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Error Investigation of a Finite Element Approximation for a Nonlinear Sturm–Liouville Problem

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Abstract—A positive definite differential eigenvalue problem with coefficients depending nonlinearly on the spectral parameter is studied. The problem is formulated as a variational eigenvalue problem in a Hilbert space with bilinear forms depending nonlinearly on the spectral parameter. The variational problem has an increasing sequence of positive simple eigenvalues that correspond to a normalized system of eigenfunctions. The variational problem is approximated by a finite element mesh scheme on a uniform grid with Lagrangian finite elements of arbitrary order. Error estimates for approximate eigenvalues and eigenfunctions are proved depending on the mesh size and the eigenvalue size. The results obtained are generalizations of well-known results for differential eigenvalue problems with linear dependence on the spectral parameter.

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INTRODUCTION

We study the positive definite differential eigenvalue problem $-(p(\lambda,x)u'(x))'+q(\lambda,x)u(x)=\lambda r(\lambda,x)u(x)$, $\lambda\in(0,\infty)$, $x\in(0,1)$, $u(0)=u(1)=0$, with given coefficients $p(\mu,x)$, $q(\mu,x)$, $r(\mu,x)$, $\mu\in(0,\infty)$, $x\in[0,1]$. For a fixed $x\in[0,1]$, the functions $p(\mu,x)$, $q(\mu,x)$, $\mu\in(0,\infty)$, are nonincreasing, while the function $r(\mu,x)$, $\mu\in(0,\infty)$, is nondecreasing. The differential problem is equivalent to the following variational eigenvalue problem: $\lambda\in(0,\infty)$, $u\in V\setminus\{0\}$, $a(\lambda,u,v)=\lambda b(\lambda,u,v)$ for any function $v\in V$. Here, $V=\{v:\ v\in W_2^1(0,1),\ v(0)=v(1)=0\}$ is a Hilbert space with norm $|\cdot|_1$. According to [1], this problem has an increasing sequence of positive simple eigenvalues λ_k, $k=1,2,\ldots$, with a limit point at infinity: $0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots$, $\lambda_k\to\infty$ as $k\to\infty$. This sequence of eigenvalues is associated with a system of normalized eigenfunctions u_k, $k=1,2,\ldots$.

The variational eigenvalue problem is approximated by the following finite element mesh scheme: $\lambda^h\in(0,\infty)$, $u^h\in V_h\setminus\{0\}$, $a(\lambda^h,u^h,v^h)=\lambda^h b(\lambda^h,u^h,v^h)$ for any function $v^h\in V_h$. Here, V_h is the space of Lagrangian finite elements of order n. For sufficiently small h, the mesh eigenvalue problem has N_h positive simple eigenvalues λ_k^h, where $N_h=\dim V_h$, $k=1,2,\ldots,N_h$, and $0<\lambda_1^h<\lambda_2^h<\cdots<\lambda_{N_h}^h$. The eigenvalues λ_k^h, $k=1,2,\ldots,N_h$, are associated with a system of normalized eigenfunctions u_k^h, $k=1,2,\ldots,N_h$. For sufficiently small h, we prove the error estimates

$$0\le\lambda_k^h-\lambda_k\le ch^{2n}\lambda_k^{n+1},\quad |u_k^h-u_k|_1\le ch^n\lambda_k^{(n+1)/2},$$

where c is a constant independent of h and λ_k, the signs of the normalized eigenfunctions u_k^h and u_k are chosen according to the conditions $b(\lambda_k,u_k^h,u_k)>0$, $b(\lambda_k^h,u_k^h,u_k^h)=1$, $b(\lambda_k,u_k,u_k)=1$.

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Nonlinear eigenvalue problems arise in various areas of science and engineering, for example, in plasma physics, structural mechanics, numerical algorithms for discrete equations, and the theory of dielectric waveguides [1–5]. Computational methods for solving nonlinear matrix eigenvalue problems were investigated in [5]. As applied to nonlinear differential eigenvalue problems, the finite element method was addressed in [6], and the influence of numerical integration in finite element schemes was studied in [1, 7, 8] with the help of the results from [9–12]. In [13] errors in approximate methods for solving nonlinear eigenvalue problems in a Hilbert space were analyzed using general results for the linear case [14–16]. Approximate methods for solving nonlinear boundary value problems and variational inequalities arising in applications were investigated in [17–23].

1. VARIATIONAL FORMULATION OF THE PROBLEM

Let $\Omega = (0,1)$, $\overline{\Omega} = [0,1]$, G be an interval of the number line $\mathbb{R}$, and $\Lambda = (0,\infty)$. As usual, $L_2(G)$ and $W_2^m(G)$ denote the real Lebesgue and real Sobolev spaces, respectively, equipped with the norms

$$|u|_{0,G} = \left(\int\limits_G (u(x))^2 dx\right)^{1/2}, \quad ||u||_{m,G} = \left(\sum_{i=0}^m |u|_{i,G}^2\right)^{1/2}$$

and seminorms $|u|_{i,G} = |u^{(i)}|_{0,G}$, $i = 0,1,\ldots,m$, where $u^{(i)} = d^i u(x)/dx^i$, $i = 1,2,\ldots,m$, $u^{(0)} = u$, and m is a positive integer. Additionally, let $W_2^0(G) = L_2(G)$. If $G = \Omega$, then, for brevity, the domain index is omitted from the notation of the norms and seminorms.

Let $p(\mu,x)$, $q(\mu,x)$, $r(\mu,x)$, $\mu \in \Lambda$, $x \in \overline{\Omega}$, be given infinitely continuously differentiable functions. Assume that $p(\mu,x)$ and $r(\mu,x)$, $\mu \in \Lambda$, $x \in \overline{\Omega}$, are positive, while the function $q(\mu,x)$, $\mu \in \Lambda$, $x \in \overline{\Omega}$, is nonnegative. For fixed $x \in \overline{\Omega}$, the functions $p(\mu,x)$ and $q(\mu,x)$, $\mu \in \Lambda$, are nonincreasing, while the function $r(\mu,x)$, $\mu \in \Lambda$, is nondecreasing. Assume that there exist positive constants p_1, p_2, p_3, p_4, q_2, q_3, r_1, r_2, and r_3 such that

$$p_1 \le p(\mu,x) \le p_2, \quad 0 \le q(\mu,x) \le q_2, \quad r_1 \le r(\mu,x) \le r_2,$$
$$\left|\frac{\partial p(\mu,x)}{\partial \mu}\right| \le p_3, \quad \left|\frac{\partial q(\mu,x)}{\partial \mu}\right| \le q_3, \quad \left|\frac{\partial r(\mu,x)}{\partial \mu}\right| \le r_3,$$
$$\left|\frac{\partial^i p(\mu,x)}{\partial x^i}\right| \le p_4, \quad \left|\frac{\partial^i q(\mu,x)}{\partial x^i}\right| \le q_4, \quad \left|\frac{\partial^i r(\mu,x)}{\partial x^i}\right| \le r_4$$

for any $\mu \in \Lambda$, $x \in \overline{\Omega}$, $i = 1,2,\ldots$.

Consider the following differential eigenvalue problem: find numbers $\lambda \in \Lambda$ and nonzero functions $u(x)$, $x \in \overline{\Omega}$, such that

$$-(p(\lambda,x)u'(x))' + q(\lambda,x)u(x) = \lambda r(\lambda,x)u(x), \quad x \in \Omega, \quad u(0) = u(1) = 0.$$

We introduce the Hilbert space $V = \{v : \ v \in W_2^1(\Omega),\ v(0) = v(1) = 0\}$ with norm $|\cdot|_1$. It is easy to see that the Friedrichs inequality $|v|_0 \le |v|_1$ holds for any $v \in V$. For $\mu \in \Lambda$ and $u,v \in V$, we define the bilinear forms

$$a(\mu,u,v) = \int\limits_0^1 (p(\mu,x)u'v' + q(\mu,x)uv)dx, \quad b(\mu,u,v) = \int\limits_0^1 r(\mu,x)uvdx.$$

The generalized formulation of the differential eigenvalue problem is as follows: find $\lambda \in \Lambda$ and $u \in V \setminus \{0\}$ such that

$$a(\lambda,u,v) = \lambda b(\lambda,u,v) \quad \forall v \in V. \tag{1}$$

Consider an auxiliary linear eigenvalue problem for a fixed $\mu \in \Lambda$: find functions $\gamma = \gamma(\mu) \in \Lambda$ and $y = y(\mu) \in V \setminus \{0\}$ such that

$$a(\mu,y,v) = \gamma b(\mu,y,v) \quad \forall v \in V. \tag{2}$$

According to [24], problem (2) has a sequence of positive simple eigenvalues $\gamma_k = \gamma_k(\mu)$, $k = 1, 2, \ldots$, arranged in increasing order: $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_k < \cdots$, $\gamma_k \to \infty$ as $k \to \infty$. These eigenvalues are associated with an orthonormal system of eigenfunctions $y_k = y_k(\mu)$, $k = 1, 2, \ldots$, such that $a(\mu, y_i, y_j) = \gamma_i \delta_{ij}$, $b(\mu, y_i, y_j) = \delta_{ij}$, $i, j = 1, 2, \ldots$. The eigenfunctions y_k, $k = 1, 2, \ldots$, form a complete system in the space V. It is true that $\gamma_k(\mu) \geq \gamma_k(\eta)$ for $\mu < \eta$, where $\mu, \eta \in \Lambda$.

Lemma 1. *For any functions $v \in V$ and any numbers $\mu, \eta \in \Lambda$, it holds that*

$$\alpha_1 |v|_1^2 \leq a(\mu, v, v) \leq \alpha_2 |v|_1^2, \quad |a(\mu, v, v) - a(\eta, v, v)| \leq \alpha_3 |\mu - \eta| |v|_1^2,$$
$$\beta_1 |v|_0^2 \leq b(\mu, v, v) \leq \beta_2 |v|_0^2, \quad |b(\mu, v, v) - b(\eta, v, v)| \leq \beta_3 |\mu - \eta| |v|_0^2,$$

where $\alpha_1 = p_1$, $\alpha_2 = p_2 + q_2$, $\beta_1 = r_1$, $\beta_2 = r_2$, $\alpha_3 = p_3 + q_3$, $\beta_3 = r_3$.

Proof. The required inequalities follow from the definitions of the bilinear forms and the properties of the coefficients of the problem. The lemma is proved. □

Theorem 1. *Problem* (1) *has a sequence of positive simple eigenvalues λ_k, $k = 1, 2, \ldots$, arranged in increasing order: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. Each eigenvalue λ_i, $i \geq 1$, is a unique root of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $\quad i \geq 1$. The sequence of eigenvalues λ_k, $k = 1, 2, \ldots$, is associated with a sequence of eigenfunctions u_k, $k = 1, 2, \ldots$. The eigenfunction u_k coincides with the eigenfunction y_k corresponding to the eigenvalue $\gamma_k(\mu)$ of the linear parametric eigenvalue problem (2) with $\mu = \lambda_k$. The eigenfunctions u_k, $k = 1, 2, \ldots$, are infinitely continuously differentiable and satisfy the estimates $|u_k|_i \leq s_i \lambda_k^{i/2}$, $k = 1, 2, \ldots$, $i = 0, 1, 2, \ldots$, where s_i, $i = 0, 1, 2, \ldots$, are constants independent of λ_k, $k = 1, 2, \ldots$.*

Proof. The theorem is proved using the results of [1, 13, 24]. The theorem is proved. □

2. MESH APPROXIMATION OF THE PROBLEM

The interval $\overline{\Omega}$ is divided by equidistant points $x_i = ih$, $i = 0, 1, \ldots, m$, into elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, $h = 1/m$. Let V_h denote the space of Lagrangian finite elements consisting of continuous functions v^h on $\overline{\Omega}$ that are polynomials of degree at most n on each element e_i, $i = 1, 2, \ldots, m$, $v^h(0) = v^h(1) = 0$, $N_h = \dim V_h = mn - 1$.

The variational problem (1) is approximated by the following finite element mesh scheme: find $\lambda^h \in \Lambda$ and $u^h \in V_h \setminus \{0\}$ such that

$$a(\lambda^h, u^h, v^h) = \lambda^h b(\lambda^h, u^h, v^h) \quad \forall v^h \in V_h. \tag{3}$$

Consider the following auxiliary linear eigenvalue problem with a fixed $\mu \in \Lambda$: find functions $\gamma^h = \gamma^h(\mu) \in \Lambda$, $y^h = y^h(\mu) \in V_h \setminus \{0\}$ such that

$$a(\mu, y^h, v^h) = \gamma^h b(\mu, y^h, v^h) \quad \forall v^h \in V_h. \tag{4}$$

Problem (4) has N_h positive simple eigenvalues $\gamma_k^h = \gamma_k^h(\mu)$, $k = 1, 2, \ldots, N_h$, arranged in increasing order: $0 < \gamma_1^h < \gamma_2^h < \cdots < \gamma_{N_h}^h$. These eigenvalues are associated with an orthonormal system of eigenfunctions $y_k^h = y_k^h(\mu)$, $k = 1, 2, \ldots, N_h$, such that $a(\mu, y_i^h, y_j^h) = \gamma_i^h \delta_{ij}$, $b(\mu, y_i^h, y_j^h) = \delta_{ij}$, $i, j = 1, 2, \ldots, N_h$. The eigenfunctions y_k^h, $k = 1, 2, \ldots, N_h$, form a complete system in the space V_h. The inequalities $\gamma_k^h(\mu) \geq \gamma_k^h(\eta)$ hold for $\mu < \eta$, where $\mu, \eta \in \Lambda$.

Theorem 2. *Problem* (3) *has N_h positive simple eigenvalues λ_k^h, $k = 1, 2, \ldots, N_h$, arranged in increasing order*: $0 < \lambda_1^h < \lambda_2^h < \cdots < \lambda_{N_h}^h$. Each eigenvalue λ_i^h, $i \geq 1$, is a unique root of the equation $\mu - \gamma_i^h(\mu) = 0$, $\mu \in \Lambda$, $i \geq 1$. *The eigenvalues λ_k^h, $k = 1, 2, \ldots, N_h$, are associated with eigenfunctions u_k^h, $k = 1, 2, \ldots, N_h$. The eigenfunction u_k^h coincides with the eigenfunction y_k^h corresponding to the eigenvalue $\gamma_k^h(\mu)$ of the linear parametric eigenvalue problem* (4) *with $\mu = \lambda_k^h$.*

Proof. It is similar to the proof of Theorem 1 taking into account that problems (3) and (4) are finite-dimensional. The theorem is proved. □

3. ERROR ANALYSIS OF THE MESH SCHEME

Let u_k be the eigenfunction of problem (1) corresponding to the eigenvalue λ_k, and let u_k^h be the eigenfunction of problem (3) corresponding to the eigenvalue λ_k^h. Denote by c different positive constants independent of h and λ_k.

For $\mu \in \Lambda$, the orthogonal projector $P_h(\mu) : V \to V_h$ is defined by the formula $a(\mu, u - P_h(\mu)u, v^h) = 0$ for any $v^h \in V_h$, where $u \in V$. Define $P_h = P_h(\lambda_k)$.

Lemma 2. *The error of the orthogonal projector satisfies the estimates*

$$|u_k - P_h u_k|_i \le c h^{n+1-i} \lambda_k^{(n+1)/2}, \quad i = 0, 1.$$

Proof. The error estimates from the lemma are derived using the results of [25]. The lemma is proved. □

Theorem 3. *For sufficiently small h, we have the error estimate*

$$0 \le \lambda_k^h - \lambda_k \le c h^{2n} \lambda_k^{n+1}.$$

Proof. The required estimate follows from the relations

$$0 \le \lambda_k^h - \lambda_k = \gamma_k^h(\lambda_k^h) - \gamma_k(\lambda_k) \le \gamma_k^h(\lambda_k) - \gamma_k(\lambda_k),$$

the error estimate for approximate eigenvalues of the linear eigenvalue problem [12], and Lemmas 1 and 2. The theorem is proved. □

Theorem 4. *For sufficiently small h, the following error estimate holds:*

$$|u_k^h - u_k|_1 \le c h^n \lambda_k^{(n+1)/2},$$

where $b(\lambda_k, u_k^h, u_k) > 0$, $b(\lambda_k^h, u_k^h, u_k^h) = 1$, *and* $b(\lambda_k, u_k, u_k) = 1$.

Proof. Let $\beta_i^h = b(\lambda_k^h, P_h u_k, y_i^h)$, $i = 1, 2, \ldots, N_h$, where y_i^h, $i = 1, 2, \ldots, N_h$, are the eigenfunctions of problem (4) with $\mu = \lambda_k^h$ and $b(\lambda_k, u_k, u_k) = 1$. Note that $y_k^h = u_k^h$. Since the eigenfunctions y_i^h, $i = 1, 2, \ldots, N_h$, form an orthonormal basis in V_h, the element $P_h u_k \in V_h$ can be represented in the form of the decomposition $P_h u_k = \beta_k^h u_k^h + v_k^h + w_k^h$, where

$$v_k^h = \sum_{i=1}^{k-1} \beta_i^h y_i^h, \quad w_k^h = \sum_{i=k+1}^{N_h} \beta_i^h y_i^h.$$

Let

$$\xi_k^h = \sup_{v^h \in V_h \setminus \{0\}} \frac{|a(\lambda_k^h, P_h u_k, v^h) - \lambda_k b(\lambda_k^h, P_h u_k, v^h)|}{|v^h|_1}.$$

It is true that $\xi_k^h \le c h^{n+1} \lambda_k^{(n+3)/2}$.

For $k \ge 1$ and $\lambda_0 = 0$, define

$$\rho_k = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} + \frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda_k}.$$

Theorem 3 implies that $\lambda_i^h \to \lambda_i$ as $h \to 0$, $i = 1, 2, \ldots, k+1$. Therefore, for sufficiently small h, we have $\lambda_k - \lambda_{k-1}^h > 0$ and $\lambda_{k+1}^h - \lambda_k > 0$, where $k \ge 1$ and $\lambda_0^h = 0$; moreover, for a positive constant c,

$$\frac{\lambda_{k-1}^h}{\lambda_k - \lambda_{k-1}^h} \le c\rho_k, \quad \frac{\lambda_{k+1}^h}{\lambda_{k+1}^h - \lambda_k} \le c\rho_k.$$

For $k \ge 1$ we prove the estimate $|v_k^h|_1 \le c\rho_k \xi_k^h$. Obviously, this estimate holds for $k = 1$. Let $k \ge 2$. Then

$$a(\lambda_k^h, P_h u_k, v_k^h) = a(\lambda_k^h, v_k^h, v_k^h), \quad b(\lambda_k^h, P_h u_k, v_k^h) = b(\lambda_k^h, v_k^h, v_k^h),$$

$$a(\lambda_k^h, v_k^h, v_k^h) \le \lambda_{k-1}^h b(\lambda_k^h, v_k^h, v_k^h).$$

Thus, we obtain the chain of inequalities

$$|v_k^h|_1 \xi_k^h \ge -a(\lambda_k^h, P_h u_k, v_k^h) + \lambda_k b(\lambda_k^h, P_h u_k, v_k^h) = -a(\lambda_k^h, v_k^h, v_k^h) + \lambda_k b(\lambda_k^h, v_k^h, v_k^h)$$
$$\ge (\lambda_k - \lambda_{k-1}^h) b(\lambda_k^h, v_k^h, v_k^h) \ge \frac{\lambda_k - \lambda_{k-1}^h}{\lambda_{k-1}^h} a(\lambda_k^h, v_k^h, v_k^h) \ge \frac{1}{c\rho_k} |v_k^h|_1^2, \quad k \ge 2,$$

which yield the required estimate.

For $k \ge 1$ we prove the estimate $|w_k^h|_1 \le c\rho_k \xi_k^h$. It is easy to see that

$$a(\lambda_k^h, P_h u_k, w_k^h) = a(\lambda_k^h, w_k^h, w_k^h), \quad b(\lambda_k^h, P_h u_k, w_k^h) = b(\lambda_k^h, w_k^h, w_k^h),$$
$$a(\lambda_k^h, w_k^h, w_k^h) \ge \lambda_{k+1}^h b(\lambda_k^h, w_k^h, w_k^h).$$

Then we have the relations

$$|w_k^h|_1 \xi_k^h \ge a(\lambda_k^h, P_h u_k, w_k^h) - \lambda_k b(\lambda_k^h, P_h u_k, w_k^h)$$
$$= a(\lambda_k^h, w_k^h, w_k^h) + \lambda_k b(\lambda_k^h, w_k^h, w_k^h) \ge \frac{\lambda_{k+1}^h - \lambda_k}{\lambda_{k+1}^h} a(\lambda_k^h, w_k^h, w_k^h) \ge \frac{1}{c\rho_k} |w_k^h|_1^2, \quad k \ge 1,$$

which lead to the required estimate.

Now, using the above-derived estimates, we have

$$|P_h u_k - \beta_k^h u_k^h|_1 \le |v_k^h|_1 + |w_k^h|_1 \le c\rho_k \xi_k^h \le c\rho_k h^{n+1} \lambda_k^{(n+3)/2}$$

for sufficiently small h.

Define $||v||_b^2 = b(\lambda_k, v, v)$ and $||v^h||_{b_h}^2 = b(\lambda_k^h, v^h, v^h)$ for any $v \in V$ and $v^h \in V_h$. Then the relations

$$\beta_k^h = ||\beta_k^h u_k^h||_{b_h} \le 1 + ||u_k - P_h u_k||_b + ||P_h u_k - \beta_k^h u_k^h||_{b_h} + |||P_h u_k||_{b_h} - ||P_h u_k||_b|,$$
$$\beta_k^h = ||\beta_k^h u_k^h||_{b_h} \ge 1 - ||u_k - P_h u_k||_b - ||P_h u_k - \beta_k^h u_k^h||_{b_h} - |||P_h u_k||_{b_h} - ||P_h u_k||_b|$$

yield

$$|\beta_k^h - 1| \le ||u_k - P_h u_k||_b + ||P_h u_k - \beta_k^h u_k^h||_{b_h} + c|||P_h u_k||_{b_h}^2 - ||P_h u_k||_b^2|,$$
$$|\beta_k^h u_k^h - u_k^h|_1 = |\beta_k^h - 1||u_k^h|_1 \le c\sqrt{\lambda_k}|\beta_k^h - 1| \le c\rho_k h^{n+1} \lambda_k^{(n+4)/2}$$

for sufficiently small h. Finally, we conclude that

$$|u_k - u_k^h|_1 \le |u_k - P_h u_k|_1 + |P_h u_k - \beta_k^h u_k^h|_1 + |\beta_k^h u_k^h - u_k^h|_1 \le c h^n \lambda_k^{(n+1)/2}.$$

The theorem is proved. □

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