ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2018, Vol. 39, No. 7, pp. 949–956. © Pleiades Publishing, Ltd., 2018.

Finite Element Approximation of the Minimal Eigenvalue of a Nonlinear Eigenvalue Problem

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(Submitted by A. V. Lapin)

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Received January 10, 2018

Abstract—The problem of finding the minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem for the ordinary differential equation with coefficients depending on a spectral parameter is investigated. This problem arises in modeling the plasma of radio-frequency discharge at reduced pressures. A necessary and sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem is established. The original differential eigenvalue problem is approximated by the finite element method on a uniform grid. The convergence of approximate eigenvalue and approximate positive eigenfunction to exact ones is proved. Investigations of this paper generalize well known results for eigenvalue problems with linear dependence on the spectral parameter.

DOI: 10.1134/S199508021807020X

Keywords and phrases: *Radio-frequency induction discharge, eigenvalue, positive eigenfunction, nonlinear eigenvalue problem, ordinary differential equation, finite element method.*

1. INTRODUCTION

In the present paper, we investigate the following differential nonlinear eigenvalue problem: find minimal eigenvalue $\lambda \in \Lambda$, $\Lambda = [0, \infty)$, corresponding to a positive eigenfunction $u(x)$, $x \in \Omega$, $\Omega = (0, \pi)$, $\overline{\Omega} = [0, \pi]$, satisfying the following equations

$$-(p(\lambda s(x))u')' = r(\lambda s(x))u, \quad x \in \Omega, \quad u(0) = u(\pi) = 0. \qquad (1)$$

We assume that $p(\mu)$, $r(\mu)$, $\mu \in \Lambda$, and $s(x)$, $x \in \overline{\Omega}$ are continuous positive functions. We also assume that the function $p(\mu)$, $\mu \in \Lambda$ is bounded and the function $r(\mu)$, $\mu \in \Lambda$ is unbounded. Note that the differential equation of problem (1) is treated in the weak sense.

Nonlinear eigenvalue problems of the form (1) arise in modeling the plasma of radio-frequency discharge at reduced pressures. An inductive coupled radio-frequency discharge has found broad applications in diverse technological plasma processes, such as processing textiles and leather-fur half-finished products, metals, hydrogen accumulation by silicon powders, synthesis of oxygen-free ceramic materials, and obtaining carbide and boride materials for nuclear and processing industry [1–5]. A more effective and qualitative choice of constructive solutions in designing inductive coupled radio-frequency devices requires mathematical models, because some technological characteristics of the plasma cannot be measured.

In the present paper, a necessary and sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem is established. The original nonlinear differential eigenvalue problem is approximated by the finite element method with numerical integration on a uniform grid. The convergence of approximate minimal eigenvalue and

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approximate positive eigenfunction to exact ones is proved. Investigations of this paper generalize well known results for eigenvalue problems with linear dependence on the spectral parameter.

Nonlinear eigenvalue problems also arise in various fields of science and technology [6–23]. Numerical methods for solving matrix eigenvalue problems with nonlinear dependence on the parameter were constructed and investigated in the papers [24–36]. Error of the finite difference methods for solving differential nonlinear eigenvalue problems was studied in [37, 38]. Finite element method for solving nonlinear eigenvalue problems was investigated in [2, 11, 39], and estimations of the effect of numerical integration in finite element eigenvalue approximations were established in [40–42] with help the results [43–46]. The investigations of approximate methods for solving nonlinear eigenvalue problems in a Hilbert space were carried out in the papers [11, 47] with using general results for linear eigenvalue problems [48–52]. In the papers [53–58], numerical methods for solving applied nonlinear boundary value problems and variational inequalities have been studied.

2. VARIATIONAL STATEMENT OF THE PROBLEM

Let $H = L_2(\Omega)$ be the real Lebesgue space with norm

$$|v|_0 = \left(\int\limits_0^\pi (v(x))^2 dx\right)^{1/2} \quad \forall v \in H.$$

By $V = \{v : v, v' \in H, u(0) = u(\pi) = 0\}$ we denote the real Sobolev space with norm

$$|v|_1 = \left(\int\limits_0^\pi (v'(x))^2 dx\right)^{1/2} \quad \forall v \in V.$$

Put $K = \{v : v \in V, u(x) > 0, x \in \Omega\}$. For fixed $\mu \in \Lambda$, we introduce the following bilinear forms

$$a(\mu, u, v) = \int\limits_0^\pi p(\mu s(x))u'v'dx, \quad b(\mu, u, v) = \int\limits_0^\pi r(\mu s(x))uvdx,$$

where $u, v \in V$. For fixed $\mu \in \Lambda$, we define the Rayleigh functional by

$$R(\mu, v) = \frac{a(\mu, v, v)}{b(\mu, v, v)} \quad \forall v \in V \setminus \{0\}.$$

The differential nonlinear eigenvalue problem (1) is equivalent to the following variational nonlinear eigenvalue problem: find the minimal number $\lambda \in \Lambda$ and a function $u \in K$, $b(\lambda, u, u) = 1$, such that

$$a(\lambda, u, v) = b(\lambda, u, v) \quad \forall v \in V. \tag{2}$$

For fixed $\mu \in \Lambda$, we introduce the linear variational parametric eigenvalue problem: find the minimal number $\gamma(\mu) \in \Lambda$ and a function $u = u_\mu \in K$, $b(\mu, u, u) = 1$, such that

$$a(\mu, u, v) = \gamma(\mu) b(\mu, u, v) \quad \forall v \in V. \tag{3}$$

The minimal eigenvalue of problem (3) satisfies the following variational representation

$$\gamma(\mu) = \min_{v \in V \setminus \{0\}} R(\mu, v).$$

Hence, the minimal eigenvalue λ of problem (2) is the minimal root of the equation

$$\gamma(\mu) = 1, \quad \mu \in \Lambda. \tag{4}$$

Put

$$(u, v)_0 = \int\limits_0^\pi u(x)v(x)dx, \quad (u, v)_1 = \int\limits_0^\pi u'(x)v'(x)dx \quad \forall u, v \in V.$$

Formulate the auxiliary linear variational eigenvalue problem: find the minimal number $\varkappa \in \Lambda$ and a function $u \in K$, $(u,u)_0 = 1$, such that

$$(u,v)_1 = \varkappa (u,v)_0 \quad \forall v \in V. \tag{5}$$

The eigenvalue and eigenfunction of problem (5) is defined by $\varkappa = 1$, $u(x) = \sqrt{\pi/2}\sin x$, $x \in \overline{\Omega}$. Moreover, the following variational property holds $\varkappa = \min_{v\in V\setminus\{0\}} \frac{(v,v)_1}{(v,v)_0}$. For $\mu,\eta \in \Lambda$, we denote

$$\delta_p(\mu,\eta) = \max_{x\in\overline{\Omega}} |p(\mu s(x)) - p(\eta s(x))|, \quad \delta_r(\mu,\eta) = \max_{x\in\overline{\Omega}} |r(\mu s(x)) - r(\eta s(x))|.$$

We also set $p_1 = \inf_{\mu\in\Lambda} p(\mu)$, $p_2 = \sup_{\mu\in\Lambda} p(\mu)$, $r_1 = \inf_{\mu\in\Lambda} r(\mu)$.

Theorem 1. *For $\mu,\eta \in \Delta$, the following estimate is valid $|\gamma(\mu) - \gamma(\eta)| \le c(\delta_p(\mu,\eta) + \delta_r(\mu,\eta))$, where c is a positive constant independent of $\mu,\eta \in \Delta$, $\Delta = [\alpha,\beta] \subset \Lambda$.*

Proof. Using the definition of bilinear forms, we obtain

$$|a(\mu,v,v) - a(\eta,v,v)| = \left|\int_0^\pi (p(\mu s(x)) - p(\eta s(x)))(v')^2 dx\right| \le \delta_p(\mu,\eta)|v|_1^2$$

for $\mu,\eta \in \Delta$, $v \in V$,

$$|b(\mu,v,v) - b(\eta,v,v)| = \left|\int_0^\pi (r(\mu s(x)) - r(\eta s(x)))v^2 dx\right| \le \delta_r(\mu,\eta)|v|_0^2$$

for $\mu,\eta \in \Delta$, $v \in H$. Consequently, for $v = u_\eta$, $\mu,\eta \in \Delta$, we have

$$|R(\mu,v) - R(\eta,v)| \le \frac{|a(\mu,v,v) - a(\eta,v,v)|}{b(\mu,v,v)} + \frac{|b(\eta,v,v) - b(\mu,v,v)|}{b(\mu,v,v)}\,\frac{a(\eta,v,v)}{b(\eta,v,v)}$$

$$\le \delta_p(\mu,\eta)\frac{1}{r_1}\frac{|v|_1^2}{|v|_0^2} + \delta_r(\mu,\eta)\frac{1}{r_1}R(\eta,v) \le \delta_p(\mu,\eta)\frac{\sigma_2}{r_1p_1}R(\eta,v) + \delta_r(\mu,\eta)\frac{1}{r_1}R(\eta,v)$$

$$= \delta_p(\mu,\eta)\frac{\sigma_2}{r_1p_1}\gamma(\eta) + \delta_r(\mu,\eta)\frac{1}{r_1}\gamma(\eta) \le \delta_p(\mu,\eta)\frac{p_2\sigma_2}{r_1^2p_1} + \delta_r(\mu,\eta)\frac{p_2}{r_1^2} \le c(\delta_p(\mu,\eta) + \delta_r(\mu,\eta)),$$

where $c = (p_2/r_1^2)\,(\sigma_2/p_1 + 1)$, $\sigma_2 = \max_{\mu\in\Delta,\ x\in\overline{\Omega}} r(\mu s(x))$,

$$\gamma(\mu) = \min_{v\in V\setminus\{0\}} R(\mu,v) = \min_{v\in V\setminus\{0\}} \left\{\left[\int_0^\pi p(\mu s(x))(v')^2dx\right] / \left[\int_0^\pi r(\mu s(x))v^2dx\right]\right\} \le p_2/r_1.$$

Thus, we derive the relations

$$\gamma(\mu) = \min_{v\in V\setminus\{0\}} R(\mu,v) \le R(\mu,u_\eta) = R(\eta,u_\eta) + R(\mu,u_\eta) - R(\eta,u_\eta)$$

$$\le \gamma(\eta) + |R(\mu,u_\eta) - R(\eta,u_\eta)| \le \gamma(\eta) + c(\delta_p(\mu,\eta) + \delta_r(\mu,\eta)),$$

which imply the desired result. □

Theorem 2. *The convergence $\delta_p(\mu,\eta) \to 0$, $\delta_r(\mu,\eta) \to 0$, as $\eta \to \mu$ holds.*

Proof. Assume that $\mu,\eta \in \Delta$, $\Delta = [\alpha,\beta] \subset \Lambda$. Let us prove $\delta_p(\mu,\eta) \to 0$ as $\eta \to \mu$. Since the function $p(\mu)$, $\mu \in \Delta$, is uniformly continuous, for any $\varepsilon > 0$ there exists δ_1 such that for any $\nu_1,\nu_2 \in \Delta$, $|\nu_1 - \nu_2| < \delta_1$, the inequality $|p(\nu_1) - p(\nu_2)| < \varepsilon$ holds.

Suppose that $x \in \overline{\Omega}$, $\mu \in \Lambda$, $\mu s(x) \in \Delta$, $s_2 = \max_{x\in\overline{\Omega}} s(x)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$, $\delta \le \delta_1/s_2$, such that for any $\eta \in \Lambda$, $\eta s(x) \in \Delta$, $|\mu - \eta| < \delta$, $|\mu s(x) - \eta s(x)| < \delta_1$, the inequality $|p(\mu s(x)) - p(\eta s(x))| < \varepsilon$ holds. Therefore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\eta \in \Lambda$, $|\mu - \eta| < \delta$, the inequality $|\delta_p(\mu,\eta)| < \varepsilon$ holds. Hence $\delta_p(\mu,\eta) \to 0$ as $\eta \to \mu$. Similarly, we prove $\delta_r(\mu,\eta) \to 0$ as $\eta \to \mu$. □

Denote $\sigma(\mu) = \min_{x\in\overline{\Omega}} r(\mu s(x))$.

Theorem 3. *A minimal simple eigenvalue of problem (2) exists if and only if $\gamma(\xi) > 1$ for some $\xi \in \Lambda$.*

Proof. According to Theorems 1 and 2, $\gamma(\mu)$, $\mu \in \Lambda$, is a continuous function. Using the variational properties of the minimal eigenvalues of the problems (3) and (5), we derive

$$\gamma(\mu) = \min_{v \in V \setminus \{0\}} \left\{ \left[\int_0^\pi p(\mu s(x))(v')^2 dx \right] / \left[\int_0^\pi r(\mu s(x)) v^2 dx \right] \right\} \leq \frac{p_2}{\sigma(\mu)} \varkappa = \frac{p_2}{\sigma(\mu)} \to 0$$

as $\mu \to \infty$, since $\sigma(\mu) = r(\mu s(x_\mu)) \to \infty$ as $\mu \to \infty$ for some $x_\mu \in \overline{\Omega}$. Hence, a minimal root $\lambda \in \Lambda$ of equation (4) exists if and only if $\gamma(\xi) > 1$ for some $\xi \in \Lambda$. The minimal root $\lambda \in \Lambda$ of equation (4) defines the minimal eigenvalue of problem (2). The eigenvalue $\lambda \in \Lambda$ is simple and corresponds to a positive eigenfunction, since $\gamma(\mu)$ is the simple eigenvalue of the parametric problem (3) for $\mu = \lambda$ corresponding to a positive eigenfunction. □

3. APPROXIMATION OF THE PROBLEM

Let us partition the interval $[0, \pi]$ by equidistant points $x_i = ih$, $i = 0, 1, \dots, N$, into the elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, N$, $h = \pi/N$. By V_h we denote the subspace of the space V, consisting of continuous functions v^h, linear on each element e_i, $i = 1, 2, \dots, N$. Set

$$K_h = \{v^h : v^h \in V_h, u^h(x) > 0, x \in \Omega\}.$$

For $\mu \in \Lambda$, u^h, $v^h \in V_h$, we introduce the approximate bilinear forms

$$a_h(\mu, u^h, v^h) = \sum_{i=1}^N hp(\mu s(x_i - h/2))(u^h(x_i - h/2))'(v^h(x_i - h/2))',$$

$$b_h(\mu, u^h, v^h) = \sum_{i=1}^N h(r(\mu s(x_{i-1}))u^h(x_{i-1})v^h(x_{i-1}) + r(\mu s(x_i))u^h(x_i)v^h(x_i))/2.$$

For fixed $\mu \in \Lambda$, we define the Rayleigh functional by

$$R_h(\mu, v^h) = \frac{a_h(\mu, v^h, v^h)}{b_h(\mu, v^h, v^h)} \quad \forall v^h \in V_h \setminus \{0\}.$$

The variational nonlinear eigenvalue problem (2) is approximated by the following finite dimensional problem: find the minimal number $\lambda^h \in \Lambda$ and a function $u^h \in K_h$, $b_h(\lambda^h, u^h, u^h) = 1$, such that

$$a_h(\lambda^h, u^h, v^h) = b_h(\lambda^h, u^h, v^h) \quad \forall v^h \in V_h. \tag{6}$$

For fixed $\mu \in \Lambda$, we introduce linear parametric eigenvalue problem: find the minimal number $\gamma^h(\mu)$ and a function $u^h = u^h_\mu \in K_h$, $b_h(\mu, u^h, u^h) = 1$, such that

$$a_h(\mu, u^h, v^h) = \gamma^h(\mu) b_h(\mu, u^h, v^h) \quad \forall v^h \in V_h. \tag{7}$$

The following variational property for the minimal eigenvalue of problem (7) is valid

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R_h(\mu, v^h).$$

Denote $y_i = u^h(x_i)$, $i = 0, 1, \dots, N$, $p_i(\mu) = p(\mu s(x_i - h/2))$, $r_i(\mu) = r(\mu s(x_i))$, $\mu \in \Lambda$, $y_{x,i} = (y_{i+1} - y_i)/h$, $y_{\overline{x},i} = (y_i - y_{i-1})/h$. Then the finite dimensional problem (6) is equivalent to the finite difference problem: find the minimal number $\lambda^h \in \Lambda$ and a positive grid function y_i, $i = 1, 2, \dots, N-1$, $\sum_{i=1}^{N-1} r_i(\lambda^h) y_i^2 = 1$, such that

$$-(p(\lambda^h) y_{\overline{x}})_{x,i} = r_i(\lambda^h) y_i, \quad i = 1, 2, \dots, N-1, \quad y_0 = y_N = 0. \tag{8}$$

Similarly, we represent the finite dimensional problem (7) for fixed $\mu \in \Lambda$ in finite difference form: find the minimal number $\gamma^h(\mu)$ and a positive grid function y_i, $i = 1, 2, \ldots, N-1$, $\sum_{i=1}^{N-1} r_i(\mu) y_i^2 = 1$, such that

$$-(p(\mu) y_{\overline{x}})_{x,i} = \gamma^h(\mu) r_i(\mu) y_i, \quad i = 1, 2, \ldots, N-1, \quad y_0 = y_N = 0. \tag{9}$$

Put

$$(u^h, v^h)_{1,h} = \sum_{i=1}^{N} h (u^h(x_i - h/2))' (v^h(x_i - h/2))',$$

$$(u^h, v^h)_{0,h} = \sum_{i=1}^{N} h (u^h(x_{i-1}) v^h(x_{i-1}) + u^h(x_i) v^h(x_i))/2.$$

Introduce the auxiliary linear eigenvalue problem: find the minimal number $\varkappa^h$ and a function $u^h \in K_h$, $(u^h, u^h)_{0,h} = 1$, such that

$$(u^h, v^h)_{1,h} = \varkappa^h (u^h, v^h)_{0,h} \quad \forall v^h \in V_h. \tag{10}$$

The following variational property holds $\varkappa^h = \min_{v^h \in V_h \setminus \{0\}} \left[(v^h, v^h)_{1,h} / (v^h, v^h)_{0,h} \right]$. As above, we write the finite dimensional problem (10) in finite difference form: find the minimal number $\varkappa^h$ and a positive grid function y_i, $i = 1, 2, \ldots, N-1$, $\sum_{i=1}^{N-1} y_i^2 = 1$, such that

$$-y_{\overline{x},x,i} = \varkappa^h y_i, \quad i = 1, 2, \ldots, N-1, \quad y_0 = y_N = 0. \tag{11}$$

The solutions of problem (11) are defined by the following formulas $\varkappa^h = \frac{4}{h^2} \sin^2 \frac{h}{2}$, $y_i = \sqrt{\pi/2} \sin x_i$, $i = 0, 1, \ldots, N$. It is easy to show that $1 \geq \varkappa^h \to 1$ as $h \to 0$. The minimal eigenvalue λ^h of the finite dimensional problem (6) (or the finite difference problem (8)) is the minimal root of the equation

$$\gamma^h(\mu) = 1, \quad \mu \in \Lambda, \tag{12}$$

where $\gamma^h(\mu)$ is the minimal eigenvalue of the finite dimensional problem (7) (or the finite difference problem (9)).

Theorem 4. *For $\mu, \eta \in \Delta$, the following estimate is valid $|\gamma^h(\mu) - \gamma^h(\eta)| \leq c(\delta_p(\mu, \eta) + \delta_r(\mu, \eta))$, where c is a positive constant independent of $\mu, \eta \in \Delta$, $\Delta = [\alpha, \beta] \subset \Lambda$.*

Proof. The proof of this theorem is similar to that of Theorem 1. □

Theorem 5. *A minimal simple eigenvalue of problem (6) exists if and only if $\gamma^h(\xi) > 1$ for some $\xi \in \Lambda$.*

Proof. By Theorems 4 and 2, $\gamma^h(\mu)$, $\mu \in \Lambda$, is a continuous function. Applying the variational properties of the minimal eigenvalues of the problems (7) and (10), we get

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R_h(\mu, v^h) \leq \frac{p_2}{\sigma(\mu)} \varkappa^h \to 0$$

as $\mu \to \infty$, since $\sigma(\mu) = r(\mu s(x_\mu)) \to \infty$ as $\mu \to \infty$ for some $x_\mu \in \overline{\Omega}$. Hence, a minimal root $\lambda^h \in \Lambda$ of equation (12) exists if and only if $\gamma^h(\xi) > 1$ for some $\xi \in \Lambda$. The minimal root $\lambda^h \in \Lambda$ of equation (12) defines the minimal eigenvalue of problem (6). The eigenvalue $\lambda^h \in \Lambda$ is simple and corresponds to a positive eigenfunction, since $\gamma^h(\mu)$ is the simple eigenvalue of the parametric problem (7) for $\mu = \lambda^h$ corresponding to a positive eigenfunction. □

By c we denote various positive constants independent of h. For fixed $\mu \in \Lambda$, we introduce the operator $P_h(\mu) : V \to V_h$ defined by the rule $a(\mu, u - P_h(\mu)u, v^h) = 0$ for any $v^h \in V_h$, where $u \in V$, $P_h(\mu)u \to u$ in V as $h \to 0$. Put $P_h = P_h(\lambda)$.

Theorem 6. *The following convergence holds $\lambda^h \to \lambda$, $u^h \to u$ in V as $h \to 0$.*

Proof. Since $\gamma^h(\mu) \to \gamma(\mu)$ as $h \to 0$ [10, 39, 44] for fixed $\mu \in \Lambda$, we derive $\lambda^h \to \lambda$ as $h \to 0$.

Taking into account the normalization $b_h(\lambda^h, u^h, u^h) = 1$, we get $p_1|u^h|_1^2 \le a_h(\lambda^h, u^h, u^h) = \gamma^h(\lambda^h) = 1$, that is $|u^h|_1 \le c$, where $c = 1/p_1^{1/2}$. Therefore, any sequence $h' \to 0$ has a subsequence $h'' \to 0$ such that $u^h \rightharpoonup w$ in V as $h = h'' \to 0$, where $w \in V$.

For any element $v \in V$ choose $v^h = P_h v$. Then we have [10, 39, 44]: $a_h(\lambda^h, u^h, v^h) \to a(\lambda, w, v)$, $b_h(\lambda^h, u^h, v^h) \to b(\lambda, w, v)$ as $h = h'' \to 0$. Passing to the limit as $h = h'' \to 0$ in the equation $a_h(\lambda^h, u^h, v^h) = b_h(\lambda^h, u^h, v^h)$, we get $a(\lambda, w, v) = b(\lambda, w, v)$ for any $v \in V$. We have $1 = b_h(\lambda^h, u^h, u^h) \to b(\lambda, w, w)$ as $h = h'' \to 0$, that is $b(\lambda, w, w) = 1$ and $w \in K$, since assuming that $-w \in K$, we obtain $w = -u$, $b(\lambda^h, u^h, u) \to -b(\lambda, u, u) = -1$ as $h = h'' \to 0$, $u \in K$, but this contradicts the inequality $b(\lambda^h, u^h, u) > 0$. Therefore, λ and $w = u$ are the minimal eigenvalue and corresponding positive eigenfunction of problem (2). Let us prove the strong convergence $u^h \to u$ in V as $h = h'' \to 0$. We have

$$p_1|u^h - P_h u|_1^2 \le a_h(\lambda^h, u^h - P_h u, u^h - P_h u)$$
$$= a_h(\lambda^h, u^h, u^h) - 2a_h(\lambda^h, u^h, P_h u) + a_h(\lambda^h, P_h u, P_h u) \to 0$$

as $h = h'' \to 0$. Here, we have taken into account the relations

$$a_h(\lambda^h, u^h, u^h) = 1, \quad a_h(\lambda^h, u^h, P_h u) \to a(\lambda, u, u) = 1, \quad a_h(\lambda^h, P_h u, P_h u) \to a(\lambda, u, u) = 1,$$

as $h = h'' \to 0$. Consequently, we obtain $|u^h - u|_1 \le |u^h - P_h u|_1 + |u - P_h u|_1 \to 0$ as $h = h'' \to 0$.

Suppose that a sequence $h' \to 0$ is such that $|u^h - u|_1 \ge c$ as $h = h' \to 0$. Then, as above, there exists a subsequence $h'' \to 0$ such that $|u^h - u|_1 \to 0$ as $h = h'' \to 0$. But this contradicts the preceding inequality. Thus, for any sequence $h' \to 0$ we have $u^h \to u$ in V as $h = h' \to 0$, that is $u^h \to u$ in V as $h \to 0$. □

ACKNOWLEDGMENTS

This work was supported by Russian Science Foundation, project no. 16-11-10299

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