\, U = \psi(U)$.

Corollary 3. *Let D be a smooth Jordan domain in $\mathbb{C}$ and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there exists a weak solution U of the quasilinear Poisson equation*

$$\triangle U(z) = |U(z)|^{q-1}U(z), \quad 0 < q < 1, \tag{22}$$

which is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$ and such that $U \in C^{1,\alpha}_{loc}(D)$ for all $\alpha \in (0,1)$. If in addition φ is Hölder continuous with some order $\beta \in (0,1)$, then U is also Hölder continuous in $\overline{D}$ with the same order.

In the combustion theory, see e.g. [6, 32] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} = \frac{1}{\delta}\, \triangle u \,+\, e^u, \quad t \geq 0, \quad z \in D, \tag{23}$$

occupies a special place. Here $u \geq 0$ is the temperature of the medium and δ is a certain positive parameter.

We restrict ourselves by stationary solutions of the equation and its generalizations in anisotropic and inhomogeneous media although our approach makes it possible to consider the parabolic case, see [20]. Applying Theorem 1, we come to the following statement.

Theorem 4. *Let D be a Jordan domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition. Suppose that $A \in M^{2\times 2}_K(D)$ and $\varphi : \partial D \to \mathbb{R}$ is a continuous function. Then there exists a weak solution $U : D \to \mathbb{R}$ of the semi-linear equation*

$$\text{div}\, [A(z)\nabla U(z)] = \delta\, e^{-U(z)}, \quad \delta > 0, \tag{24}$$

which is locally Hölder continuous in D, continuous in $\overline{D}$ and such that *$u|_{\partial D} = \varphi$. If in addition φ is Hölder continuous, then u is also Hölder continuous in $\overline{D}$.*

By Corollary 1, applied to the corresponding quasilinear Poisson equation, we will finish this section with the following statement.

Corollary 4. *Let D be a smooth Jordan domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a weak solution U of the equation*

$$\triangle U(z) = \delta\, e^{-U(z)}, \quad \delta > 0, \tag{25}$$

which is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$ and such that $U \in C^{1,\alpha}_{loc}(D)$ for all $\alpha \in (0,1)$. If in addition φ is Hölder continuous with some order $\beta \in (0,1)$, then U is also Hölder continuous in $\overline{D}$ with the same order.

Concluding the presentation, we want to emphasize the fact that joint use of the regularity results for the quasilinear Poisson equations (10) and the comprehensively developed theory of conformal and quasiconformal mappings in the plane, see e.g. the monographs [1, 10, 22, 23, 27, 29, 33] opens up a new approach to the study of a number of problems arising in the mathematical physics in anisotropic and inhomogeneous media.

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