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Clarkson's Inequalities for Periodic Sobolev Space

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Abstract—The validity of Clarkson's inequalities for periodic functions in the Sobolev space normed without the use of pseudodifferential operators is proved. The norm of a function is defined by using integrals over a fundamental domain of the function and its generalized partial derivatives of all intermediate orders. It is preliminarily shown that Clarkson's inequalities hold for periodic functions integrable to some power p over a cube of unit measure with identified opposite faces. The work is motivated by the necessity of developing foundations for the functional-analytic approach to evaluating approximation methods.

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INTRODUCTION

The thesis about the application of functional analysis to problems of computational mathematics advanced by Academician S.L. Sobolev in 1974 still remains topical: "...the estate of new mathematics includes also computation theory, which is presently as hard to imagine without Banach spaces as without electronic computing machines" [1, p. 7]. This paper considers questions related to the uniform convexity of periodic function spaces. Having proved this property for a space, we can solve various extremal problems requiring the proof of the uniqueness of the sought element of the space. Uniform convexity is proved by verifying Clarkson's inequalities or their generalizations [1–9]. The initial form for such a problem is the form of the norm of a function in the space under examination.

Sobolev normed the spaces $W_p^{(m)}$ by means of projection operators [1, 3] in the most general form. However, in [1], a special form of a norm was also given, which was called the simplest form and expressed in terms of integrals of the function and its higher-order derivatives. The function and its generalized partial derivatives up to order m are p-integrable on a certain domain Ω such that $\overline{G} \subset \Omega$, and

$$\left|\left|f \left| W_p^{(m)}(\Omega)\right.\right|\right| = \left[\left|\int\limits_G |f| dx\right|^p + \int\limits_\Omega \left(\sum_{|\alpha|=m} (D^\alpha f)^2\right)^{p/2} dx\right]^{1/p}.$$

Examples of norms are also found in other researches. In accordance with the theory of elliptic differential equations, Agranovich [10] considers Sobolev spaces as special cases of more general spaces. The difference between these spaces is in the order of partial differentiation, which is nonnegative and integer for Sobolev spaces and real for general spaces. The norms are defined without specifying the coefficients of the derivatives:

$$\left|\left|f \left| W_p^{(m)}(\mathbb{R}_n)\right.\right|\right| = \sum_{|\alpha|\le m} \left[\int\limits_{\mathbb{R}_n} |D^\alpha f(x)|^p \, dx\right]^{1/p}.$$

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Concerning this norm, the author of [10] made the remark that, in the integrands, it suffices to retain only $\alpha = (0, \ldots, 0)$ and $\alpha = (m, 0, \ldots, 0)$, ..., $(0, \ldots, 0, m)$, which we consider important. In this respect, Maz'ya's book [11] is also interesting for our study, in which function space normed by means of derivatives of intermediate orders were introduced. In [11], Maz'ya considered embedding theorems under various conditions imposed on the domain of integration. Spaces normed with the use of derivatives of intermediate orders, which he denoted by $V_p^m(\Omega)$, appear in all fundamental theorems together with the spaces $L_p^m(\Omega)$ and $W_p^m(\Omega)$. The notation $W_p^m(\Omega)$ Maz'ya uses for spaces with norm

$$\left|\left|f\left|W_p^m(\Omega)\right|\right|\right. = \left|\left|\nabla_m f\left|L_p^m(\Omega)\right|\right|\right. + \left|\left|f\left|L_p(\Omega)\right|\right|\right.$$
$$= \left[\int\limits_{\Omega}\left(\sum_{|\alpha|=m}|D^\alpha f(x)|^2\right)^{p/2}dx\right]^{1/p} + \left[\int\limits_{\Omega}|f(x)|^p\,dx\right]^{1/p}.$$

The norm on $V_p^m(\Omega)$ is defined by

$$\left|\left|f\left|V_p^m(\Omega)\right|\right|\right. = \sum_{k=0}^{m}\|\nabla_k f\left|L_p(\Omega)\right.\| = \sum_{k=0}^{m}\left[\int\limits_{\Omega}\left(\sum_{|\alpha|=k}|D^\alpha f(x)|^2\right)^{p/2}dx\right]^{1/p}.$$

Next, in studies concerned with the problem of estimating error functionals of cubature formulas, various normings of the spaces $W_p^{(m)}$ of both periodic and nonperiodic functions are used. Shoinzhurov [12] normed the space of periodic functions in a special way by means of the operator Laplace

$$\left|\left|f\left|\widetilde{W}_p^{(m)}(Q)\right|\right|\right. = \left[\int\limits_{Q}\left|(1-\Delta)^{m/2}f(x)\right|^p dx\right]^{1/p}.$$

This norming has made it possible of solve problems posed in [12] for any real m. For positive integer m, norms containing improper integrals over $\mathbb{R}_n$ of a function and its derivatives of all orders were used in [13, 14]:

$$\left|\left|f\left|W_p^{(m)}(\mathbb{R}_n)\right|\right|\right. = \left[\int\limits_{\mathbb{R}_n}\sum_{|\alpha|\le m}\frac{|\alpha|!}{\alpha!}|D^\alpha f(x)|^p\,dx\right]^{1/p}.$$

The normings listed above do not exhaust all normings of the Sobolev space. Note that the expression for the norm of a function must necessarily contain the function itself and all of its partial derivatives of given highest order. Derivatives of intermediate orders, as well as coefficients multiplying them, may be or be not included. It is seen from the examples given above that, in some studies, the norm can be defined in terms of the inverse Fourier transform of a fundamental solution of some differential equation.

In this paper, we norm the space $\widetilde{W}_p^{(m)}(Q)$ in the spirit of [13–15]. Here, m takes only positive integer values, but our norm is not a special case of any of the norms given above. The author applied such norms to weighted spaces in [16, 17].

Various normings of a space determine the corresponding settings of extremal problems. In [18, 19], the uniform convexity properties of Hilbert spaces were used for substantiating the uniqueness of an extremal function for a linear functional. We also mention papers [20–22] devoted to the proof of Clarkson's inequalities and their analogues in spaces with certain specific features.

In our paper, the integrability exponent $p \in (1, \infty)$ renders the Sobolev space under consideration non-Hilbert, except for the only value $p = 2$.

1. SPACES, NORMS, AND INITIAL INEQUALITIES

We norm the space $\widetilde{W}_p^{(m)}(Q)$ of periodic functions of period 1 with respect to each independent variable as follows:

$$\left|\left|f\left|\widetilde{W}_p^{(m)}(Q)\right|\right|\right| = \left[\int\limits_Q \sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}|D^\alpha f|^p\,dx\right]^{1/p}, \tag{1}$$

where Q is the unit cube with identified opposite faces and $p\in(1,\infty)$. Note the binomial coefficient multiplying the kth partial derivatives. Each derivative $D^\alpha f(x)$ of a function in $\widetilde{W}_p^{(m)}(Q)$ belongs to the periodic function space $\widetilde{L}_p(Q)$ with norm

$$\left|\left|f\left|\widetilde{L}_p(Q)\right|\right|\right| = \left[\int\limits_Q |f|^p\,dx\right]^{1/p}. \tag{2}$$

Clarkson's inequalities and their generalization were proved in [1–4] for the spaces $L_p(\Omega)$ of nonperiodic p-integrable functions defined on a bounded domain $\Omega\subset\mathbb{R}_n$ with norm

$$||f\,|L_p(\Omega)\,|| = \left[\int\limits_\Omega |f|^p\,dx\right]^{1/p}.$$

Clarkson [2] gave inequalities for the spaces $L_p(\Omega)$ of functions on a bounded domain Ω. At present, inequalities named after Clarkson are usually given in a different form (see, e.g., [3]). Clarkson's first inequality is valid for $p\ge 2$:

$$\left|\left|\frac{f+g}{2}\right|L_p(\Omega)\right|\right|^p + \left|\left|\frac{f-g}{2}\right|L_p(\Omega)\right|\right|^p \le \frac{1}{2}\left(||f\,|L_p(\Omega)\,||^p + ||g\,|L_p(\Omega)\,||^p\right), \quad p\ge 2; \tag{3}$$

Clarkson's second inequality is valid for p in the interval specified below:

$$\left|\left|\frac{f+g}{2}\right|L_p(\Omega)\right|\right|^q + \left|\left|\frac{f-g}{2}\right|L_p(\Omega)\right|\right|^q \le \left(\frac{1}{2}\,||f\,|L_p(\Omega)\,||^p + \frac{1}{2}\,||g\,|L_p(\Omega)\,||^p\right)^{q-1},$$
$$1<p\le 2, \quad q=p/(p-1). \tag{4}$$

In the present paper, we use this form.

The role of a bounded domain of integration is played in our paper by the unit n-cube with identified opposite faces. This means that the faces and edges containing the origin belong to the domain, while those opposite to them do not. The cube is open on the side of faces containing the point of n-space all of whose coordinates are 1. The translates for such a cube by integer vectors cover the entire space; therefore, this is a fundamental domain for periodic functions with unit matrix of periods.

We do not check the equivalence of norms or use embedding theorems. Our starting point in obtaining the main results are known inequalities for arbitrary variable quantities [3]. We give these inequalities below in the form of lemmas.

Lemma 1 (Sobolev). *For $p\ge 2$ and $0\le x\le 1$,*

$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \le \frac{1}{2}\left(1+x^p\right). \tag{5}$$

Lemma 2 (Sobolev). *For $p\le 2$ and any x and y,*

$$\left(\left|\frac{x+y}{2}\right|^{p/(p-1)} + \left|\frac{x-y}{2}\right|^{p/(p-1)}\right)^{p-1} \le \frac{1}{2}\left(|x|^p+|y|^p\right). \tag{6}$$

The proofs of the main statements also use the inverse Minkowski inequalities

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{m}|x_{ik}|\right)^{r}\right)^{1/r} \geq \sum_{k=1}^{m}\left(\sum_{i=1}^{n}|x_{ik}|^{r}\right)^{1/r}, \quad 0<r<1 \tag{7}$$

(for sums) and

$$\left(\int\limits_{\Omega}\left(\sum_{k=1}^{m}|f_k(x)|\right)^{r}dx\right)^{1/r} \geq \sum_{k=1}^{m}\left(\int\limits_{\Omega}|f_k(x)|^{r}\,dx\right)^{1/r}, \quad 0<r<1 \tag{8}$$

(for integrals).

2. THE PERIODIC SOBOLEV SPACE

Sobolev identification [1] is the following equivalence relation between points of Euclidean n-space $\mathbb{R}_n$: two points x' and x'' are considered equivalent if the differences between the respective coordinates of these points are integer, i.e., $x_1'-x_1''\in\mathbb{Z},\dots,x_n'-x_n''\in\mathbb{Z}$. This relation is indeed an equivalence, because it is reflexive ($x\sim x$), symmetric ($x'\sim x''\Rightarrow x''\sim x'$), and transitive ($x'\sim x'', x''\sim x'''\Rightarrow x'\sim x'''$). The quotient $\{\mathbb{R}_n/\sim\}$ by this relation is, geometrically, the n-torus Θ_n.

Given $x\in\mathbb{R}_n$, the points equivalent to (or identified with) x have the new coordinates $t_j=\{x_j\}=x_j-[x_j]$, $j=1,\dots,n$, $t=U(x)$, where $U(x)$ is the fractional part operator. The inverse map is one-to-infinite: $U^{-1}(t)=\{x:x_j=t_j+\beta_j\}$, where $0\leq t_j<1$ and the β_j are the components of any integer vector for $j=1,\dots,n$. The manifold thus obtained is the n-torus $\Theta_n=\{t:0\leq t_j<1, j=1,\dots,n\}$, which can be treated as the unit n-cube with identified opposite faces.

The spaces of periodic functions on $\mathbb{R}_n$ are related to functions decreasing at infinity faster than any negative degree of $|x|$ and to spaces of functions on the torus Θ_n. Namely, let $D=D(\Omega)$ be the space of infinitely differentiable compactly supported functions on Ω with convergence $f_k\to f$, $k\to\infty$, defined as the uniform convergence of derivatives: $D^\alpha f_k(x)\rightrightarrows D^\alpha f(x)$, $k\to\infty$, $\alpha=0,1,2,\dots$, $x\in\Omega$, $\mathrm{supp} f_k\subset\Omega'$, $\Omega':\overline{\Omega}\subset\Omega$. Next, let $S=S(\mathbb{R}_n)$ be the Schwartz space of infinitely differentiable functions on $\mathbb{R}_n$ such that all their derivatives (including the functions themselves) satisfy the condition $|D^\alpha f|\leq K\dfrac{1}{1+|x|^s}$, $s>0$. Convergence in S is defined as the convergence of products: $x^sD^\alpha f_k(x)\rightrightarrows x^sD^\alpha f(x)$, $s>0$, $k\to\infty$, $\alpha=0,1,2,\dots$, $x\in\mathbb{R}_n$. The spaces S and D with these convergences are complete, that is, any fundamental sequence in each of these spaces converges to an element of the space. The functions from D belong to S; thus, $D\subset S$.

According to a definition in [1], a function of n variables is said to be periodic with period matrix A if it satisfies the condition $f(x+A\beta)=f(x)$ for any integer vector β. Let $\widetilde{S}(Q)$ be the space of infinitely differentiable periodic functions with unit period matrix and convergence $D^\alpha f_k(x)\rightrightarrows D^\alpha f(x)$, $k\to\infty$, $\alpha=0,1,2,\dots$, $x\in Q$, where $Q=\{x:0\leq x_j<1, j=1,\dots,n\}$ is the unit cube.

The map of Euclidean n-space $\mathbb{R}_n$ to the n-torus Θ_n that was considered above takes the space $\widetilde{S}(Q)$ of periodic functions to the space $T=T(\Theta_n)$ of infinitely differentiable periodic functions on the torus Θ_n. Convergence in T is again defined as the uniform convergence of all derivatives: $D^\alpha f_k(t)\rightrightarrows D^\alpha f(t)$, $k\to\infty$, $\alpha=0,1,2,\dots$, $t\in\Theta_n$.

Given any function $f\in S$, we can construct an infinitely differentiable periodic function in the form of a series as $\widetilde{\phi}(x)=\sum\limits_{\beta}f(x+E\beta)$. A direct verification proves that $\widetilde{\phi}(x)$ is periodic. The definition of a periodic function with unit period matrix implies

$$\begin{aligned}\widetilde{\phi}(x+E\gamma)&=\sum_{\beta}f(x+E\gamma+E\beta)\\&=\sum_{\beta}f(x+E(\gamma+\beta))=\sum_{\gamma'}f(x+E\gamma')=\widetilde{\phi}(x).\end{aligned}\tag{9}$$

In the next to last relation, we made the change $\gamma' = \gamma + \beta$, where γ' and γ are integer vectors. Each term of the series is a function belonging to the Schwartz space and, therefore, infinitely differentiable. Moreover, the function series $\sum_\beta f(x + E\beta)$ is bounded above by the convergent number series $\sum_{|\beta'|\neq 0} \dfrac{1}{|E\beta'|^s}$ with $s > 1$:

$$|f(x+E\beta)| \le K\frac{1}{1+|x+E\beta|^s} \le K\frac{1}{1+|E\beta'|^s} \le K\frac{1}{|E\beta'|^s},$$

$$x \in Q, \quad \beta' = (\beta'_1, \ldots, \beta'_n)^T : \ \beta'_j = \begin{cases} 1+\beta_j, & \beta_j < 0, \\ \beta_j, & \beta_j \ge 0, \end{cases} \quad j = 1, \ldots, n. \tag{10}$$

Therefore, series (9) converges to an infinitely differentiable function, which we denote by $\widetilde{\phi}(x)$. Thus, $\widetilde{\phi}(x) \in \widetilde{S}$. To the periodic function $\widetilde{\phi}(x)$ of period 1 with respect to each variable x_j, $j = 1, \ldots, n$, there corresponds a smooth function $\phi(t) \in T(\Theta)$ such that $\widetilde{\phi}(x) = \phi(t)$ for $x = t$. Moreover, $x \in \mathbb{R}_n$ and $t \in \Theta_n$ are equal up to an integer multiplier E.

Our main space is the space of periodic functions with unit period matrix integrable to the pth power together with their derivatives of order up to m. Differentiation is understood in the generalized sense. Such functions form the Sobolev space $\widetilde{W}_p^{(m)}(Q)$. This is the closure of the set of smooth periodic functions in norm (1). The limit elements in the sense of convergence in this norm cannot be smooth functions; therefore, adding them to the space extends the stock of main functions. The functions from $\widetilde{S}(Q)$ are included in $\widetilde{W}_p^{(m)}(Q)$, because the generalized derivatives of functions infinitely differentiable in the ordinary sense are p-integrable.

3. CLARKSON'S INEQUALITIES FOR PERIODIC P-INTEGRABLE FUNCTIONS

Let us prove the validity of Clarkson's inequalities for the space $\widetilde{L}_p(Q)$ with norm (2). We begin with the first inequality. Following [3], given functions $f, g \in \widetilde{L}_p(Q)$, $p \ge 2$ (for definiteness, we assume that $|g| \le |f|$ and, hence, $|g/f| \le 1$), we transform the sum

$$\left|\frac{f+g}{2}\right|^p + \left|\frac{f-g}{2}\right|^p = |f|^p \left[\left(\frac{1+\left|\frac{g}{f}\right|}{2}\right)^p + \left(\frac{1-\left|\frac{g}{f}\right|}{2}\right)^p\right]. \tag{11}$$

According to Lemma 1, for the expression in brackets in (11), inequality (5) with $x = |g/f|$ holds:

$$\left(\frac{1+\left|\frac{g}{f}\right|}{2}\right)^p + \left(\frac{1-\left|\frac{g}{f}\right|}{2}\right)^p \le \frac{1}{2}\left(1+\left|\frac{g}{f}\right|^p\right), \quad 2 \le p < \infty. \tag{12}$$

Therefore, the sum on the left-hand side of (11) satisfies the inequality

$$\left|\frac{f+g}{2}\right|^p + \left|\frac{f-g}{2}\right|^p \le |f|^p \left[\frac{1}{2}\left(1+\left|\frac{g}{f}\right|^p\right)\right] = \frac{1}{2}\left(|f|^p + |g|^p\right), \quad 2 \le p < \infty. \tag{13}$$

Integrating both sides of (13) over Q (the integrals exist by assumption), we obtain

$$\int\limits_Q \left|\frac{f+g}{2}\right|^p dx + \int\limits_Q \left|\frac{f-g}{2}\right|^p dx \le \frac{1}{2}\left(\int\limits_Q |f|^p dx + \int\limits_Q |g|^p dx\right), \quad 2 \le p < \infty.$$

Imposing the condition of periodicity with respect to each variable and passing to an expression in terms of norms (2), we obtain Clarkson's first inequality for the space $\widetilde{L}_p(Q)$:

$$\left\|\frac{f+g}{2}\middle|\widetilde{L}_p(Q)\right\|^p + \left\|\frac{f-g}{2}\middle|\widetilde{L}_p(Q)\right\|^p$$

$$\leq \frac{1}{2}\left(\left\|f\left|\widetilde{L}_p(Q)\right\|^p + \left\|g\left|\widetilde{L}_p(Q)\right\|^p\right), \quad 2 \leq p < \infty. \tag{14}$$

In order to prove Clarkson's second inequality in the space $\widetilde{L}_p(Q)$ of periodic functions p-integrable on the period, to the left-hand side of inequality (6) in Lemma 2, that is,

$$\left(\left|\frac{f+g}{2}\right|^{p/(p-1)} + \left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} \leq \frac{1}{2}\left(|f|^p + |g|^p\right), \quad 1 < p \leq 2, \tag{15}$$

we apply the inverse Minkowski inequality (8), which is valid for the exponents $r = p-1$ satisfying the condition $0 < p-1 < 1$:

$$\left[\int\limits_Q \left(\left|\frac{f+g}{2}\right|^{p/(p-1)} + \left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} dx\right]^{1/(p-1)}$$

$$\geq \left[\int\limits_Q \left(\left|\frac{f+g}{2}\right|^{p/(p-1)}\right)^{p-1} dx\right]^{1/(p-1)} + \left[\int\limits_Q \left(\left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} dx\right]^{1/(p-1)}.$$

After simplification, we obtain

$$\left[\int\limits_Q \left(\left|\frac{f+g}{2}\right|^{p/(p-1)} + \left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} dx\right]^{1/(p-1)}$$

$$\geq \left[\int\limits_Q \left|\frac{f+g}{2}\right|^p dx\right]^{1/(p-1)} + \left[\int\limits_Q \left|\frac{f-g}{2}\right|^p dx\right]^{1/(p-1)}. \tag{16}$$

Now, let us integrate inequality (15):

$$\int\limits_Q \left(\left|\frac{f+g}{2}\right|^{p/(p-1)} + \left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} dx \leq \frac{1}{2}\left(\int\limits_Q |f|^p dx + \int\limits_Q |g|^p dx\right), \quad 1 < p \leq 2.$$

We raise both sides to the power $1/(p-1) > 0$:

$$\left[\int\limits_Q \left(\left|\frac{f+g}{2}\right|^{p/(p-1)} + \left|\frac{f-g}{2}\right|^{p/(p-1)}\right)^{p-1} dx\right]^{1/(p-1)}$$

$$\leq \left[\frac{1}{2}\left(\int\limits_Q |f|^p dx + \int\limits_Q |g|^p dx\right)\right]^{1/(p-1)}, \quad 1 < p \leq 2.$$

Taking into account (16), we obtain

$$\left[\int\limits_Q \left|\frac{f+g}{2}\right|^p dx\right]^{1/(p-1)} + \left[\int\limits_Q \left|\frac{f-g}{2}\right|^p dx\right]^{1/(p-1)}$$

$$\leq \left[\frac{1}{2}\left(\int\limits_Q |f|^p dx + \int\limits_Q |g|^p dx\right)\right]^{1/(p-1)}, \quad 1 < p \leq 2.$$

Passing to an expression in terms of norms (2), we obtain Clarkson's second inequality for the space $\widetilde{L}_p(Q)$:

$$\left\|\frac{f+g}{2}\,\middle|\,\widetilde{L}_p(Q)\right\|^{p/(p-1)}+\left\|\frac{f-g}{2}\,\middle|\,\widetilde{L}_p(Q)\right\|^{p/(p-1)}$$
$$\leq\left(\frac{1}{2}\left\|f\,\middle|\,\widetilde{L}_p(Q)\right\|^{p}+\frac{1}{2}\left\|g\,\middle|\,\widetilde{L}_p(Q)\right\|^{p}\right)^{1/(p-1)},\quad 1<p\leq 2. \tag{17}$$

Thus, we have obtained Clarkson's first inequality (14) and second inequality (17) for the periodic function space $\widetilde{L}_p(Q)$, which are similar to inequalities (3) and (4). The space $\widetilde{L}_p(Q)$ contains the derivatives of the functions considered in the next section.

4. CLARKSON'S INEQUALITIES FOR PERIODIC FUNCTIONS IN THE SOBOLEV SPACE

First, we prove Clarkson's first inequality. By definition, periodic functions $f,g\in\widetilde{W}_p^{(m)}(Q)$ in the Sobolev space satisfy the condition $D^\alpha f, D^\alpha g\in\widetilde{L}_p(Q)$, $|\alpha|\leq m$. Since the derivatives belong to the space $\widetilde{L}_p(Q)$, it follows that each of them satisfies the inequality

$$\int\limits_Q\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p dx+\int\limits_Q\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p dx$$
$$\leq\frac{1}{2}\left(\int\limits_Q|D^\alpha f|^p\,dx+\int\limits_Q|D^\alpha g|^p\,dx\right),\quad 2\leq p<\infty,\quad |\alpha|\leq m.$$

This inequality remains valid after the summation of all derivatives with a common constant multiplier:

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p dx+\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p dx$$
$$\leq\frac{1}{2}\left(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q|D^\alpha f|^p\,dx+\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q|D^\alpha g|^p\,dx\right),$$
$$2\leq p<\infty,\quad k=0,1,\ldots,m.$$

Summation over k with the binomial coefficients $\binom{m}{k}$ preserves this inequality as well:

$$\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p dx$$
$$+\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p dx\leq\frac{1}{2}\left[\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q|D^\alpha f|^p\,dx\right.$$
$$\left.+\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int\limits_Q|D^\alpha g|^p\,dx\right],\quad 2\leq p<\infty. \tag{18}$$

The passage to expression (18) in terms of norm (1) yields Clarkson's first inequality for the periodic Sobolev space $\widetilde{W}_p^{(m)}(Q)$:

$$\left\|\frac{f+g}{2}\,\middle|\,\widetilde{W}_p^{(m)}(Q)\right\|^{p}+\left\|\frac{f-g}{2}\,\middle|\,\widetilde{W}_p^{(m)}(Q)\right\|^{p}$$

$$\leq \frac{1}{2}\left(\left\|f\left|\widetilde{W}_p^{(m)}(Q)\right\|^p+\right\|g\left|\widetilde{W}_p^{(m)}(Q)\right\|^p\right), \quad 2\leq p<\infty. \tag{19}$$

To derive Clarkson's second inequality, we introduce derivatives of functions into inequality (15):

$$\left(\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^{p/(p-1)}+\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^{p/(p-1)}\right)^{p-1}$$
$$\leq \frac{1}{2}\left(|D^\alpha f|^p+|D^\alpha g|^p\right), \quad |\alpha|\leq m, \quad 1<p\leq 2. \tag{20}$$

We sum both sides of (20) with a common multiplier and binomial coefficients, as it was done for (18):

$$\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^{p/(p-1)}+\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^{p/(p-1)}\right)^{p-1}$$
$$\leq \frac{1}{2}\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(|D^\alpha f|^p+|D^\alpha g|^p\right), \quad 1<p\leq 2.$$

Then, we integrate over Q:

$$\int\limits_Q\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^{p/(p-1)}+\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^{p/(p-1)}\right)^{p-1}dx$$
$$\leq \frac{1}{2}\int\limits_Q\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(|D^\alpha f|^p+|D^\alpha g|^p\right)dx, \quad 1<p\leq 2.$$

Let us raise this expression to the power $1/(p-1)>0$:

$$\left(\int\limits_Q\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^{p/(p-1)}+\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^{p/(p-1)}\right)^{p-1}dx\right)^{1/(p-1)}$$
$$\leq \left(\frac{1}{2}\int\limits_Q\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(|D^\alpha f|^p+|D^\alpha g|^p\right)dx\right)^{1/(p-1)}, \quad 1<p\leq 2. \tag{21}$$

To the sum over the multi-index α under the integral sign on the left-hand side of (21) we apply the inverse Minkowski inequality (7) with exponent $r=p-1$:

$$\left(\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^{p/(p-1)}+\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^{p/(p-1)}\right)^{p-1}\right)^{1/(p-1)}$$
$$\geq\left(\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p\right)^{1/(p-1)}$$
$$+\left(\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p\right)^{1/(p-1)}.$$

Next, we return the sum under the sign of integral and apply the inverse Minkowski inequality (8) to the whole integral:

$$\left(\int\limits_Q\left(\left(\sum_{k=0}^m\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p\right)^{1/(p-1)}\right.\right.$$

$$+\left(\sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p\right)^{\frac{1}{p-1}}\Bigg)^{p-1}dx\Bigg)^{1/(p-1)}$$

$$\geq\left(\int\limits_Q\sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f+D^\alpha g}{2}\right|^p dx\right)^{1/(p-1)}$$

$$+\left(\int\limits_Q\sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|\frac{D^\alpha f-D^\alpha g}{2}\right|^p dx\right)^{1/(p-1)}.$$

Since the differentiation operator D^α is linear, we can pass to an expression in terms of norms (4):

$$\left(\int\limits_Q\sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|D^\alpha\left(\frac{f+g}{2}\right)\right|^p dx\right)^{1/(p-1)}$$

$$+\left(\int\limits_Q\sum_{k=0}^{m}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\left|D^\alpha\left(\frac{f-g}{2}\right)\right|^p dx\right)^{1/(p-1)}$$

$$=\left\|\frac{f+g}{2}\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^{p/(p-1)}+\left\|\frac{f-g}{2}\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^{p/(p-1)}.$$

We have obtained Clarkson's second inequality for the periodic Sobolev space $\widetilde{W}_p^{(m)}(Q)$:

$$\left\|\frac{f+g}{2}\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^{p/(p-1)}+\left\|\frac{f-g}{2}\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^{p/(p-1)}$$

$$\leq\left(\frac{1}{2}\left\|f\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^p+\frac{1}{2}\left\|g\left|\widetilde{W}_p^{(m)}(Q)\right.\right\|^p\right)^{1/(p-1)},\quad 1<p\leq 2. \tag{22}$$

Thus, we have obtained inequalities (19) and (22), which are similar to (3) and (4), for the space $\widetilde{W}_p^{(m)}(Q)$.

CONCLUSION

The validity of Clarkson's inequalities between the norms of any two elements of the periodic Sobolev space means that the unit sphere fulfillment property uniform convexity in the space with non-Hilbert integrability exponent $p\in(1,\infty)$. The proof is based on number inequalities applied to the values of functions at points of the unit n-cube with identified opposite faces, over which integration was subsequently performed. The result is intended for application to problems in which it is necessary to represent periodic generalized functions in terms of integrable periodic functions on spaces normed without the use of pseudodifferential operators.

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REFERENCES

1. S. L. Sobolev, *Introduction to the Theory of Cubature Formulas* (Nauka, Moscow, 1974) [in Russian].
2. J. A. Clarkson, "Uniformly convex spaces," Trans. Am. Math. Soc. **40**, 396–414 (1936). doi 10.2307/1989630
3. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics* (Nauka, Moscow, 1988) [in Russian].
4. O. Hanner, "On the uniform convexity of L_p and l_p," Ark. Mat. **3**, 239–244 (1956). doi 10.1007/BF02589410
5. P. Enflo, "Banach spaces which can be given an equivalent uniformly convex norm," Isr. J. Math. **13**, 281–288 (1972). doi 10.1007/BF02762802
6. R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces* (Longman, Harlow, 1993).
7. V. R. Portnov, "Sobolev projection operators for seminorms with infinite-dimensional kernels," Tr. MIAN SSSR **140**, 252–263 (1976).
8. V. R. Portnov, "On some integral inequations," in *Embedding Theorems and Their Applications* (Nauka, Moscow, 1970), pp. 195–203 [in Russian].
9. V. R. Portnov, "On a certain projection operator of Sobolev type," Dokl. AN SSSR **189**, 258–260 (1969).
10. M. S. Agranovich, *Sobolev Spaces, their Generalizations and Elliptic Problems in Domains with Smooth and Lipschitz Boundary* (MTsNMO, Msocow, 2013) [in Russian].
11. V. G. Maz'ya, *Sobolev Spaces* (Leningr. Gos. Univ., Leningrad, 1985) [in Russian].
12. Ts. B. Shoinzhurov, "The theory of cubature formulas in function spaces with the norm depending on the function and its derivatives," Doctoral (Phys. Math.) Dissertation (Ulan-Ude, 1977).
13. Ts. B. Shoinzhurov, *Estimation of Norm of Cubature Formula Error Functional in Various Functional Spaces* (Buryat. Nauch. Tsentr SO RAN, Ulan-Ude, 2005) [in Russian].
14. Ts. B. Shoinzhurov, *Cubature Formulae in Sobolev Space* W_p^m (Vost. Sib. Gos. Tekh. Univ., Ulan-Ude, 2002) [in Russian].
15. I. V. Korytov, "Function representing error functional of a cubature formula in Sobolev space," Izv. Tomsk. Politekh. Univ. **323** (2), 21–25 (2013).
16. I. V. Korytov, "The extreme function of a linear functional at the weighted Sobolev space," Vestn. Tom. Univ., Mat. Mekh., No. 2, 5–15 (2011).
17. I. V. Korytov, "Representation of error functional of cubature formula at weighted Sobolev space," Vychisl. Tekhnol. **11**, 59–66 (2006).
18. S. L. Sobolev and V. L. Vaskevich, *Cubature Formulas* (Inst. Matem., Novosibirsk, 1996) [in Russian].
19. V. L. Vaskevich, "Errors, condition numbers, and guaranteed accuracy of higher-dimensional spherical cubatures," Sib. Math. J. **53**, 996–1010 (2012). doi 10.1134/S0037446612060043
20. A. Mbarki, A. Ouahab, and I. E. Hadi, "Convexity and fixed point properties in spaces of Bochner integrals nuclear-valued functions," Appl. Math. Sci. **8** (84), 4179–4186 (2014).
21. H. Mizuguchi and K. S. Saito, "A note on Clarkson's inequality in the real case," J. Math. Inequalities **4**, 29–132 (2010). doi 10.7153/jmi-04-13
22. T. Formisano and E. Kissin, "Clarkson–McCarthy inequalities for l_p-spaces of operators in Schatten ideals," Integr. Equations Oper. Theory **79**, 151–173 (2014). doi 10.1007/s00020-014-2145-x