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On Convergence of Combinatorial Ricci Flow on Surfaces with Negative Weights

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Abstract—Chow and Lou in 2003 had shown that the analogue of the Hamilton Ricci flow on surfaces in the combinatorial setting converges to the Thruston's circle packing metric. The combinatorial setting includes weights defined for edges of a triangulation. Crucial assumption in the paper of Chow and Lou was that the weights are nonnegative. We show that the same results on convergence of Ricci flow can be proved under weaker condition: some weights can be negative and should satisfy certain inequalities. As a consequence we obtain theorem of existence of Thurston's circle packing metric for a wider range of weights.

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1. INTRODUCTION

We start from the basic definitions. Consider a closed surface M with a triangulation T. Let $V = \{A_1, \ldots, A_N\}$ be the set of verticies of T. Denote by E and F the sets of all edges and faces of the triangulation T. A *weight* is a function $w : E \to [-1, 1], w(A_iA_j) = w_{ij} = w_{ji}$. For a fixed triple (M, T, w) define a metric on M in the following way: (1) the metric is flat on each face of T; (2) the metric depends on parameters $r = \{r_i > 0 | i = 1, \ldots, N\} \in \mathbb{R}_+^N$; (3) the length l_{ij} of an edge $A_iA_j \in E$ is given by

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_jw_{ij}}. \tag{1}$$

Clearly these conditions determine a metric on M in a unique way provided for any face $A_iA_jA_k \in F$ the lengths l_{ij}, l_{jk}, l_{ik} satisfy the triangle inequalities. Hence it is natural to consider the subspace $\mathcal{R} \subseteq \mathbb{R}_+^N$ of those r which define $\{l_{ij} | A_iA_j \in E\}$ satisfying the triangle inequalities on each face of F. This combinatorial setting has very simple geometrical meaning. Namely, consider on the Euclidean plane circles C_i, C_j of radii r_i, r_j. Assume θ_{ij} is the intersection angle of the circles (θ_{ij} is chosen in such a way that $\theta_{ij} = 0$ for externally tangent circles). Then the distance between A_i and A_j is given by (1) where $w_{ij} = \cos\theta_{ij}$.

The curvature of such a metric is concetrated in the vertices of the triangulation. The *curvature* at the vertex A_i is defined to be $K_i = 2\pi - \sum_{A_iA_jA_k \in F} \angle A_jA_iA_k$. The *combinatorial Ricci flow* is the differential equation $dr_i/dt = -K_ir_i$ which defines the evolution of the metric in terms of the evolution of the parameters $r = \{r_i > 0 | i = 1, \ldots, N\}$. Hence it is natural to consider the Ricci flow on the space $\mathcal{R}$. This definition of a combinatorial Ricci flow is taken from [1] and is inspired by papers [2, 3].

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For surfaces of genus $g \geq 2$ it is also natural to discuss metrics on (M, T, w) which are not flat on each face but has constant negative curvature. So we consider another type of metrics which are defined in the following way:

(1) the metric on each face of the triangulation T has constant curvature -1;

(2) the metric depends on parameters $r = \{r_i > 0 | i = 1, \dots, N\} \in \mathbb{R}_+^N$;

(3) the length l_{ij} of an edge $A_iA_j \in E$ is given by $\cosh l_{ij} = \cosh r_i \cosh r_j + w_{ij} \sinh r_i \sinh r_j$.

It has the same geometrical meaning as described above but instead of circles on the Eucledean plane one considers circles on the hyperbolic plane. In this case the *combinatorial Ricci flow* is the differential equation $dr_i/dt = -K_i \sinh r_i$.

In [1] Chow and Lou proved that under certain conditions on weights and combinatorics of a triangulation T the Ricci flow for any initial $r(0) = (r_1^0, \dots, r_N^0)$ converges exponentially fast to the metric of constant curvature $K_i = 2\pi\xi(M)/N$ in the euclidean background and to the metric with $K_i = 0$ in the hyperbolic background. For the precise statements see Theorems 1.1 and 1.2 in [1].

Most important assumption on weights $\{w_{ij}\}$ in the paper [1] is the inequality $w_{ij} \geq 0$ for all i, j such that $A_iA_j \in E$. For other conditions (on combinatorics of the triangulation T) and for discussion of the Ricci flow for spherical background also see [1]. Detailed analysis of the arguments from [1] shows that assumptions $w_{ij} \geq 0$ were used to prove two statements on which other arguments are based.

(A) *The space $\mathcal{R}$ coincides with $\mathbb{R}_+^N$.*

(B) *For any triangle $\triangle A_iA_jA_k \in F$ denote by θ_i, θ_j and θ_k the internal angles at the corresponding vertices. Then one has $\partial\theta_i/\partial r_i < 0$, $\partial\theta_j/\partial r_i > 0$ for all $i \neq j$. Also $\partial(\theta_i + \theta_j + \theta_k)/\partial r_i = 0$ in the Eucledean plane and $\partial(\theta_i + \theta_j + \theta_k)/\partial r_i < 0$ in the hyperbolic plane.*

The purpose of our paper is to weaken assumptions on the weights $w = \{w_{ij}\}$ in such a way that these two statements still hold. As a consequence we obtain theorem of existence of Thurston's circle packing metric for a wider range of weights than it was proved for example in [1, 4].

Let us make important remark on extremal values of weights ($|w_{ij}| = 1$). The case $w_{ij} = w_{ik} = w_{jk} = 1$ correspond to pairwise external tangent circles C_i, C_j, C_k. This case was considered in [1, 5].

In other cases of $|w_{ij}| = |w_{ik}| = |w_{jk}| = 1$ points A_i, A_j, A_k belong to some line hence the triangle $\triangle A_iA_jA_k$ degenerates to a line segment for any r_i, r_j, r_k. Though this case could have some geometrical meaning we exclude it from our consideration. So in what follows the case when $|w_{ij}| = |w_{ik}| = |w_{jk}| = 1$ and at least one of the weights is negative is not considered and we will not mention it in assumptions of statements.

2. EUCLIDEAN GEOMETRY

First of all we investigate the space $\mathcal{R}$. Fix a triple (M, T, w). Consider a triangle of the triangulation T. To simplify notation suppose this is the triangle $\triangle A_1A_2A_3$. For positive numbers r_1, r_2, r_3 define $l_3 = l_{12}, l_1 = l_{23}, l_2 = l_{13}$ by formula (1). In [6, Lemma 13.7.2] it was shown that for $w \geq 0$ numbers l_1, l_2, l_3 satisfy triangle inequalities for any positive r_1, r_2 and r_3. We generalize this statement in two following Lemmas for more general assumptions. Three triangle inequalities on l_1, l_2, l_3 are equivalent to one inequality $(l_1 + l_2 + l_3)(-l_1 + l_2 + l_3)(l_1 - l_2 + l_3)(l_1 + l_2 - l_3) > 0$, which can easily be written in squares of l_1, l_2, l_3:

$$-(l_1^2 + l_2^2 + l_3^2)^2 + 4l_1^2l_2^2 + 4l_1^2l_3^2 + 4l_2^2l_3^2 > 0. \tag{2}$$

Lemma 1. *Let $w_{13} = \beta, w_{12} = \gamma$ and $w_{23} = \alpha$. Assume $\beta = \gamma \geq 0 > \alpha$. Then for any $r_1, r_2, r_3 > 0$ lengths l_1, l_2, l_3 satisfy the inequality (2).*

Proof. Expressing squares of l_1, l_2, l_3 by (1) we obtain the condition on r_1, r_2, r_3:

$$4(1-\gamma^2)r_1^2r_2^2 + 4(1-\gamma^2)r_1^2r_3^2 + 4(1-\alpha^2)r_2^2r_3^2 + 8r_1r_2r_3((\alpha+\gamma^2)r_1 + (\gamma+\alpha\gamma)r_2 + (\alpha\gamma+\gamma)r_3) > 0.$$

All the summands are positive except $8r_1r_2r_3(\alpha + \gamma^2)r_1$. Divide the inequality by 4 and transform it to

$$(1-\alpha^2)r_2^2r_3^2 + 2(1+\alpha)\gamma r_1r_2r_3(r_2+r_3) + r_1^2((1-\gamma^2)r_2^2 + 2(\alpha+\gamma^2)r_2r_3 + (1-\gamma^2)r_3^2) > 0.$$

The left hand side can be transformed to

$$(1-\alpha^2)r_2^2r_3^2+2(1+\alpha)\gamma r_1r_2r_3(r_2+r_3)+r_1^2(1-\gamma^2)(r_2-r_3)^2+2r_1^2(1+\alpha)r_2r_3,$$

which is obviously positive. □

Lemma 2. *Let* $w_{13}=\beta$, $w_{12}=\gamma$ *and* $w_{23}=\alpha$. *Assume* β and γ *are positive,* α *is negative, and* $\beta\gamma+\alpha\geq 0$, *and* γ or β *or* $|\alpha|\neq 1$. *Then for any* $r_1,r_2,r_3>0$ *numbers* l_1, l_2, l_3 *satisfy the inequality (2).*

Proof. Expressing in (2) squares of l_1, l_2, l_3 by (1) we obtain

$$4(1-\gamma^2)r_1^2r_2^2+4(1-\beta^2)r_1^2r_3^2+4(1-\alpha^2)r_2^2r_3^2+8r_1r_2r_3((\alpha+\beta\gamma)r_1 + (\beta+\alpha\gamma)r_2+(\alpha\beta+\gamma)r_3)>0.$$

This inequality is satisfied for any $r_1>0$, $r_2>0$, $r_3>0$, since from our assumption $\beta\gamma+\alpha\geq 0$ it follows that $\beta+\alpha\gamma\geq 0$ and $\alpha\beta+\gamma\geq 0$. □

Remark 1. *It is not difficult to show that (2) cannot be proved for any positive* r_1, r_2, r_3 *for positive* β, γ *and negative* α *without additional assumptions. Namely for* $\alpha=-0.6$, $\beta=0.9$, $\gamma=0.1$ *the inequality (2) is satisfied not for all positive* r_1, r_2, r_3.

Now we address the question (B) from the introduction. Consider the triangle $\triangle A_1A_2A_3$. For given positive numbers r_1, r_2, r_3 denote by $\theta_1, \theta_2, \theta_3$ the internal angles at the corresponding vertices of the triangle (provided l_1, l_2, l_3 satisfy the inequality (2)).

Theorem 1. *Assume the weights* α,β,γ *satisfy one of the following conditions:* (i) *all three weights are nonnegative;* (ii) *two weights* β *and* γ *are nonnegative,* α *is negative, and* $\beta\gamma+\alpha>0$ (or $\beta\gamma+\alpha=0$ *but* β *and* γ *are not equal to 1). Then* (a) $\partial\theta_i/\partial r_i<0$ *for all* i; (b) $\partial\theta_j/\partial r_i>0$ *for all* $j\neq i$; (c) $\partial(\theta_i+\theta_j+\theta_k)/\partial r_i=0$.

Proof. The statements under the assumption (i) were proved in [6, Lemma 13.7.3].

We address the assumptions (ii). Note that our arguments also apply to the assumption (i). From Thurston's proof [6] it follows that the inequalities (a) and (b) can be deduced from the fact that radical center of three circles with radii r_1, r_2, r_3 and the centers in the corresponding vertices of the triangle $\triangle A_1A_2A_3$ belongs to interior of the triangle.

Consider in the Euclidean plane the cartesian coordinate system with origin at A_1. Choose the axes in such a way that the vertex A_2 has coordinates $(l_3,0)$, and the vertex A_3 has coordinates $(l_2\cos\theta_1, l_2\sin\theta_1)$. Then coordinates of the radical center can be found from the system of equations

$$x^2+y^2-r_1^2=(x-l_3)^2+y^2-r_2^2,\quad x^2+y^2-r_1^2=(x-l_2\cos\theta_1)^2+(y-l_2\sin\theta_1)^2-r_3^2.$$

From the first equation one has $x=(r_1^2+r_1r_2\gamma)/l_3$. Substitute it for x in the second equation and obtain the equation on y: $2yl_2\sin\theta_1=2r_1^2+2r_1r_3\beta-\dfrac{l_2^2+l_3^2-l_1^2}{l_3^2}(r_1^2+r_1r_2\gamma)$. We need to prove than under assumptions (ii) one has $y>0$. This means that the radical center and the vertex A_3 belong to the same halfplane with respect to the line A_1A_2. For this purpose we need to check that for all $r_1,r_2,r_3>0$ one has $l_3^2(2r_1^2+2r_1r_3\beta)-(l_2^2+l_3^2-l_1^2)(r_1^2+r_1r_2\gamma)>0$. Substituting for l_1,l_2,l_3 their expressions in terms of weights and r_1,r_2,r_3 we reduce the inequality to

$$2r_1r_2((1-\gamma^2)r_1r_2+(\alpha+\beta\gamma)r_1r_3+(\beta+\alpha\gamma)r_2r_3)>0. \tag{3}$$

From the assumptions (ii) it follows that $\beta+\alpha\gamma\geq 0$. Hence (3) is satisfied for all $r_1,r_2,r_3>0$.

The radical center and the vertex A_2 are checked to belong to the same halfplane with respect to the line A_1A_3 in the same way. It is left to check that the radical center and the vertex A_1 belong to the same halfplane with respect to the line A_2A_3. In previous calculations change the indices $(1,2,3)\to(2,3,1)$ and correspondingly change the weights: $(\alpha,\beta,\gamma)\to(\beta,\gamma,\alpha)$. Then instead of (3) we need to check that

$$2r_2r_3((1-\alpha^2)r_2r_3+(\beta+\gamma\alpha)r_2r_1+(\gamma+\beta\alpha)r_2r_1) \tag{4}$$

is positive for all $r_2,r_2,r_3>0$. Under assumptions (ii) one has $\beta+\gamma\alpha\geq 0$ and $\gamma+\beta\alpha\geq 0$, hence (4) is positive for all $r_1,r_2,r_3>0$. □

Remark 2. *It is not possible to prove the statements of Theorem 1 for* $\gamma = \beta \geq 0 > \alpha$ *without assumption* $\gamma^2 + \alpha > 0$. *Indeed, in this case the inequality (4) is satisfied for all* r_1, r_2, r_3. *Nevertheless the inequality (3) can be reduced to*

$$2r_1r_2((1-\gamma^2)r_1r_2 + (\alpha+\gamma^2)r_1r_3 + (\gamma+\alpha\gamma)r_2r_3) > 0,$$

and for $\alpha + \gamma^2 < 0$ *one can easily find positive* r_1, r_2, r_3 *which do not satisfy this inequality. Indeed, one should fix some* r_1 *and* r_3, *and choose sufficiently small* $r_2 > 0$.

3. HYPERBOLIC GEOMETRY

In hyperbolic geometry let $w_{23} = \alpha$, $w_{13} = \beta$ and $w_{12} = \gamma$. When for lengths l_{12}, l_{13}, l_{23} one has

$$\cosh l_{12} = \cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2, \quad \cosh l_{13} = \cosh r_1 \cosh r_3 + \beta \sinh r_1 \sinh r_3,$$
$$\cosh l_{23} = \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3 \tag{5}$$

Lemma 3. *Assume* $\beta = \gamma \geq 0 \geq \alpha$. *Then for any* $r_1, r_2, r_3 \in \mathbb{R}^3_+$ *there exists a triangle with edges* l_{12}, l_{13}, l_{23} *defined by formulas (5).*

Proof. It is enough to prove the inequalities $l_{13} + l_{12} > l_{23}$, $l_{13} + l_{23} > l_{12}$. Indeed, arguments for $l_{13} + l_{23} > l_{12}$ are easily applied for $l_{12} + l_{23} > l_{13}$ by interchanging weights β and γ, and the indices 1 and 2. Since $\cosh t$ is strictly increasing for $t \in [0, \infty)$ one needs to prove the inequalities

$$\cosh(l_{13} + l_{12}) > \cosh l_{23}, \tag{6}$$
$$\cosh(l_{13} + l_{23}) > \cosh l_{12}. \tag{7}$$

We start with the inequality (6).

Expanding the lefthand side we obtain $\cosh l_{13} \cosh l_{12} + \sinh l_{13} \sinh l_{12} > \cosh l_{23}$. We are going to prove stronger inequality

$$\cosh l_{13} \cosh l_{12} > \cosh l_{23} \tag{8}$$

Substitute l_{12}, l_{13}, l_{23} from (5) and reduce (8) to

$$(\cosh r_1 \cosh r_3 + \gamma \sinh r_1 \sinh r_3)(\cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2)$$
$$> \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3. \tag{9}$$

Using assumption on α and γ now we obtain (9):

$$(\cosh r_1 \cosh r_3 + \gamma \sinh r_1 \sinh r_3)(\cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2)$$
$$\geq (\cosh r_1 \cosh r_3)(\cosh r_1 \cosh r_2) > \cosh r_1 \cosh r_3 \geq \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3.$$

The inequality (7) is more involved. Rewrite it as

$$\cosh l_{13} \cosh l_{23} + \sinh l_{13} \sinh l_{23} > \cosh l_{12}. \tag{10}$$

First of all we need to estimate $\cosh l_{13} \cosh l_{23} - \cosh l_{12}$:

$$\cosh l_{13} \cosh l_{23} - \cosh l_{12} = \cosh l_{13}(\cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3) - \cosh l_{12}$$
$$\geq \cosh l_{13}(\cosh r_2 \cosh r_3 - \sinh r_2 \sinh r_3) - \cosh l_{12} = (\cosh r_1 \cosh r_3 + \gamma \sinh r_1 \sinh r_3)$$
$$\times (\cosh r_2 \cosh r_3 - \sinh r_2 \sinh r_3) - (\cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2)$$
$$= (\sinh r_1 \cosh r_3 + \gamma \sinh r_3 \cosh r_1)(\sinh r_1 \cosh r_2 - \cosh r_1 \sinh r_2)$$
$$= (\cosh r_1 \sinh r_3 + \gamma \cosh r_3 \sinh r_1) \sinh(r_3 - r_2). \tag{11}$$

If $r_2 \leq r_3$ then (10) clearly follows from (11). For $r_3 > r_2$ we obtain (10) from two estimates:

$$\sinh l_{13} > \cosh r_1 \sinh r_3 + \gamma \cosh r_3 \sinh r_1, \tag{12}$$
$$\sinh l_{23} > |\sinh(r_3 - r_2)|. \tag{13}$$

For (12) note that

$$\cosh^2 l_{13} - (\cosh r_1 \sinh r_3 + \gamma \cosh r_3 \sinh r_1)^2 = \cosh^2 r_1 - \gamma^2 \sinh^2 r_1 \geq \cosh^2 r_1 - \sinh^2 r_1 = 1,$$

hence $\cosh r_1 \sinh r_3 + \gamma \cosh r_3 \sinh r_1 \leq \sqrt{\cosh^2 l_{13} - 1} = \sinh l_{13}$. Further

$$\cosh l_{23} = \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3 \geq \cosh r_2 \cosh r_3 - \sinh r_2 \sinh r_3 = \cosh(r_2 - r_3).$$

Hence $\sinh l_{23} = \sqrt{\cosh^2 l_{23} - 1} \geq \sqrt{\cosh^2(r_2 - r_3) - 1} = |\sinh(r_2 - r_3)|$, and (13) is proved. □

Lemma 4. *Let $\beta \geq 0$, $\gamma \geq 0$, and $0 > \alpha$. Assume $\beta\gamma + \alpha > 0$ (or $\beta\gamma + \alpha = 0$ but $\beta \neq 1$ and $\gamma \neq 1$). Then for any positive r_1, r_2, r_3 there exists a triangle with edges l_{12}, l_{13}, l_{23} defined by formulas (5).*

Proof. We need to check two inequalities

$$\cosh(l_{13} + l_{12}) > \cosh l_{23}, \tag{14}$$

$$\cosh(l_{13} + l_{23}) > \cosh l_{12}. \tag{15}$$

since the equality $\cosh(l_{12} + l_{23}) > \cosh l_{13}$ can be proved by the same argument as the last one.

We start with the inequality (14). As in the proof of previous Lemma 3 we are going to prove the stronger inequality

$$\cosh l_{13} \cosh l_{12} > \cosh l_{23}. \tag{16}$$

Substitute for l_{12}, l_{13}, l_{23} their expressions from (5) and reduce (16) to

$$(\cosh r_1 \cosh r_3 + \beta \sinh r_1 \sinh r_3)(\cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2)$$
$$> \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3. \tag{17}$$

Using assumption on α, β and γ one has

$$(\cosh r_1 \cosh r_3 + \beta \sinh r_1 \sinh r_3)(\cosh r_1 \cosh r_2 + \gamma \sinh r_1 \sinh r_2)$$
$$\geq (\cosh r_1 \cosh r_3)(\cosh r_1 \cosh r_2) > \cosh r_2 \cosh r_3 \geq \cosh r_2 \cosh r_3 + \alpha \sinh r_2 \sinh r_3.$$

The inequality (15) is more difficult. Assume $\beta \geq \gamma$ and substitute γ for β in (15). Then the lefthand side of the inequality decreases but is still greater than righthand side by the proof of (7), see the proof of Lemma 3.

Now assume $\gamma > \beta$. Then it is enough to check the inequality (15) for $\gamma = 1$ and $\alpha = -\beta$, which can be written as

$$\sinh l_{13} \sinh l_{23} > \cosh r_1 \cosh r_2 + \sinh r_1 \sinh r_2$$
$$- (\cosh r_1 \cosh r_3 + \beta \sinh r_1 \sinh r_3)(\cosh r_2 \cosh r_3 - \beta \sinh r_2 \sinh r_3). \tag{18}$$

The righthand side can be transformed to

$$A = -\cosh r_1 \cosh r_2 \sinh^2 r_3 + \sinh r_1 \sinh r_2 (1 + \beta^2 \sinh^2 r_3) - \beta \cosh r_3 \sinh r_3 (\sinh(r_1 - r_2)).$$

Note that for (18) it is enough to prove stronger inequality $B^2 > A^2$, where $B = \sinh l_{13} \sinh l_{23}$. Let $u_j = e^{r_j}$, $j = 1, 2, 3$. Then one has

$$16 u_1 u_2 u_3^2 A = -(u_1^2 + 1)(u_2^2 + 1)(u_3^2 - 1)^2 + (u_1^2 - 1)(u_2^2 - 1)(4u_3^2 + \beta^2 (u_3^2 - 1)^2)$$
$$+ \beta (u_3^2 + 1)(u_3^2 - 1)((u_1^2 + 1)(u_2^2 - 1) - (u_1^2 - 1)(u_2^2 + 1)).$$

Also one has

$$B^2 = (\cosh^2 l_{13} - 1)(\cosh^2 l_{23} - 1) = \frac{[(u_1^2 + 1)(u_3^2 + 1) + \beta (u_1^2 - 1)(u_3^2 - 1)]^2 - 16 u_1^2 u_3^2}{16 u_1^2 u_3^2}$$
$$\times \frac{[(u_2^2 + 1)(u_3^2 + 1) - \beta (u_2^2 - 1)(u_3^2 - 1)]^2 - 16 u_2^2 u_3^2}{16 u_2^2 u_3^2}.$$

To finish the proof one can show by cumbersome but straightforward computation that

$$(16 u_1 u_2 u_3^2)^2 (B^2 - A^2) = 16(1 - \beta^2)(u_1^2 u_2^2 - 1)^2 u_3^2 (u_3^2 - 1)^2 > 0.$$

□

Before we address the question (B) for the hyperbolic geometry we need to remind some useful notions and facts. Consider on the hyperbolic plane a point P and a circle C of radius r with the center in the point O. Consider a geodesic line which contains P and intersects the circle in the points A and B. The *degree of the point P with respect to the circle C* is $\deg_C P = d(l_{AP})d(l_{BP})$. Here l_{MN} is the distance between points M and N of the hyperbolic plane, and $d(l) = (e^l - 1)/(e^l + 1)$. It can be shown that the degree $\deg_C P$ does not depend on the choice of the line. Hence if the distance from P to the center of the circle C is equal to l then

$$\deg_C P = d(l+r)d(l-r) = \frac{e^{l+r}-1}{e^{l+r}+1}\frac{e^{l-r}-1}{e^{l-r}+1}. \tag{19}$$

For circles C_1 and C_2 the set of all points P such that $\deg_{C_1} P = \deg_{C_2} P$ is a geodesic line, which is called the *radical axis* of the circles C_1 and C_2. For two intersecting (but not coinciding) circles C_1 and C_2 the radical axis contains their common points. For any three circles the radical axes of every pair of the circles pass through certain point. This point is called the *radical center* of three circles. For details of the proofs of these results see for example [7].

It is suitable to perform calculations in the Klein model of the hyperbolic plane. Remind that in this model the points are the points of the open unit disk $\Lambda = \{z \in \mathbb{C} : |z| < 1\}$, lines are the chords and the diameters of the disk Λ. The distance between two points $A, B \in \Lambda$ is defined as one half of the logarithm of the cross ratio:

$$l_{AB} = \frac{1}{2}\ln\left(\frac{B-P}{B-Q} : \frac{A-P}{A-Q}\right), \tag{20}$$

where P and Q are the points of intersection of the line AB and the unit circle $|z| = 1$ such that the order of the points on the line AB is $PABQ$.

We need a particular case of this formula then A coincides with the origin $(0,0)$ and $B = (x,0)$ is a point on the real axis. Then $P = (-1,0), Q = (1,0)$, and

$$l_{AB} = \frac{1}{2}\ln\left(\frac{x+1}{x-1} : \frac{0+1}{0-1}\right) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad x = \frac{e^{2l}-1}{e^{2l}+1} = \tanh l. \tag{21}$$

Lemma 5. *Consider the circle C_1 of a radius r_1 with center in the origin and the circle C_2 of a radius r_2 with the center at a point $(x,0)$. Then the radical axis of the circles C_1 and C_2 is given by the equation $x = p$, where* $p = \frac{\cosh r_1 \cosh l - \cosh r_2}{\cosh r_1 \sinh l}$.

Proof. It is clear that the radical axis is perpendicular to the real axis Ox. Hence it is enough to find the point $P = (p,0)$ on the real axis such that $\deg_{C_1} P = \deg_{C_2} P$. Calculate $\deg_{C_1} P$ by formulas (21) and (19):

$$\deg_{C_1} P = \frac{e^{\frac{1}{2}\ln\left(\frac{1+p}{1-p}\right)+r_1}-1}{e^{\frac{1}{2}\ln\left(\frac{1+p}{1-p}\right)+r_1}+1} \cdot \frac{e^{\frac{1}{2}\ln\left(\frac{1+p}{1-p}\right)-r_1}-1}{e^{\frac{1}{2}\ln\left(\frac{1+p}{1-p}\right)-r_1}+1} = \frac{1-\sqrt{1-p^2}\cosh r_1}{1+\sqrt{1-p^2}\cosh r_1}. \tag{22}$$

Now calculate $\deg_{C_2} P$ by formulas (19) and (20):

$$\begin{aligned}\deg_{C_2} P &= \frac{e^{\frac{1}{2}\ln\left(\frac{1+x}{1-x}\cdot\frac{1-p}{1+p}\right)+r_2}-1}{e^{\frac{1}{2}\ln\left(\frac{1+x}{1-x}\cdot\frac{1-p}{1+p}\right)+r_2}+1}\frac{e^{\frac{1}{2}\ln\left(\frac{1+x}{1-x}\cdot\frac{1-p}{1+p}\right)-r_2}-1}{e^{\frac{1}{2}\ln\left(\frac{1+x}{1-x}\cdot\frac{1-p}{1+p}\right)-r_2}+1}\\ &= \frac{(1-xp)-\cosh r_2\sqrt{1-x^2}\sqrt{1-p^2}}{(1-xp)+\cosh r_2\sqrt{1-x^2}\sqrt{1-p^2}}.\end{aligned} \tag{23}$$

From (22) and (23) we obtain the equation on p:

$$\frac{1-\sqrt{1-p^2}\cosh r_1}{1+\sqrt{1-p^2}\cosh r_1} = \frac{(1-xp)-\cosh r_2\sqrt{1-x^2}\sqrt{1-p^2}}{(1-xp)+\cosh r_2\sqrt{1-x^2}\sqrt{1-p^2}}.$$

From this one can deduce that $p = \frac{\cosh r_1 \cosh l - \cosh r_2}{\cosh r_1 \sinh l}$. □

Theorem 2. *Assume the weights α, β and γ satisfy one of the conditions (i) all three weights are nonnegative; (ii) two weights β and γ are nonnegative, the weight α is negative, $\beta\gamma + \alpha > 0$*

(or $\beta\gamma+\alpha=0$, but $\beta\neq 1$ and $\gamma\neq 1$). *Then* (a) $\frac{\partial\theta_i}{\partial r_i}<0$ *for all* i; (b) $\frac{\partial\theta_j}{\partial r_i}>0$ *for all* $j\neq i$; (c) $\frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_i}<0$ for all i.

Proof. For (c) note that l_{jk} does not depend on r_i, while $\frac{\partial\theta_j}{\partial r_i}>0$ and $\frac{\partial\theta_k}{\partial r_i}>0$. Hence $\frac{\partial Area(\triangle A_iA_jA_k)}{\partial r_i}>0$, so

$$\frac{\partial(\theta_i+\theta_j+\theta_k)}{\partial r_i}=\frac{\partial(\pi-Area(\triangle A_iA_jA_k))}{\partial r_i}<0.$$

The statements of Lemma under assumptions (i) were proved in [6].

We address the assumptions (ii) but our arguments also apply for (i). As in proof of Lemma 1 the inequalities (a) and (b) follow from the fact that radical center Ω of the circles C_1, C_2, C_3 of radii r_1,r_2,r_3 with centers at the corresponding vertices of the triangle $\triangle A_1A_2A_3$ belongs to the interior of $\triangle A_1A_2A_3$. Hence we need to check that Ω and the vertex A_k belong to the same halfplane with respect to the line A_iA_j for $k=1,2,3$ and $\{i,j\}=\{1,2,3\}\setminus\{k\}$. We start from $k=3$. In the Klein model we consider the following configuration of points A_1, A_2, A_3. Namely, the vertex A_2 coincides with the origin, the vertex A_1 belongs to the positive direction of the Ox axis, and the vertex A_3 belongs to the upper halfplane. By Lemma 5 the radical axis of C_2 and C_1 is given by the equation $x=p$, where $p=\frac{\cosh r_2\cosh l_{12}-\cosh r_1}{\cosh r_2\sinh l_{12}}$. The angle $\angle A_1A_2A_3=\theta$ can be found from the cosine theorem: $\cos\theta=\frac{\cosh l_{23}\cosh l_{12}-\cosh l_{13}}{\sinh l_{12}\sinh l_{23}}$.

The radical axis of the circles C_2 and C_3 can be obtained by rotating the line $x=q$, where by Lemma 5

$$q=\frac{\cosh r_2\cosh l_{23}-\cosh r_3}{\cosh r_2\sinh l_{23}},$$

by angle θ counterclockwise. Hence the radical axis of the circles C_2 and C_3 is given by the equation $x\cos\theta+y\sin\theta=q$. Thus the coordinates of the radical center Ω can be found from the system of equations $x=p,\quad x\cos\theta+y\sin\theta=q$. Now it is clear that $y>0$ if and only if $q-p\cos\theta>0$.

In the inequality $q-p\cos\theta>0$ substitute for p,q and $\cos\theta$ their expressions in terms of r_i, l_{ij}:

$$\frac{\cosh r_2\cosh l_{23}-\cosh r_3}{\cosh r_2\sinh l_{23}}-\frac{\cosh r_2\cosh l_{12}-\cosh r_1}{\cosh r_2\sinh l_{12}}\frac{\cosh l_{23}\cosh l_{12}-\cosh l_{13}}{\sinh l_{12}\sinh l_{23}}>0.$$

Multiply by positive expression $\cosh r_2\sinh l_{23}\sinh^2 l_{12}=\cosh r_2\sinh l_{23}(\cosh^2 l_{12}-1)$ and reduce the inequality to $(\cosh r_2\cosh l_{23}-\cosh r_3)(\cosh^2 l_{12}-1)-(\cosh r_2\cosh l_{12}-\cosh r_1)(\cosh l_{23}\cosh l_{12}-\cosh l_{13})>0$. This one can be transformed to

$$\cosh r_1(\cosh l_{23}\cosh l_{12}-\cosh l_{13})+\cosh r_2(\cosh l_{12}\cosh l_{13}-\cosh l_{23})$$
$$+\cosh r_3(1-\cosh^2 l_{12})>0.$$

Using expressions (5) for l_{12}, l_{13}, l_{23} reduce the inequality to

$$\sinh r_1\sinh r_2((1-\gamma^2)\cosh r_3\sinh r_1\sinh r_2$$
$$+(\alpha+\beta\gamma)\cosh r_2\sinh r_1\sinh r_3+(\beta+\alpha\gamma)\cosh r_1\sinh r_2\sinh r_3)>0.$$

Under our restriction on α,β and γ all the summands are positive for all $r_1,r_2,r_3\in\mathbb{R}_+$.

The case $k=2$ is similar. To be precise one needs to interchange the indices 2 and 3 in the previous formulas.

For $k=1$ we need to check the inequality $p-q\cos\theta>0$. Substitute for p,q and $\cos\theta$ their expressions in terms of r_i and l_{ij}:

$$\frac{\cosh r_2\cosh l_{12}-\cosh r_1}{\cosh r_2\sinh l_{12}}-\frac{\cosh r_2\cosh l_{23}-\cosh r_3}{\cosh r_2\sinh l_{23}}\cdot\frac{\cosh l_{12}\cosh l_{23}-\cosh l_{13}}{\sinh l_{23}\sinh l_{12}}>0.$$

Multiply it by the positive quantity $\cosh r_2\sinh l_{12}\sinh^2 l_{23}=\cosh r_2\sinh l_{12}(\cosh^2 l_{23}-1)$ and reduce the inequality to

$$(\cosh r_2\cosh l_{12}-\cosh r_1)(\cosh^2 l_{13}-1)-(\cosh r_2\cosh l_{23}-\cosh r_3)$$
$$\times(\cosh l_{23}\cosh l_{12}-\cosh l_{13})>0.$$

This one can be transformed to the inequality

$$\cosh r_2(\cosh l_{23}\cosh l_{13} - \cosh l_{12}) + \cosh r_3(\cosh l_{12}\cosh l_{23} - \cosh l_{13}) + \cosh r_1(1 - \cosh^2 l_{23}) > 0.$$

Using expressions (5) for l_{12}, l_{13}, l_{23} reduce the inequality to

$$\sinh r_2 \sinh r_3((\alpha\gamma + \beta)\cosh r_3 \sinh r_1 \sinh r_2 + (\alpha\beta + \gamma)\cosh r_2 \sinh r_1 \sinh r_3 + (1 - \alpha^2)\cosh r_1 \sinh r_2 \sinh r_3).$$

Under our assumptions on α, β and γ all the summands are positive for all $r_1, r_2, r_3 \in \mathbb{R}_+$. □

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