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About Orthogonal Systems of One Kind of Functions

A. N. Barmenkov* **and N. A. Barmenkov**

(Submitted by A. M. Elizarov)

National Research Nuclear University "MEPhI", Kashirskoe sh. 31, Moscow, 115409 Russia

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Abstract—We present an algorithm for construction complete, orthogonal sequences special kind. This systems of functions depend on parameter and may be used for modeling of physical processes.

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1. INTRODUCTION

In 1953 K. Shaidukov [1] proved by the method of theory of functions of real variable completeness in $L_2[0;2\pi]$ of sequence

$$\Big\{\cos(nt+bt);\ \sin(nt+bt)\Big\}_{n=0}^{\infty}, \tag{1}$$

when $b \le 2/3$ and start of a whole direction of research of such systems. In [2], [3] by the methods of the monograph [4] it is shown that completeness and minimality in $L_p[0;2\pi]$ in a more general system of functions

$$\Big\{\cos(nt+\alpha(t));\ \sin(nt+\alpha(t))\Big\}_{n=0}^{\infty} \tag{2}$$

depends only on the difference of values of the $\alpha(t)$ at the endpoints of the segment $[0;2\pi]$, $\alpha(t) \in Lip_\nu[0;2\pi] \cap Var[0;2\pi]$, $0 < \nu \le 1$. Where $f(t) \in Lip_\nu[0;2\pi]$ if $|f(t_1)-f(t_2)| \le L|t_1-t_2|^\alpha$, $0<\alpha\le 1$, for arbitrary $t_1, t_2 \in [0;2\pi]$ and any constant L; $f(t) \in Var[0;2\pi]$ if $\sup(\sum_{k=1}^{n-1}|f(t_{k+1})-f(t_k)|) < \infty$ for arbitrary $0 < t_1 < t_2 < ... < t_n < 2\pi$. So that the sequence (2) is complete in $L_p[0;2\pi]$, $p>1$, when $(2\pi)^{-1}[\alpha(2\pi)-\alpha(0)] \le 1/2 + 1/(2p)$. That system (1) is complete in $L_2[0;2\pi]$ for $\alpha(t) = bt$ when $b \le 3/4$. Moreover if $\alpha(t) \ne const$, $t \in [0;2\pi]$, then system (2) is complete and minimality in $L_p[0;2\pi]$, $1<p<\infty$, if and only if $(2p)^{-1} < (2\pi)^{-1}[\alpha(2\pi)-\alpha(0)] \le 1/2 + (2p)^{-1}$. It is interesting to describe the complete and orthogonal sequence of the following type

$$\Big\{e^{\beta(t)}\cos[nt+\alpha(t)];\ e^{\beta(t)}\sin[nt+\alpha(t)]\Big\}_{n=0}^{\infty}, \tag{3}$$

where $\alpha(t)$, $\beta(t)$ are real functions on $[0;2\pi]$.

*E-mail: anbarmenkov@mail.ru

2. BASIC CONCEPTS

The definition of completeness and minimality of the sequence is given in ([5], p. 73). It is also shown there that this sequence is a basis in some space.

Theorem 1. *Let* $\alpha(t)$, $\beta(t)$ *are real functions of bounded variation on the segment* $[0;2\pi]$, $(\alpha(t),\beta(t)\in Lip_\nu[0;2\pi]\cap Var[0;2\pi]$, $0<\nu\le 1$, *and* $\alpha(t)$ *is different from constant. Let*

$$B_a(z)=(|a|/a)(a-z)/(1-\bar{a}z)\quad \text{if}\quad a\in\mathbb{C},|a|<1\quad\text{and}\quad a\neq 0;\quad B_0(z)=z.$$

Then there is not complete and orthogonal sequence of type (3) in $L_2[0;2\pi]$ *which is different from system*

$$\{R\cos\left[nt+(1/2)\arg[B_a(e^{it})]\right]);\ R\sin\left[nt+(1/2)\arg[B_a(e^{it})]\right]\}_{n=0}^{\infty},\tag{4}$$

when $1/4<(2\pi)^{-1}[\alpha(2\pi)-\alpha(0)]<3/4$, $R>0$ *is arbitrary real number. If* $a=0$ *then this sequence has the form* $\{R\cos(nt+t/2+r)\,;\ R\sin(nt+t/2+r)\}_{n=0}^{\infty}$, *where* r *is arbitrary real number.*

The result of Theorem 1 leads to one of the methods for constructing complete orthogonal sequences of the form (3) in the space $L_2[0;2\pi]$.

Theorem 2. *Let real function* $\alpha(t)\in Lip_\nu[0;2\pi]\cap Var[0;2\pi]$, $0<\nu\le 1$. *The system (2) forms orthonormal basis in* $L_2[0;2\pi]$

$$\Big\{[\pi(1+|a|)]^{-1/2}\cos\alpha(t);\ [\pi(1-|a|)]^{-1/2}\sin\alpha(t)\Big\}\bigcup\Big\{\pi^{-1/2}\cos[nt+\alpha(t)];$$
$$\pi^{-1/2}\sin[nt+\alpha(t)]\Big\}_{n=1}^{\infty},$$

if and only if $\alpha(t)=\arg[B_a(e^{it}))]/2$ *for* $a\neq 0$, $a\in\mathbb{C}$, $|a|<1$; $\alpha(t)=t/2+r$ *for* $a=0$.

Proof. Orthogonality of complete and minimal sequence (3) in $L_2[0;2\pi]$ for $n=1,2,3,\ldots$ to function $e^{\beta(t)}\cos\alpha(t)$ equals to

$$\int_0^{2\pi} e^{2\beta(t)}\cos\alpha(t)\cos[nt+\alpha(t)]dt=0,\quad \int_0^{2\pi} e^{2\beta(t)}\cos\alpha(t)\sin[nt+\alpha(t)]dt=0.$$

Hence we obtain equality $\int_0^{2\pi} e^{2\beta(t)+i[\alpha(t)+nt]}\cos\alpha(t)dt=0,\quad n=1,2,\ldots.$ Similarly for $e^{\beta(t)}\sin\alpha(t)$ we get

$$\int_0^{2\pi} e^{2\beta(t)+i[\alpha(t)+nt]}\sin\alpha(t)dt=0,\quad \int_0^{2\pi} e^{2\beta(t)+i[2\alpha(t)+nt]}dt=0,\quad n=1,2,\ldots.\tag{5}$$

The orthogonality of the pair $e^{\beta(t)}\cos\alpha(t)$; $e^{\beta(t)}\sin\alpha(t)$ leads to the equality

$$\int_0^{2\pi} e^{2\beta(t)}\sin 2\alpha(t)dt=0.\tag{6}$$

Equation (5) leads to the existence of a function $\phi(z)$ of Hardy H_1 class ([5], p. 102), for angular bounded values of which is right equation

$$\exp 2[\beta(t)+i\alpha(t)]=\phi(e^{it}),\quad t\in[0;2\pi].\tag{7}$$

Since $\alpha(t),\beta(t)\in Lip_\nu[0;2\pi]\cap Var[0;2\pi]$, $0<\nu\le 1$, then by Theorem F. and M. Riesz (see [5], p. 103), it follows from (5) that $\Phi(z)$ is continuous in the disk $|z|\le 1$ and absolutely continuous on the unit circle (that is, for $z=1$ the point is not a jump). This means that the change in the function argument $\phi(z)$ on the unit circle should be integer multiple of 2π.

As it shown in [2, 3] for functions $\alpha(t)\in Lip_\nu[0;2\pi]\cap Var[0;2\pi]$, $0<\nu\le 1$, conditions of completeness and minimality in spaces $L_p[0;2\pi]$, $p>1$, of sequences (2) and (3) and some of their generalizations depends only on values $\alpha(t)$ on the ends of segment $[0;2\pi]$, $(2\pi)^{-1}Var_{[0;2\pi]}\arg(\phi(e^{it}))=$

$(\pi)^{-1}[\alpha(2\pi)-\alpha(0)]=n$, where n is integer. According to Theorem 1 we have $1/2<\pi^{-1}[\alpha(2\pi)-\alpha(0)]<3/2$ what is possible only if $n=1$. Therefore, according to argument principle, function $\phi(z)$ analytical in $|z|<1$ and continuous in $|z|\le 1$ has only one zero at some point $a\in\mathbb{C}$, $|a|<1$ and $\phi(a)=0$. Then there exists function $\psi(z)$, analytic in $|z|<1$ and continuous in $|z|\le 1$, $\psi(z)\ne 0$ in $|z|<1$, for which equation (7) takes the form $e^{2[\beta(t)+i\alpha(t)]}/B_a(e^{it})=\psi(e^{it})$. So, if the sequence (3) is complete and orthogonal in $L_2[0,2\pi]$, then there is a point $a\in\mathbb{C}$, $|a|<1$, and there is a function $\psi(z)$ analytical in $|z|<1$ and continuous in $|z|\le 1$, and $\psi(z)\ne 0$, where $|z|<1$, and right the equation $e^{2[\beta(t)+i\alpha(t)]}=\phi(e^{it})=B_a(e^{it})\psi(e^{it})$, $t\in[0;2\pi]$. Equation (6) means $\int_0^{2\pi}e^{2\beta(t)}\sin 2\alpha(t)dt=0$. Then we obtain

$$\int_0^{2\pi}\phi(e^{it})dt=2\pi\phi(0),\quad \phi(e^{it})=B_a(e^{it})\psi(e^{it}),\quad B_a(0)=|a|.$$

It means that $\psi(0)$ is real value. Similarly to (5), we have

$$0=\int_0^{2\pi}e^{2\beta(t)+i[\alpha(t)+int]}[\cos\alpha(t)-i\sin\alpha(t)]dt=\int_0^{2\pi}e^{2\beta(t)+int}dt,\quad n=1,2,\ldots$$

Since $e^{2\beta(t)}$ is real function, it means that $e^{\beta(t)}=const$, $t\in[0;2\pi]$. Taking $e^{\beta(t)}=R$, $t\in[0;2\pi]$, in the ratio (7), we get $R^2e^{2i\alpha(t)}=\phi(e^{it})=R^2B_a(e^{it})\tilde{\psi}(e^{it})$, $t\in[0;2\pi]$. Consequently, $\tilde{\psi}(e^{it})=e^{2i\alpha(t)}/B_a(e^{it})$, where the function $\tilde{\psi}(z)$ analytic in $|z|<1$ and continuous in $|z|\le 1$, $\tilde{\psi}(z)\ne 0$, $|z|<1$. Obviously, $|\tilde{\psi}(e^{it})|=1$, $t\in[0;2\pi]$ ($\alpha(t)$ is real function) then $\ln[e^{2i\alpha(t)}/B_a(e^{it})]=\ln\tilde{\psi}(e^{it})$. Therefore $\ln\tilde{\psi}(z)$ is analytic function in $|z|<1$ and in view the fact, that $\ln\tilde{\psi}(z)$ on the unit circle takes only imaginary values. It implies $\ln\tilde{\psi}(z)\equiv ir$, r is arbitrary real number. Since $B_a(0)=|a|$, we get $R^{-2}\phi(0)=B_a(0)\tilde{\psi}(0)=|a|e^{ir}$. On the other hand $\phi(0)$ must be real value. It is obvious that if $a\ne 0$, then $r=\pi k$ (k is integer number, $e^{i\pi k}=(-1)^k$) from periodicity of the sequence (4) shout be put $r=0$. If $a=0$ then $B_a(0)=0$ and condition, that $\phi(0)$ is real value, according to the ratio $R^{-2}\phi(0)=B_a(0)e^{ir}$ is performed when r is arbitrary real number. The ratio $e^{2i\alpha(t)}/B_a(e^{it})=e^{ir}$, $t\in[0;2\pi]$, means that $\alpha(t)=\arg(B_a(e^{it}))/2$ if $a\ne 0$, $a\in C$, $|a|<1$ and $\alpha(t)=t/2+r$ if $a=0$ for arbitrary real r.

The proof of orthogonality of system (4) follows from computing:

$$\int_0^{2\pi}\cos(nt+\alpha(t))\cos(kt+\alpha(t))dt=\frac{1}{2}\int_0^{2\pi}\cos(n-k)tdt+\frac{1}{2}\int_0^{2\pi}\cos((n+k)t+2\alpha(t))dt=D_{n,k}\delta_n^k,$$

$n,k\ge 0$, $n+k\ge 1$, $\delta_n^k=1$, if $n=k$ and $\delta_n^k=0$, if $n\ne k$, $D_{n,k}>0$ are any const. This equality occurs, because the function $B_a(z)$ is analytic in $|a|<1$ and continuous in $|z|\le 1$:

$$\int_0^{2\pi}\cos((n+k)t+2\alpha(t))dt=\Re\int_0^{2\pi}e^{2i\alpha(t)}e^{i(n+k)t}dt=\Re\int_0^{2\pi}B_a(e^{it})e^{i(n+k)t}dt=0,$$

where $\Re z$ is the real part of complex number z. Similarly test the other pairs of functions. This means that complete and orthogonal sequences of kind (3) in $L_2[0;2\pi]$ it is systems (4). This completely proves the Teorem 1. □

For the proof of Theorem 2 let us calculate the norms of elements of systems (2) in case orthogonality: $D_{0,0}=\int_0^{2\pi}\cos^2\alpha(t)dt=\pi+(1/2)\int_0^{2\pi}\cos 2\alpha(t)dt$. Because the condition orthogonality of the sequence (2) is $e^{2i\alpha(t)}=B_a(e^{it})$, we obtain $\int_0^{2\pi}\cos 2\alpha(t)dt=\Re\int_0^{2\pi}B_a(e^{it})dt=2\pi|a|$. Then we get $D_{0,0}=\|\cos\alpha(t)\|^2_{L_2[0;2\pi]}=\pi(1+|a|)$. Similarly

$$\|\sin\alpha(t)\|^2_{L_2[0;2\pi]}=\pi-(1/2)\int_0^{2\pi}\cos 2\alpha(t)dt,\quad \|\sin\alpha(t)\|^2_{L_2[0;2\pi]}=\pi(1-|a|).$$

For $n = 1, 2, ...$ we get $D_{n,n} = \| \cos(nt + \alpha(t)) \|^2_{L_2[0;2\pi]} = \pi + (1/2) \int_0^{2\pi} \cos[2nt + 2\alpha(t)]dt = \pi$, $\int_0^{2\pi} \cos^2[nt + \alpha(t))]t = \pi$, i.e. $\| \cos(nt + \alpha(t)) \|^2_{L_2[0;2\pi]} = \pi$. Similarly $\| \sin(nt + \alpha(t)) \|^2_{L_2[0;2\pi]} = \pi$.

So it is proved that sequence of the Theorem 2 is complete, orthogonal and its elements have norms in $L_2[0; 2\pi]$ equal one. It means that system is orthonormal basis in this space, according [6] (see p. 85). This completely proves the Theorem 2.

3. APPLICATION

The previously obtained relation for $\alpha(t)$ allow to obtain simples examples of complete and orthogonal sequences of the form (2) depending on the choice of the parameter a, $a \in C$, $|a| < 1$.

Example 1. If $a \neq 0$ then $\alpha(t) = (1/2)\arg[B_a(e^{it})]$ and complete orthogonal sequence of the form (2) in $L_2[0; 2\pi]$ is $\{\cos\left[nt + (1/2)\arg[B_a(e^{it})]\right];\ \sin\left[nt + (1/2)\arg[B_a(e^{it}))]\right]\}_{n=0}^{\infty}$.

Example 2. If $a = 0$ (in $B_a(z)$) then $\alpha(t) = t/2$ and complete orthogonal sequence of the form (2) in $L_2[0; 2\pi]$ is $\{\cos(nt + t/2 + r);\ \sin(nt + t/2 + r)\}_{n=0}^{\infty}$ for arbitrary real number r.

Moreover, these examples are described all complete and orthogonal sequences of the form (2) in $L_2[0; 2\pi]$, if real function $\alpha(t) \sqsubseteq Lip_\nu[0; 2\pi] \cap Var[0; 2\pi]$, $0 < \nu \leq 1$.

REFERENCES

1. K. Shaidukov, "About the completeness of trigonometric system," Usp. Mat. Nauk **6**, 143–153 (1953).
2. A. Barmenkov and J. Kazmin, "About the completeness of two systems of functions," in *Theory of Mappings, its Display and Applications, Collection of Scientific Works* (Naukova Dumka, Kiev, 1982), p. 29–43 [in Russian].
3. A. Barmenkov, "About approximation properties of some systems of functions," Cand. Sci. (Phys. Math.) Dissertation (Moscow, 1983).
4. I. Danyluk, *Irregular Boundary Value Problems on the Plane* (Nauka, Moscow,1975) [in Russian].
5. I. Privalov, *Boundary Properties of Analytic Functions* (Gostekhizdat, Moscow, 1950) [in Russian].
6. *Functional Analysis,* Ed. by C. Krein (Nauka, Moscow, 1964) [in Russian].