ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2018, Vol. 39, No. 9, pp. 1478–1483. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © L.U. Sultanov, 2017, published in Uchenye Zapiski Kazanskogo Universiteta, Seriya Fiziko-Matematicheskie Nauki, 2017, Vol. 159, No. 4, pp. 509–517.

Analysis of Finite Elasto-Plastic Strains: Integration Algorithm and Numerical Examples

L. U. Sultanov*

Kazan (Volga Region) Federal University, Kazan, 420008 Russia

Received August 29, 2017

Abstract—The paper is devoted to the development of a calculation technique for elasto-plastic solids with regard to finite strains. The kinematics of elasto-plastic strains is based on the multiplicative decomposition of the total deformation gradient into elastic and inelastic (plastic) components. The stress state is described by the Cauchy stress tensor. Physical relations are obtained from the equation of the second law of thermodynamics supplemented with a free energy function. The free energy function is written in an invariant form of the left Cauchy—Green elastic strain tensor. An elasto-plasticity model with isotropic strain hardening is considered. Based on an analog of the associated rule of plastic flows and the von Mises yield criterion, we develop the method of stress projection onto the yield surface (known as the radial return method) with an iterative refinement of the current stress-strain state. The iterative procedure is based on the introduction of additional virtual stresses to the resolving power equation. The constitutive relations for the rates and increments of the true Cauchy stresses are constructed. In terms of the incremental loading method, the variational equation is obtained on the basis of the principle of possible virtual powers. Spatial discretization is based on the finite element method; an octanodal finite element is used. We present the solution to the problem of tension of a circular bar and give a comparison with results of other authors.

DOI: 10.1134/S1995080218090056

Keywords and phrases: *nonlinear elasticity, finite strains, plasticity.*

INTRODUCTION

The present paper is a continuation of [1], where the principal kinematic and constitutive relations were given. The multiplicative decomposition of the deformation gradient into elastic and inelastic (plastic) components is used. To separate elastic and inelastic strains, the method of stress projection onto the yield surface, referred to as the radial return algorithm, is applied. An analog of the associated flow rule is obtained. The problem is solved in terms of the incremental loading method using the variational equation of the principle of virtual powers. The numerical discretization is based the finite element method.

1. KINEMATICS

The kinematics of finite elasto-plastic strains is described by means of the deformation gradient $\mathbf{F}$, which can be represented in the form of the multiplicative expansion [1–8]:

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p,$$

where $\mathbf{F}_e$ is the gradient of elastic deformations and $\mathbf{F}_p$ is the gradient of inelastic (plastic) deformations. The stress state is described by the Cauchy stress tensor $\boldsymbol{\sigma}$ [1, 9]:

$$\boldsymbol{\sigma} = 2\rho \mathbf{B}_e \cdot \frac{\partial \psi}{\partial \mathbf{B}_e} = \frac{2\rho_0}{J} \mathbf{B}_e \cdot \frac{\partial \psi}{\partial \mathbf{B}_e}, \tag{1}$$

*E-mail: Lenar.Sultanov@kpfu.ru

where $\mathbf{B}_e = {\mathbf{F}_e}^{\mathrm{T}} \cdot \mathbf{F}_e$ is the left Cauchy–Green elastic strain tensor, $J = \rho_0/\rho$ is the relative variation of the volume, ρ_0 is the density of a medium in the initial state, ρ is the density of the medium in the current state, ψ is the free energy function, which is defined as a scalar function of the tensor $\mathbf{B}_e$ of the elastic strain measure.

As the plastic yield condition, we take

$$\Phi_p(\boldsymbol{\sigma}, \chi) = 0, \tag{2}$$

where Φ_p the yield function and χ is the hardening parameter. We use the expression for the generalized rate $\overset{\nabla}{\mathbf{B}}_e$, which is obtained from the condition for the minimum of the generalized functional [1]:

$$\overset{\nabla}{\mathbf{B}}_e = -2\dot{\lambda}\frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}} \cdot \mathbf{B}_e. \tag{3}$$

Linearizing the constitutive relations (1) and using (3), we obtain the Cauchy stress rate tensor [1]:

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \boldsymbol{\Lambda}_e \cdot\cdot \mathbf{d} + \mathbf{h}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{h}^{\mathrm{T}} - I_{1d}{}^k\boldsymbol{\sigma} \\ &- \dot{\lambda}4\rho\left\{\frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}} \cdot \mathbf{B}_e \cdot \frac{\partial \psi}{\partial \mathbf{B}_e} + \mathbf{B}_e \cdot \frac{\partial^2 \psi}{\partial \mathbf{B}_e \partial \mathbf{B}_e} : \frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}}\mathbf{B}_e\right\} \\ &= \boldsymbol{\Lambda}_e \cdot\cdot \mathbf{d} + \mathbf{h}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{h}^{\mathrm{T}} - \dot{\lambda}\left\{2\frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}}\cdot\boldsymbol{\sigma} + \boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}}\right\} = \dot{\boldsymbol{\sigma}}_e + \dot{\boldsymbol{\sigma}}_p,\end{aligned} \tag{4}$$

where

$$\dot{\boldsymbol{\sigma}}_e = \boldsymbol{\Lambda}_e \cdot\cdot \mathbf{d} + \mathbf{h}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{h}^{\mathrm{T}} - I_{1d}{}^k\boldsymbol{\sigma}, \quad \dot{\boldsymbol{\sigma}}_p = -\dot{\lambda}\left\{2\frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}}\cdot\boldsymbol{\sigma} + \boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial \boldsymbol{\sigma}}\right\},$$

$$\boldsymbol{\Lambda}_e = \frac{4\rho_0}{J}\mathbf{B}_e \cdot \frac{\partial^2 \psi}{\partial \mathbf{B}_e \partial \mathbf{B}_e} \cdot \mathbf{B}_e.$$

2. ALGORITHM OF INTEGRATION OF THE CONSTITUTIVE RELATIONS

We will use the method of stress projection onto the yield surface [6, 10–14]. To this end, we write the Cauchy stress rate (4) in the kth state

$$\begin{aligned}{}^k\dot{\boldsymbol{\sigma}} &= {}^k\boldsymbol{\Lambda}_e \cdot\cdot {}^k\mathbf{d} + {}^k\mathbf{h}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\sigma}\cdot{}^k\mathbf{h}^{\mathrm{T}} - I_{1^k d}{}^k\boldsymbol{\sigma} \\ &- {}^k\dot{\lambda}\left\{2\frac{\partial \Phi_p}{\partial {}^{k+1}\boldsymbol{\sigma}}\cdot{}^{k+1}\boldsymbol{\sigma} + {}^{k+1}\boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial {}^{k+1}\boldsymbol{\sigma}}\right\}.\end{aligned} \tag{5}$$

Knowing the parameters of the kth state and using (5), we determine the $(k+1)$th state by the following formula:

$$\begin{aligned}{}^{k+1}\boldsymbol{\sigma} &= {}^k\boldsymbol{\sigma} + {}^k\dot{\boldsymbol{\sigma}}\Delta t \\ &= {}^k\boldsymbol{\sigma} + \left[{}^k\boldsymbol{\Lambda}_e \cdot\cdot {}^k\mathbf{d} + {}^k\mathbf{h}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\sigma}\cdot{}^k\mathbf{h}^{\mathrm{T}} - I_{1d}{}^k\boldsymbol{\sigma} - {}^k\dot{\lambda}\left(2\frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\right)\right]\Delta t \\ &= {}^k\boldsymbol{\sigma} + \left[{}^k\boldsymbol{\Lambda}_e \cdot\cdot {}^k\mathbf{d} + {}^k\mathbf{h}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\sigma}\cdot{}^k\mathbf{h}^{\mathrm{T}} - I_{1d}{}^k\boldsymbol{\sigma}\right]\Delta t - \dot{\lambda}\left(2\frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\right)\Delta t \\ &= {}^{k+1}\tilde{\boldsymbol{\sigma}} - \dot{\lambda}\left(2\frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial {}^k\boldsymbol{\sigma}}\right)\Delta t.\end{aligned} \tag{6}$$

Introducing the trial stress tensor into consideration

$${}^{k+1}\tilde{\boldsymbol{\sigma}} = {}^k\boldsymbol{\sigma} + \left[{}^{k+1}\boldsymbol{\Lambda}_e \cdot\cdot {}^k\mathbf{d} + {}^k\mathbf{h}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\sigma}\cdot{}^k\mathbf{h}^{\mathrm{T}} - I_{1^k\mathbf{d}}{}^k\boldsymbol{\sigma}\right]\Delta t,$$

from (6) we obtain the equation for determining the true stresses

$${}^{k+1}\boldsymbol{\sigma} = {}^{k+1}\tilde{\boldsymbol{\sigma}} - {}^k\dot{\lambda}\left(2\frac{\partial \Phi_p}{\partial {}^{k+1}\boldsymbol{\sigma}}\cdot\tilde{\boldsymbol{\sigma}} + {}^{k+1}\boldsymbol{\Lambda}_e \cdot\cdot \frac{\partial \Phi_p}{\partial {}^{k+1}\boldsymbol{\sigma}}\right)\Delta t. \tag{7}$$

The calculation algorithm consists in finding the tensor of true stresses and the plastic strain rate from the yield criterion (2) and relation (7).

3. INTEGRATION OF THE MOTION EQUATION

We represent the deformation process in the form of a sequence of stress-strain states of the studied body, which are realized at certain points of time (time layers). A similar strategy for solving nonlinear problems is prevalent at present and is successfully used in problems of both statics (the incremental loading method) and dynamics (a step-by-step integration method) [13–15]. In accordance with this technique, we assume that we know all parameters of the process at a certain time ${}^k t$, including the configuration, stress state, the magnitude of the elastic and plastic strains, and so on. A problem is to determine the stress-strain states at the time ${}^{k+1}t = {}^k t + \Delta t$.

As the starting point, we take the equation of virtual powers in the current state [9, 16, 17]. We write it as the operator equation $\mathbf{G} = 0$. The equation ${}^k\mathbf{G} = 0$ must be satisfied in the kth time layer. A similar equation in the next time layer can be represented in the form ${}^{k+1}\mathbf{G} = {}^k\mathbf{G} + {}^k\dot{\mathbf{G}}\Delta t = 0$. In our case, the equation of virtual powers in the current state (at $t = {}^{k+1}t$) takes the form

$$\left\{\int\limits_{V_k} {}^k\boldsymbol{\sigma}\cdot\cdot\delta^k\mathbf{d}dV - \int\limits_{S_k^\sigma} {}^k\mathbf{t}_n\delta\boldsymbol{v}dS - \int\limits_{V_k} {}^k\mathbf{F}\delta\boldsymbol{v}dV\right\}$$

$$+\frac{d}{dt}\left\{\int\limits_{V_k} {}^k\boldsymbol{\sigma}\cdot\cdot\delta^k\mathbf{d}dV - \int\limits_{S_k^\sigma} {}^k\mathbf{t}_n\cdot\delta\boldsymbol{v}dS - \int\limits_{V_k} {}^k\mathbf{F}\cdot\delta\boldsymbol{v}dV\right\}\Delta t = 0. \qquad (8)$$

Linearizing (8) and taking into account relation (5), we obtain the resolving equation

$$\int\limits_{V_k}\left\{{}^k\mathbf{d}\cdot\cdot{}^k\Lambda_e\cdot\cdot\delta\mathbf{d} + \frac{1}{2}{}^k\boldsymbol{\sigma}\cdot\cdot\left[\delta\mathbf{h}^{\mathrm{T}}\cdot{}^k\mathbf{h}+{}^k\mathbf{h}^{\mathrm{T}}\cdot\delta\mathbf{h}\right] - \left[{}^k\nabla\cdot{}^k\boldsymbol{v}\right]{}^k\mathbf{F}\cdot\delta\boldsymbol{v}\right.$$

$$\left.-\dot{\lambda}\left[2\frac{\partial\Phi_p}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma} + \boldsymbol{\Lambda}_e : \frac{\partial\Phi_p}{\partial\boldsymbol{\sigma}}\right]\right\}dV - \int\limits_{S_k^\sigma}\left\{{}^k\mathbf{t}_n\cdot{}^k\mathbf{h}^{\mathrm{T}} - \left[{}^k\nabla\cdot{}^k\boldsymbol{v}\right]{}^k\mathbf{t}_n\right\}\delta\boldsymbol{v}dS$$

$$= \int\limits_{S_k^\sigma} {}^k\Delta\mathbf{t}_n\cdot\delta\boldsymbol{v}dS + \int\limits_{V_k} {}^k\Delta\mathbf{F}\cdot\delta\boldsymbol{v}dV$$

$$-\frac{1}{\Delta t}\left\{\int\limits_{V_k} {}^k\boldsymbol{\sigma}\cdot\cdot\delta^k\mathbf{d}dV - \int\limits_{S_k^\sigma} {}^k\mathbf{t}_n\cdot\delta\boldsymbol{v}dS - \int\limits_{V_k} {}^k\mathbf{F}\cdot\delta\boldsymbol{v}dV\right\}. \qquad (9)$$

Since quasistatic problems are solved, we may pass from rates to increments, for example, by putting $\boldsymbol{v} = \Delta\mathbf{u}/\Delta t$.

Solving equation (9), we obtain the displacement vector $\mathbf{u}$ that determines the configuration at the next step of loading

$${}^{k+1}\mathbf{x} = {}^k\mathbf{x} + \boldsymbol{v}\Delta t.$$

In statics problems, the time increment is a loading parameter, which is usually assumed to be equal to unity. When plastic strains occur, the method of stress projection with iterative refinement is used, which consists in the following. By solving the elastic problem (9), we find the increment of the displacement field along which we determine the trial stress field at the next step of loading

$$\tilde{\boldsymbol{\sigma}} = {}^k\boldsymbol{\sigma} + \left[{}^{k+1}\boldsymbol{\Lambda}_e\cdot\cdot{}^k\mathbf{d} + {}^k\mathbf{h}\cdot{}^k\boldsymbol{\sigma} + {}^k\boldsymbol{\sigma}\cdot{}^k\mathbf{h}^{\mathrm{T}} - I_{1^k\mathbf{d}}{}^k\boldsymbol{\sigma}\right].$$

Then we apply the method of projection of stresses onto the yield surface, which consists in solving the system of nonlinear equations

$${}^{k+1}\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} - \Delta\lambda\left(2\frac{\partial\Phi_p}{\partial^{k+1}\boldsymbol{\sigma}}\cdot\tilde{\boldsymbol{\sigma}} + {}^{k+1}\boldsymbol{\Lambda}_e\cdot\cdot\frac{\partial\Phi_p}{\partial^{k+1}\boldsymbol{\sigma}}\right) = 0, \qquad (10)$$

$$\Phi_p\left({}^{k+1}\boldsymbol{\sigma}, \chi\left(\Delta\lambda\right)\right) = 0. \tag{11}$$

Having solved the system of nonlinear equations (10), (11) by the Newton method, we obtain the tensor of true stresses ${}^{k+1}\boldsymbol{\sigma}$ and the increment of the rate of plastic strains $\Delta\lambda$. Due to the fact that the true stress state thus obtained does not satisfy the equilibrium equation (9), we construct an iterative procedure for refinement of the obtained stress-strain states, which is based on the introduction, into the resolving system of equations (9), of the power of additional stresses on the possible rate distortions

$$\int\limits_{V_k}\left\{{}^k_m\mathbf{d}\cdot\cdot{}^k\Lambda_e\cdot\cdot\delta\mathbf{d} + \frac{1}{2}{}^k\boldsymbol{\sigma}\cdot\cdot\left[\delta\mathbf{h}^{\mathrm{T}}\cdot{}^k_m\mathbf{h} + {}^k_m\mathbf{h}^{\mathrm{T}}\cdot\delta\mathbf{h}\right] - \left[{}^k\nabla\cdot{}^k\boldsymbol{v}\right]{}^k\mathbf{F}\cdot\delta\boldsymbol{v}\right.$$

$$\left.-\ \Delta^k_m\lambda\left[2\frac{\partial\Phi_p}{\partial^k_m\boldsymbol{\sigma}}\cdot{}^k_m\boldsymbol{\sigma} + {}^k\Lambda_e\cdot\cdot\frac{\partial\Phi_p}{\partial^k_m\boldsymbol{\sigma}}\right]\right\}dV - \int\limits_{S^\sigma_k}\left\{{}^k\mathbf{t}_n\cdot{}^k_m\mathbf{h}^{\mathrm{T}} - \left[{}^k_m\nabla\cdot{}^k_m\boldsymbol{v}\right]{}^k\mathbf{t}_n\right\}\delta\boldsymbol{v}dS$$

$$= \int\limits_{S^\sigma_k}\Delta^k\mathbf{t}_n\cdot\delta\boldsymbol{v}dS + \int\limits_{V_k}\Delta^k\mathbf{F}\cdot\delta\boldsymbol{v}dV$$

$$-\left\{\int\limits_{V_k}{}^k\boldsymbol{\sigma}\cdot\cdot\delta^k\mathbf{d}dV - \int\limits_{S^\sigma_k}{}^k\mathbf{t}_n\cdot\delta\boldsymbol{v}dS - \int\limits_{V_k}{}^k\mathbf{F}\cdot\delta\boldsymbol{v}dV\right\} + \int\limits_{V_k}{}^k_m\boldsymbol{\sigma}_{ad}\cdot\cdot\delta^k\mathbf{d}dV,$$

where ${}^{k+1}_m\boldsymbol{\sigma}_{ad} = {}^{k+1}_m\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}$.

4. NUMERICAL EXAMPLE

Consider the problem of tension of a circular bar with the following characteristics: cross-section radius $R = 6.413$ mm and bar length $L = 53.334$ mm. Note that to specify the location of the neck in the center of the bar, we reduce the radius by 1.8% [2–4]. The free energy function is given as follows:

$$\rho_0\psi = \frac{\lambda + 2\mu}{8}\left(I_{1\mathbf{B}} - 3\right)^2 + \mu\left(I_{1\mathbf{B}} - 3\right) - \frac{\mu}{2}\left(I_{2\mathbf{B}} - 3\right),$$

where λ and μ are the Lame coefficients. As a plastic yield criterion, we take the Huber-Mises condition, which admits the following generalization for an isotropic medium:

$$\Phi = \sigma_{\text{int}} - \sigma_T\left(\chi\right) \le 0.$$

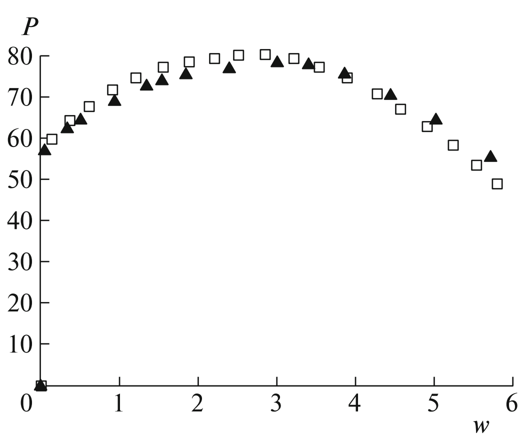

Fig. 1. Force–displacement graph. Solid line is the dependence obtained with the help of the developed algorithm, □ — [2], ▲ — [3].

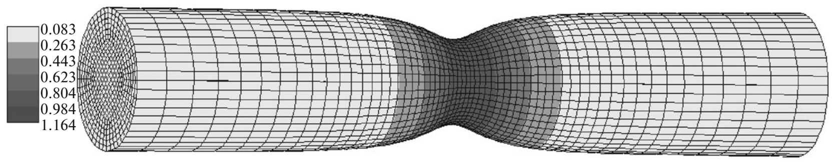

Fig. 2. Plastic strain intensity.

Here $\sigma_{\text{int}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' \cdot \cdot \boldsymbol{\sigma}'}$ is the stress intensity, $\sigma_T(\chi)$ is the hardening function, χ is the hardening parameter, and $\boldsymbol{\sigma}'$ is the deviator of the stress tensor.

The nonlinear isotropic hardening function has the form [2–4]

$$\sigma_T(\chi) = \sigma_T + h\chi + (\sigma_\infty - \sigma_T)(1 - e^{-\delta\chi}).$$

The parameters of a material are as follows: $E = 206900$ MPa, $\nu = 0.29$, $\sigma_\infty = 715$ MPa, $\sigma_T = 450$ MPa, $h = 0.129$, and $\delta = 16.93$. The numerical implementation is based on the finite element method. An octanodal 3D finite element is used. Figure 1 displays a graph of the dependence of the force arising at the end of the bar upon the displacement. The solid line shows the dependence obtained by the above-described technique, tiny $square$ — by [2], $blacktriangle$ —by [3]. It is seen that our results are in good agreement with the results of other authors. Figure 2 depicts the deformed state of the bar with the field of intensity of plastic strains. The intensity of plastic strains takes the maximum value in the neck formation region.

CONCLUSION

The paper presents a technique for solving deformation problems with finite elasto-plastic strains. The multiplicative decomposition of the strain gradient is used. The constitutive relation is written in the real state in the form of the dependence of the Cauchy stress tensor on the left Cauchy–Green strain tensor. To take plastic strains into account, we use the method of stress projection onto the yield surface with an iterative refinement of the current stress-strain state. In terms of the incremental loading method, the resolving equation is obtained. The numerical implementation is based on the finite element method. As a demonstration of the working capacity of the developed technique, the problem of tension of a bar with the neck formation has been solved and the results of the solution have been compared with the solutions of other authors.

ACKNOWLEDGMENTS

This work was financially supported by the Russian Science Foundation under the research project no. 16-11-10299.

REFERENCES

1. L. U. Sultanov, "Analysis of finite elastoplastic deformations. Kinematics and constitutive equations," Uchen. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki **157** (4), 158–165 (2015).
2. B. Eidel and F. Gruttmann, "Elastoplastic orthotropy at finite strains: multiplicative formulation and numerical implementation," Comput. Mater. Sci. **28**, 732–742 (2003). doi 10.1016/j.commatsci.2003.08.027
3. J. Schröder, F. Gruttmann, and J. Löblein, "A simple orthotropic finite elasto-plasticity model based on generalized stress-strain measures," Comput. Mech. **30**, 48–64 (2002). doi 10.1007/s00466-002-0366-3
4. J. S. Simo, "A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation," Comput. Methods Appl. Mech. Eng. **66**, 199–219 (1988). doi 10.1016/0045-7825(88)90076-X
5. C. Miehe, "A theory of large-strain isotropic thermoplasticity based on metric transformation tensors", Arch. Appl. Mech. **66**, 45–64 (1995). doi 10.1007/BF00786688
6. Y. Basar and M. Itskov, "Constitutive model and finite element formulation for large strain elasto-plastic analysis of shell," Comput. Mech. **23**, 466–481 (1999). doi 10.1007/s004660050426

7. A. Meyers, P. Schievbe, and O. T. Bruhns, "Some comments on objective rates of symmetric Eulerian tensors with application to Eulerian strain rates," Acta Mech. **139**, 91–103 (2000). doi 10.1007/BF01170184
8. H. Xiao, O. T. Bruhns, and A. Meyers, "A consistent finite elastoplasticity theory combining additive and multiplicative decomposition of the stretching and deformation gradient", Int. J. Plasticity **16**, 143–177 (2000). doi 10.1016/S0749-6419(99)00045-5
9. J. Bonet and R. D. Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis* (Cambridge Univ. Press, Cambridge, 1997).
10. M. Rouainia and D. M. Wood, "Computational aspects in finite strain plasticity analysis of geotechnical materials," Mech. Res. Commun. **33**, 123–133 (2006). doi 10.1016/j.mechrescom.2005.06.014
11. J. S. Simo and M. Ortiz, "A unified approach to finite deformation elastoplastic analysis lased on the use of hyperelastic constitutive equations," Comput. Methods. Appl. Mech. Eng. **49**, 221–245 (1985). doi 10.1016/0045-7825(85)90061-1
12. A. L. Eterovic and K.-J. Bathe, "A hyperelastic-based large strain elasto-plastic constitutive formulation with combined isotropic-kinematic hardening using the logarithmic stress and strain measures," Int. J. Numer. Meth. Eng. **30**, 1099–1114 (1990). doi 10.1002/nme.1620300602
13. R. L. Davydov and L. U. Sultanov, "Numerical algorithm of solving the problem of large elastic-plastic deformation by FEM," Vestn. Perm. Politekh. Univ., Mekh., No. 1, 81–93 (2013).
14. R. L. Davydov and L. U. Sultanov, "Numerical algorithm for investigating large elasto-plastic deformations", J. Eng. Phys. Thermophys. **88**, 1280–1288 (2015). doi 10.1007/s10891-015-1310-7
15. A. I. Golovanov and L. U. Sultanov, "Numerical investigation of large elastoplastic strains of three-dimensional bodies," Int. Appl. Mech. **41**, 614–620 (2005). doi 10.1007/s10778-005-0129-x
16. A. I. Abdrakhmanova and L. U. Sultanov, "Numerical modelling of deformation of hyperelastic incompressible solids," Mater. Phys. Mech. **26**, 30–32 (2016).
17. A. I. Golovanov, Yu. G. Konoplev, and L. U. Sultanov, "Numerical investigation of finite deformations of hyperelastic bodies. IV. Finite-element implementation. Examples of the solution of problems," Uch. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki **152**, 115–126 (2010).

Translated by I. Tselishcheva