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The Cauchy Singular Integral on Non-Smooth Curve

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Abstract—We consider the Cauchy singular integral with Hölder's density on non-smooth curves: various versions of its definition, conditions for its existence, boundedness and so on.

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Let Γ be a simple Jordan curve on the complex plane $\mathbb{C}$. The original definition of the Cauchy singular integral over this curve is

$$\mathcal{S}_\Gamma f(t) := \lim_{\epsilon\to 0}\frac{1}{\pi i}\int\limits_{\Gamma\setminus\{|\tau-t|\le\epsilon\}}\frac{f(\tau)d\tau}{\tau-t},\quad t\in\Gamma. \tag{1}$$

Of course, the existence of the limit needs the proof.

This integral operator has a lot of applications. In particular, it is of importance for theory of boundary value problems for analytic functions, for aero and hydrodynamics, and for theory of elasticity, see [1–3]. There exists a great body of publications on this subject. Here we restrict our references by the classical monograph [4] and recent survey [5].

It is well known [1–3], that $\mathcal{S}_\Gamma f$ exists if the curve Γ is smooth or piecewise-smooth, and the density f satisfies the Hölder condition

$$\sup\left\{\frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^\nu}: t_{1,2}\in\Gamma, t_1\ne t_2\right\} := h_\nu(f;\Gamma)<\infty;$$

with any exponent $\nu\in(0;1]$. We denote the class of all that functions $H_\nu(\Gamma)$. The norm $\|f\|_\nu = \sup\{|f(t)| : t\in\Gamma\} + h_\nu(f;\Gamma)$ turns this class into Banach space.

If $f\in H_\nu(\Gamma)$ and Γ is smooth, then the function $\mathcal{S}_\Gamma f(t)$ is continuous and satisfies the Hölder condition with the same exponent ν for $\nu<1$, and with arbitrarily close to unit exponent for $\nu=1$. But this result is not valid at the corners of a piecewise-smooth curve. For example, if Γ is boundary of a square with counterclockwise circuit and $f\equiv 1$, then function $\mathcal{S}_\Gamma f$ equals 1 at all points of Γ excluding the vertices, where it is equal to $3/2$.

In this connection there arises another definition of the Cauchy singular integral:

$$\mathcal{S}^*_\Gamma f(t) := f(t) + \frac{1}{\pi i}\int\limits_\Gamma\frac{f(\tau)-f(t)}{\tau-t}d\tau,\quad t\in\Gamma, \tag{2}$$

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where the integral is understood as improper. At the points of smoothness of the curve the functions $\mathcal{S}_\Gamma f(t)$ and $\mathcal{S}^*_\Gamma f(t)$ coincide, but at the corners the second one keeps continuity. For instance, $\mathcal{S}^*_\Gamma 1(t) = 1$ at all points of the boundary of the square, including its vertices.

The formula (2) keeps validity for certain non-smooth rectifiable curves. For instance, the integral (2) converges if $f \in H_\nu(\Gamma)$ for $\nu > 0$, and rectifiable curve Γ is $AD-$regular, i.e. the sum of lengths of its arcs covered by any disc of radius r does not exceed Cr, where positive constant C does not depend on the center of the disk and on r.

There exists one more approach to the definition of the singular integral. Let us consider jump problem, i.e., the problem on reconstruction of analytic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it vanishes at the point at infinity, and has at every point $t \in \Gamma$ continuous limit values $\Phi^\pm(t)$ from domains $D^\pm$ satisfying relation

$$\Phi^+(t) - \Phi^-(t) = f(t), \tag{3}$$

where f ia a given function defined on Γ. If curve Γ is smooth and $f \in H_\nu(\Gamma)$, then unique solution of this problem (see [1–3]) is the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(t)\,dt}{t - z}, \tag{4}$$

and $\Phi^\pm(t) = \frac{1}{2}(\pm f(t) + \mathcal{S}_\Gamma f(t))$. For piecewise-smooth curves the last formula keeps its validity at the angular points if we replace there $\mathcal{S}_\Gamma f(t)$ by $\mathcal{S}^*_\Gamma f(t)$. A solution of the jump problem is unique for any rectifiable curve. This fact is implied by the following Painlevé theorem (see [6]): if a function is continuous in domain D and analytic in $D \setminus \Gamma$, where a curve Γ is rectifiable, then this function is analytic in D, too. Therefore, we are able to define an analog of the singular integral for non-smooth rectifiable curve as sum

$$\mathcal{S}^\Gamma f(t) = \Phi^+(t) + \Phi^-(t), \quad t \in \Gamma, \tag{5}$$

where Φ is a unique solution of the jump problem (3), if it exists.

This definition is expandable on non-rectifiable curves of the following class. Let a curve Γ contain a finite set of points E such that for any its neighborhood $N(E)$ the difference $\Gamma \setminus N(E)$ is union of finite number of rectifiable arcs. We call that curve E^c-rectifiable. For example, arc $\{z = x + iy : -1 \le x \le +1, y = x \sin x^{-p}\}$ is 0^c-rectifiable, but it is not rectifiable for $p \ge 1$. Clearly, the Painleve theorem is valid for E^c-rectifiable curves, what enables us to apply the last definition of the Cauchy integral to that curves.

All three definitions lead to the same result at the points of smoothness of the curve.

In the present paper we study the singular integral (5) for non-smooth and non-rectifiable curves Γ. Let us describe our class of curves.

Let Γ' and Γ be simple closed curves bounding finite domains D' and D relatively. The set $D' \triangle \overline{D}$ consists of finite or infinite set of mutually disjoint domains Δ_j, $j = \pm 1, \pm 2, \ldots$. Each of these domains has signature s_j equaling $+1$ for $\Delta_j \subset D'$, and -1 for $\Delta_j \subset D$. We say that the curve Γ is perturbation of Γ' of type $A(t_0)$ if

– all boundaries γ_j of domains Δ_j are simple piecewise-smooth curves;

– family of domains Δ_j is infinite and has unique condensation point t_0.

Example 1. *$Q' = \{z = x + iy : -1 < x < 1, -2 < y < 0\}$, and Γ is boundary of domain $Q = Q \cup (\bigcup\limits_{j=1}^{\infty} P_j) \setminus (\bigcup\limits_{j=-1}^{-\infty} P_j)$, where $P_j = \{z = x + iy : b_j < x < a_j, 0 < y < h_j\}$, $a_1 > b_1 > a_2 > b_2 > \cdots > 0$, $h_j > 0$ for $j > 0$, and $P_j = \{z = x + iy : a_j < x < b_j, 0 > y > h_j\}$, $a_1 < b_1 < a_2 < b_2 < \cdots < 0$, $h_j < 0$ for $j < 0$. These rectangles play role of domains Δ_j, and signature s_j equals to the sign of j. If limits $\lim\limits_{j\to\pm\infty} a_j$ and $\lim\limits_{j\to\pm\infty} h_j$ vanish, then the origin is unique condensation point of these domains, i.e., Γ is perturbation of Γ' of type $A(0)$. Clearly, curve Γ is 0^c-rectifiable, but it is not rectifiable if at least one of series $\sum\limits_{j=1}^{+\infty} h_j$, $\sum\limits_{j=-1}^{-\infty} h_j$ diverges.*

We can replace here the square Q' by any domain of the lower half-plane with smooth boundary Γ' containing segment $[-1;1]$ of the real axis.

Let a t_0^c-rectifiable curve Γ be perturbation of type $A(t_0)$ of a smooth curve Γ', and $f \in H_\nu(\Gamma)$. We extend f up to a function $F \in H_\nu(\mathbb{C})$ by means of the Whitney extension operator, see, for instance, [4]. We seek a solution of the jump problem on Γ as sum $\Phi = \sum\limits_{j=-\infty}^{+\infty} s_j\phi_j$, where ϕ_0 is solution of the jump problem

$$\phi^+(t) - \phi^-(t) = F(t)$$

on the curve Γ', $s_0 = +1$, and ϕ_j, $j \neq 0$, are solutions of the same problem on curves γ_j. The solution ϕ_0 for any $\nu > 0$ is determined by the integral of Cauchy type. This conclusion is valid also for the solutions ϕ_j. It remains to study the convergence of the series.

Clearly,

$$\phi_j(z) = \frac{1}{2\pi i}\int\limits_{\gamma_j} \frac{F(t)\,dt}{t-z} \tag{6}$$

(for $j = 0$ we put $\gamma_0 = \Gamma'$), and by the Borel—Pompeju formula

$$\phi_j(z) = \chi_j(z)F(z) - \frac{1}{2\pi i}\iint\limits_{\Delta_j} \frac{\partial F}{\partial \overline{w}}\frac{dw\,d\overline{w}}{w-z}, \tag{7}$$

where $\chi_j(z)$ is characteristic function of domain Δ_j; for $j = 0$ we put $\Delta_0 = D'$. The properies of the integral term of the last formula are well known (see, for instance, [7]). In particular, the series

$$\sum_{j=-\infty}^{+\infty} s_j \iint\limits_{\Delta_j} \frac{\partial F}{\partial \overline{w}}\frac{dw\,d\overline{w}}{w-z}$$

converges to a continuous in the whole complex plane function if derivative $\dfrac{\partial F}{\partial \overline{w}}$ is integrable in the union of all domains with degree $p > 2$.

By virtue of the known properties of the Whitney extension [4]

$$\left|\frac{\partial F}{\partial \overline{w}}\right| \leq \frac{h_\nu(f;\Gamma)}{\operatorname{dist}^{1-\nu}(w;\Gamma)}.$$

Hence, there is valid

Theorem 1. *Let a t_0^c-rectifiable curve Γ be perturbation of type $A(t_0)$ of a smooth curve Γ', and $f \in H_\nu(\Gamma)$. Then singular integral $\mathcal{S}^\Gamma f$ exists at all points of the curve including the condensation points t_0 if the series*

$$\sum_{j=1}^{+\infty} \iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)}, \quad \sum_{j=-1}^{-\infty} \iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)} \tag{8}$$

converge for certain $q > 2(1-\nu)$.

Note that the integral $\iint\limits_{\Delta_j} \frac{dx\,dy}{\operatorname{dist}^q(x+iy;\Gamma)}$ diverges for $q \geq 1$ if set $\Gamma \cap \gamma_j$ contains a continuum.Therefore, the assumptions of Theorem imply inequality $\nu > 1/2$.

Example 2. *We consider the curve from the previous example. The immediate evaluation shows that the series* (8) *converge simultaneously with the series*

$$\sum_{j=1}^{+\infty}(a_j - b_j)^{1-q}h_j, \quad \sum_{j=-1}^{-\infty}(b_j - a_j)^{1-q}h_j.$$

Hence, the singular integral $\mathcal{S}^\Gamma f$ on this curve exists if the last series converge for certain $q \in (2(1-\nu);1)$.

Then we apply the following property of the integral term of formula (7): if derivative $\dfrac{\partial F}{\partial \overline{w}}$ is integrable in the union of all domains Δ_j with degree $p > 2$, then the integral over the union satisfies in the whole complex plane the Hölder condition with exponent $1 - 2p^{-1}$ (see, for instance, [7]). Therefore, under assumptions of Theorem 1 the function $\mathcal{S}^\Gamma f$ satisfies on Γ the Hölder condition with any exponent $\mu < 2\nu - 1$. In addition, the Hölder coefficient $h_\mu(\mathcal{S}^\Gamma f;\Gamma)$ is bounded by a value depending on $h_\nu(f;\Gamma)$ only. Thus, we obtain

Theorem 2. *Let a t_0^c-rectifiable curve Γ be perturbation of type $A(t_0)$ of a smooth curve Γ', and the series* (8) *converge for certain $q > 2(1-\nu)$. Then the singular integral $\mathcal{S}^\Gamma$ is bounded operator from $H_\nu(\Gamma)$ to $H_\mu(\Gamma)$ for $1 > \nu > 1/2$, $1 - 2(1-\nu)q^{-1} > \mu > 0$.*

In connection with the applications of the singular integral we are interested in spaces X such that $\mathcal{S}^\Gamma X = X$. An example of space satisfying this equality is space $H^*(\Gamma)$, which consists of functions f satisfying the Hölder condition with any exponent $\nu < 1$. We equip this space with topology generated by infinite sequence of semi-norms $\sup\{|f(t)| : t \in \Gamma\}$, $h_{\nu_1}(f;\Gamma)$, $h_{\nu_2}(f;\Gamma)$, $h_{\nu_3}(f;\Gamma), \ldots$, where $\nu_1, \nu_2, \nu_3, \ldots$ is increasing sequence of positive values such that $1 > \nu_1 > 1/2$ and $\lim\limits_{k\to+\infty} \nu_k = 1$. As a result, we obtain

Theorem 3. *Let a t_0^c-rectifiable curve Γ be perturbation of type $A(t_0)$ of a smooth curve Γ', and the series (8) converge for certain $q > 0$. Then the singular integral $\mathcal{S}^\Gamma$ is continuous mapping of $H^*(\Gamma)$ onto itself.*

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