ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2017, Vol. 38, No. 4, pp. 719–723. © Pleiades Publishing, Ltd., 2017.

Algebraic Sets of Universal Algebras and Algebraic Closure Operator

A. G. Pinus*

(Submitted by M. M. Arslanov)

Novosibirsk State Technical University, ul. K. Marksa 20, Novosibirsk, 630073 Russia

Received June 2, 2016

Abstract—The paper is a brief survey of the author's results connected with the lattices of algebraic sets of universal algebras and with the operator of algebraic closure on the subsets of direct powers of basic sets of algebras.

DOI: 10.1134/S1995080217040163

Keywords and phrases: *Algebraic geometry of universal algebras, algebraic sets, algebraic closure operator.*

Dedicated to Boris Isaakovich Plotkin

1. INTRODUCTION

In universal algebraic geometry some primal notions of classical algebraic geometry are applied to arbitrary universal algebras and its varieties.

The foundations of algebraic geometry of universal algebras laid down in a series of works [1–7] by B. Plotkin, V. Remeslennikov et al. They are connected with the ideas of the transfer of the basic concepts and constructions of the classical algebraic geometry (see [8], for example) to arbitrary universal algebra.

The notion of algebraic set (or, as otherwise called, algebraically closed set) in a universal algebra is one of the main concepts in the algebraic geometry of these algebras. There exist various equivalent definitions of this notion. Let us recall some of them having the model-theoretical flavor.

Let $\mathfrak{A} = \langle A; \sigma \rangle$ be a universal algebra of signature σ with the base set A. Recall that $B \subseteq A^n$, where $n \in \omega$, is called an *algebraic set* of the algebra $\mathfrak{A}$ if B is the set of solutions of some (possibly infinite) system of term equations in $\mathfrak{A}$; i.e.

$$B = \{\overline{a} \in A^n | \mathfrak{A} \models \mathcal{T}(\overline{a})\},$$

where $\mathcal{T}(\overline{x}) = \{t_i^1(\overline{x}) = t_i^2(\overline{x}) | i \in I\}$ and $t_j^i(\overline{x})$ are terms of the signature σ.

In this case, we say that B is an *n-dimensional algebraic set* of the algebra $\mathfrak{A}$. The family of all n-dimensional algebraic sets of the algebra is a complete lattice under set-theoretical inclusion $\subseteq$ denoted in this paper by $\mathrm{Alg}_n\mathfrak{A}$.

The sequence of lattices $\mathrm{Alg}\mathfrak{A} = \langle \mathrm{Alg}_n\mathfrak{A} | n \in \omega \rangle$ is called the *algebraic geometry* of the algebra $\mathfrak{A}$. We consider this sequence as a derived structure of the algebra $\mathfrak{A}$: similarly to the lattices $\mathrm{Sub}\mathfrak{A}$ (of subalgebras), $\mathrm{Con}\mathfrak{A}$ (of congruences), groups $\mathrm{Aut}\mathfrak{A}$ (of automorphisms), semigroups $\mathrm{End}\mathfrak{A}$ (of endomorphisms), $\mathrm{Iso}\mathfrak{A}$ (inner isomorphisms), $\mathrm{Ihm}\mathfrak{A}$ (inner homomorphisms), and so on.

*E-mail: ag.pinus@gmail.com

2. THE ALGEBRAIC GEOMETRY OF ALGEBRA AS ITS DERIVED STRUCTURE

Because the algebraic geometry we consider as a derived structure of algebra, then we have the problem of determines the algebra (the functional clone, respectively) by its algebraic geometry. And we have the classical problems of abstract and concrete descriptions of algebraic geometries of algebras arise.

Problem 1. *a) Let us consider a sequence of lattices* $\langle\langle D_n;\subseteq\rangle|n\in\omega\rangle$*, where* D_n *are some collections of subsets of the sets* A^n *for some set* A*. Find necessary and sufficient conditions for* $\langle\langle D_n;\subseteq\rangle|n\in\omega\rangle$ *to coincide with* $\mathrm{Alg}\mathfrak{A}$ *for some universal algebra* $\mathfrak{A}$*.*

b) Let $\langle L_n|n\in\omega\rangle$ *be a sequence of complete lattices. Find some necessary and sufficient conditions for* L_n *to be isomorphic to* $\mathrm{Alg}_n\mathfrak{A}$ $(n\in\omega)$ *for some universal algebra* $\mathfrak{A}$*.*

The following result gives a partial answer to the Problem 1b).

Theorem 1 [9]. *For any complete lattice* L *there exists a universal algebra* $\mathfrak{A}$ *such that* $\mathrm{Alg}_1\mathfrak{A}\cong L$*.*

Another natural problem regarding the algebraic sets of universal algebras is whether an algebra $\mathfrak{A}$ is characterized by its algebraic geometry $\mathrm{Alg}\mathfrak{A}$.

Notice that the lattices $\mathrm{Alg}\mathfrak{A}$ are defined through the functional clone $\mathrm{Tr}\mathfrak{A}$ of the term functions on the base set of the algebra $\mathfrak{A}$ (i.e., through the class of algebras rationally equivalent, in sense by A.I. Mal'cev, to the algebra $\mathfrak{A}$) rather than by the collection of the signature functions of the algebra $\mathfrak{A}$. From this we have the following definition [10].

Definition 1. *a) For any functional clone* F *on the set* A the algebraic geometry *of the clone* F *is the sequence of lattices* $\mathrm{Alg}F=\langle\mathrm{Alg}_nF|n\in\omega\rangle$*, for*

$$\mathrm{Alg}_nF=\langle\{B=\{\overline{a}\in A^n|t_i^1(\overline{a})=t_i^2(\overline{a}),t_i^1,t_i^2\in F^n,i\in I\};\subseteq\}\rangle,$$

where F^n *is the collection of the* n*-ary functions from* F*.*

b) Two clones F_1 *and* F_2 *on the set* A *are called algebraically equivalent* $(F_1\sim F_2)$ *if* $\mathrm{Alg}_nF_1=\mathrm{Alg}_nF_2$ *for any* $n\in\omega$*.*

Two algebras $\mathfrak{A}_1=\langle A;\sigma_1\rangle$ *and* $\mathfrak{A}_2=\langle A;\sigma_2\rangle$ *with the same base set are* geometrically equivalent *if the clones of their term functions are algebraically equivalent, i.e. if* $\mathrm{Alg}\mathfrak{A}_1=\mathrm{Alg}\mathfrak{A}_2$*.*

Notice that the algebraic geometry of any functional clone F on the set A is the algebraic geometry of the universal algebra $\mathfrak{A}_F=\langle A;F\rangle$ such that its signature functions are all the functions from F. Notice that on any set A having more than one element there exist two different clones on A which are they are algebraically equivalent.

Let us recall some results on relations between algebras (clones) having the same algebraic sets. First of all, we formulate the following necessary condition for the coincidence of the algebraic geometries of two algebras.

Theorem 2 [11]. *Let universal algebras* $\mathfrak{A}_0=\langle A;\sigma_0\rangle$ *and* $\mathfrak{A}_1=\langle A;\sigma_1\rangle$ *with the same base set* A *be such that* $\mathrm{Alg}\mathfrak{A}_0=\mathrm{Alg}\mathfrak{A}_1$ (*and* $\mathrm{Sub}\mathfrak{A}_0=\mathrm{Sub}\mathfrak{A}_1$) *then also* $\mathrm{End}\mathfrak{A}_0=\mathrm{End}\mathfrak{A}_1$ ($\mathrm{Ihm}\mathfrak{A}_0=\mathrm{Ihm}\mathfrak{A}_1$).

Recall that $L_{\omega_1\omega}$ is the generalization of the first order logic language $L_{\omega\omega}$ obtained by admitting countable conjunctions and disjunctions of sets of formulas having a common finite sets of free variables. Let $L^+_{\omega_1\omega}$ be the fragment of the language $L_{\omega_1\omega}$ consisting of the formulas without implications and negations.

Consider a formula

$$\underset{i\in I}{\&}(\Phi_i(x_1,x_2,\dots x_n))\to y=t_i(x_1,x_2,\dots,x_n),$$

where each t_i is a term and Φ_i is a finite or countable conjunctions of term equations of the signature σ. This formula said to be an *explicit* $L^+_{\omega_1\omega}$*-scheme for the algebra* $\mathfrak{A}=\langle A;\sigma\rangle$, if the formula $\forall x_1,x_2,\dots,x_n\big(\bigvee_{i\in I}\Phi_i(x_1,x_2,\dots,x_n)\big)$ and the formulas

$$\forall x_1,\dots,x_n(\Phi_i(x_1,\dots,x_n)\,\&\,\Phi_j(x_1,\dots,x_n)\to t_i(x_1,\dots,x_n)=t_j(x_1,\dots,x_n))$$

(for any $i\neq j$ in I) are true in the algebra $\mathfrak{A}$. An explicit $L^+_{\omega_1\omega}$-scheme for an algebra $\mathfrak{A}$ is called a *positive conditional term* for $\mathfrak{A}$ [12] if the set I is finite and the formulas Φ_i are finite conjunctions of term equations.

A function $f(x_1, x_2, \dots, x_n)$ on the basic set A of an algebra $\mathfrak{A} = \langle A; \sigma\rangle$ is $L^+_{\omega_1\omega}$-*definable* (*respectively: definable by explicit* $L^+_{\omega_1\omega}$-*scheme*) if there exists an $L^+_{\omega_1\omega}$-formula $\Phi(x_1, x_2, \dots, x_n)$ (respectively: there exists an explicit $L^+_{\omega_1\omega}$-scheme Φ) for the algebra $\mathfrak{A}$ such that $f(a_1, a_2, \dots, a_n) = b \Leftrightarrow \mathfrak{A} \models \Phi(a_1, a_2, \dots, a_n, b)$ for any $a_1, a_2, \dots, a_n, b \in A$.

The following propositions follows from Theorem 2.

Corollary 1 [11]. *Suppose that algebras* $\mathfrak{A}_0 = \langle A; \sigma_0\rangle$ *and* $\mathfrak{A}_1 = \langle A; \sigma_1\rangle$ *with the same base set are at most countable and their signatures are also at most countable. If these algebras have the same algebraic geometry* ($\mathrm{Alg}\mathfrak{A}_0 = \mathrm{Alg}\mathfrak{A}_1$) *then* σ_i*-functions are* $L^+_{\omega_1\omega}$*-definable on the algebra* $\mathfrak{A}_{1-i}$ *for* $i = 0, 1$.

Corollary 2 [11]. *Suppose that algebras* $\mathfrak{A}_0 = \langle A; \sigma_0\rangle$ *and* $\mathfrak{A}_1 = \langle A; \sigma_1\rangle$ *with the same base set are at most countable and their signatures are also at most countable. If these algebras have the same algebraic geometry (i.e.* $\mathrm{Alg}\mathfrak{A}_0 = \mathrm{Alg}\mathfrak{A}_1$ *and the same subalgebras lattices (i.e.* $\mathrm{Sub}\mathfrak{A}_0 = \mathrm{Sub}\mathfrak{A}_1$) *then* σ_i*-functions are definable in the algebra* $\mathfrak{A}_{1-i}$ *by explicit* $L^+_{\omega_1\omega}$*-schemes, for* $i = 0, 1$. A universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$ is called *additive* (respectively: an *equitional domain*), if any union (respectively: any finite union) of its n-dimensional algebraic sets is also its algebraic set. For additive algebras Corollary 2 can be specified as follows.

Corollary 3 [13]. *Let* $\mathfrak{A}_0 = \langle A; \sigma_0\rangle$ *be an additive universal algebra. Then for any algebra* $\mathfrak{A}_1 = \langle A; \sigma_1\rangle$ *such that* $\mathrm{Sub}\mathfrak{A}_1 = \mathrm{Sub}\mathfrak{A}_0$ *the following conditions are equivalent: a)* $\mathrm{Alg}\mathfrak{A}_1 = \mathrm{Alg}\mathfrak{A}_0$; *b) for* $i = 0, 1$ *any* σ_i*-function is defined in the algebra* $\mathfrak{A}_{1-i}$ *by some explicit* $L^+_{\omega_1\omega}$*-scheme.*

Corollary 4 [13]. *Let* $\mathfrak{A}_0 = \langle A; \sigma_0\rangle$ *be an infinitely generated free algebra. Then for any universal algebra* $\mathfrak{A}_1 = \langle A; \sigma_1\rangle$ *such that* $\mathrm{Sub}\mathfrak{A}_1 = \mathrm{Sub}\mathfrak{A}_0$ *and* $\mathrm{Alg}\mathfrak{A}_1 = \mathrm{Alg}\mathfrak{A}_0$ *the algebras* $\mathfrak{A}_0$ *and* $\mathfrak{A}_1$ *are rationally equivalent.*

Corollary 5 [13]. *Let* $\mathfrak{A}_0 = \langle A; \sigma_0\rangle$ *be a finite or uniformly locally finite (of finite signature) algebra which is an equational domain. Then for any algebra* $\mathfrak{A}_1 = \langle A; \sigma_1\rangle$ *(of finite signature* σ_1 *if* A *is infinite) such that* $\mathrm{Sub}\mathfrak{A}_1 = \mathrm{Sub}\mathfrak{A}_0$ *the following conditions are equivalent: a)* $\mathrm{Alg}\mathfrak{A}_1 = \mathrm{Alg}\mathfrak{A}_0$; *b) for* $i = 0, 1$ *any* σ_i*-function is a positive conditional term function on the algebra* $\mathfrak{A}_{1-i}$. *A functional clone* F *on a set* A *is called equationally additive* if the algebra $\mathfrak{A}_F = \langle A; F\rangle$ is additive.

Corollary 6 [14]. *For every finite* A *cardinality of algebraically non-equivalent equationally additive clones on* A *is finite.*

3. THE ALGEBRAIC CLOSURE OPERATOR ON SUBSETS OF UNIVERSAL ALGEBRAS

The concept of algebraic set for a universal algebra $\mathfrak{A}$ leads to the concept of *algebraic closure operator* on a subset of direct powers of the set A.

Given $B \subseteq A^n$ denote $\overline{B}_{\mathfrak{A}}$ the *algebraic closure* of B in $\mathfrak{A}$, i.e. the least n-dimensional algebraic set in $\mathfrak{A}$ such that this set includes B. Thus, $\overline{c} \in \overline{B}_{\mathfrak{A}}$ for $\overline{c} \in A^n$ iff, for any termal equation $t^1(\overline{x}) = t^2(\overline{x})$ such that any $\overline{b} \in B$ is its root, we have $\mathfrak{A} \models t^1(\overline{c}) = t^2(\overline{c})$. Obviously, the operator $B \to \overline{B}_{\mathfrak{A}}$ is a closure operator on a subset of the set A^n. In Theorem 2 we have mentioned some relations between the collection of algebraic sets of an algebra $\mathfrak{A}$ and the semigroups $\mathrm{End}\mathfrak{A}$ and $\mathrm{Ihm}\mathfrak{A}$.

Let us give a description of the algebraic closure operators $B \to \overline{B}_{\mathfrak{A}}$ on universal algebras in this terminology. First of all, we recall the definition of the quasiorder relation $\leqslant_{\mathrm{Ihm}_{\mathfrak{A}}}$ on the base set A of a universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$ (see [15]). One defines $a \leqslant_{\mathrm{Ihm}_{\mathfrak{A}}} b$ for $a, b \in A$ iff there exist $\varphi \in \mathrm{Ihm}\mathfrak{A}$ such that $\varphi(b) = a$, that is if and only if there exists an inner homomorphism φ of the algebra $\mathfrak{A}$ such that this homomorphism maps the algebra $\langle b\rangle_{\mathfrak{A}}$ onto the algebra $\langle a\rangle_{\mathfrak{A}}$ and $\varphi(b) = a$. Henceforth, for $C \subseteq A$ the subalgebra of the algebra $\mathfrak{A}$ generated by the subset C of A is denoted by $\langle C\rangle_{\mathfrak{A}}$ and for $a \in A$ the algebra $\langle a\rangle_{\mathfrak{A}}$ denotes with $\langle\{a\}\rangle_{\mathfrak{A}}$.

Let us show that algebraic closure operator $B \to \overline{B}_{\mathfrak{A}}$ can be described using this quasiorder on some extensions of the algebra $\mathfrak{A}$. We also define some series of quasiorders on basic sets of direct powers of the algebra $\mathfrak{A} = \langle A; \sigma\rangle$. For any algebra $\mathfrak{A} = \langle A; \sigma\rangle$ and any natural n we define the quasiorder $\leqslant_{\mathrm{Ihm}_n\mathfrak{A}}$ on A^n as follows. One defines $\overline{b} \leqslant_{\mathrm{Ihm}_n\mathfrak{A}} \overline{a}|$ for $\overline{b} = \langle b_1, b_2, \dots, b_n\rangle, \overline{a} = \langle a_1, a_2, \dots, a_n\rangle \in A^n$ iff there exists an inner homomorphism φ of the algebra $\mathfrak{A}$ such that this homomorphism maps the algebra $\langle\{b_1, b_2, \dots, b_n\}\rangle_{\mathfrak{A}}$ onto the algebra $\langle\{a_1, a_2, \dots, a_n\}\rangle_{\mathfrak{A}}$ in such a way that $\varphi(a_i) = b_i$ for $i \leqslant n$.

Let $\overline{a}$ be a tuple of elements of the algebra $\mathfrak{A}$. By $D^{+}_{\mathfrak{A},\overline{a}}(\overline{x})$ denote the positive diagram of the tuple $\overline{a}$ in $\mathfrak{A}$, i.e. the collection of all term equations which they are true on the tuple $\overline{a}$ in the algebra $\mathfrak{A}$.

Obviously the following lemma holds.

Lemma. *For $\overline{a}, \overline{b} \in A^n$ the following conditions are equivalent: 1) $\overline{b} \leqslant_{\mathrm{Ihm}_n \mathfrak{A}} \overline{a}$; 2) $\mathfrak{A} \models D^{+}_{\mathfrak{A},\overline{a}}(\overline{b})$; 3) $\overline{b} \in \overline{\{\overline{a}\}}_{\mathfrak{A}}$.*

An algebra $\mathfrak{A} = \langle A; \sigma\rangle$ is *algebraically complete*, if for any $B \subseteq A^n$ there exists an element $\overline{m}_B \in A^n$ such that $\overline{B}_{\mathfrak{A}} = \overline{\{\overline{a}\}}_{\mathfrak{A}}$. An algebra $\mathfrak{A}' = \langle A'; \sigma\rangle$ is called an *n-algebraic completion* of the algebra $\mathfrak{A}$, if $\mathfrak{A}'$ is an extension of the algebra $\mathfrak{A}$, $\mathfrak{A}'$ is algebraically complete, and for any $\overline{c} \in (A')^n$ there exists $B \subseteq A^n$such that $\overline{c} = \overline{m}_B$.

So, if $\mathfrak{A}' = \langle A'; \sigma\rangle$ is some n-algebraic completion of the algebra $\mathfrak{A} = \langle A; \sigma\rangle$, then $\mathrm{Alg}_n\mathfrak{A} = \{A^n \cap B|$ B is some principal ideal in the quasiordered set $\langle (A')^n; \leqslant_{\mathrm{Ihm}_n\mathfrak{A}'} \rangle \}$. One of the examples of an n-algebraic completion of the algebra $\mathfrak{A} = \langle A; \sigma\rangle$ is the algebra $(\mathfrak{A}^n)^{A^n}$.

We have the following assertion.

Theorem 3 [16]. *Let $\mathfrak{A} = \langle A; \sigma\rangle$ be an algebra satisfying the following conditions: 1) its cardinality $\aleph$ is not less than continuum ($\aleph \geqslant 2^{\aleph_0}$); 2) its signature is at most countable. Then for any natural n there exists an n-algebraic completion $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}'$ has the same cardinality $\aleph$.*

Notice, that the restriction $\aleph \geqslant 2^{\aleph_0}$ in this theorem is essential.

Let us give some criterion of n-algebraical completeness of a universal algebra. Let $\mathcal{K}$ be a class of n-generated algebras of a signature σ with some fixed n-tuple $\langle c_1, c_2, \ldots, c_n\rangle$ of generators of such algebra. By fixing n-tuple, one can consider the σ-algebras as algebras in the enriched signature $\sigma' = \sigma \cup \langle c_1, c_2, \ldots, c_n\rangle$. The *pseudo $\mathcal{K}$-direct product of algebras* $\mathfrak{A}_i = \langle A_i, \sigma_i\rangle$ in $\mathcal{K}$ $(i \in I)$ is the σ'-algebra $\mathfrak{A} = \langle A; \sigma'\rangle$ in $\mathcal{K}$. We denote this as $(\mathfrak{A} = \underset{i\in I}{P}{}_{\mathcal{K}}\mathfrak{A}_i)$ such that the following conditions hold:

(1) There exist homomorphisms π_i from the algebra $\mathfrak{A}$ onto the algebras $\mathfrak{A}_i$ for $i \in I$.

(2) For any algebra $\mathfrak{B}$ in $\mathcal{K}$ and any homomorphisms φ_i from $\mathfrak{B}$ onto $\mathfrak{A}_i$, where $i \in I$ there exists a homomorphism ψ from $\mathfrak{B}$ onto $\mathfrak{A}$ such that $\pi_i\psi = \phi_i$.

All these homomorphisms are homomorphisms in the signature σ'.

For an algebra $\mathfrak{A} = \langle A; \sigma\rangle$ let σ' be an enrichment of σ by fixing of its generators. By $\mathrm{Sub}_{ng}\mathfrak{A}$ denote the class of all n-generated subalgebras of $\mathfrak{A}$ in the signature σ'. We have the following criterion of n-algebraical completeness of universal algebras.

Theorem 4 [16]. *The algebra $\mathfrak{A}$ is n-algebraically complete if and only if the class $\mathrm{Sub}_{ng}\mathfrak{A}$ is closed which relative to pseudodirect products.*

Let us move to a study of $\leqslant_{\mathrm{Ihm}}$ on $\mathfrak{A}$ itself.

Definition 2. *The quasiorder $\leqslant$ on a set A is called* Ihm-*admissible* (Ihm-*forbidden, respectively) if there exists a universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$ such that the quasiorder $\leqslant$ coincides with the quasiorder $\leqslant_{\mathrm{Ihm}\mathfrak{A}}$ (if for any algebra $\mathfrak{A} = \langle A; \sigma\rangle$ its quasiorder $\leqslant_{\mathrm{Ihm}\mathfrak{A}}$ differs from the quasiorder $\leqslant$, respectively).*

The quasiorder set $\langle A; \leqslant_{\mathrm{Ihm}\mathfrak{A}}\rangle$ is a derived structure for the algebra $\mathfrak{A} = \langle A; \sigma\rangle$. So, natural problems of concrete and abstract descriptions of such quasiorders arise. The first problem is if there exist Ihm-admissible and Ihm-forbidden quasiorders on various sets A. In the case of Ihm-admissible quasiorders for any set this problem can be solved trivially while the case of Ihm-forbidden quasiorders is much more complicated. We formulate here some results for this problem.

First of all, let us observe that any quasiorder on a set of cardinality at most three is Ihm-admissible.

Theorem 5 [17]. *On any set of cardinality at least four there exists an* Ihm-*forbidden quasiorder..*

Theorem 6 [17]. *a) For any lower semilattice $\langle A; \leqslant\rangle$ the quasiorder $\leqslant$ is* Ihm-*admissible; b) Any linear quasiorder is* Ihm-*admissible; c) The direct product of* Ihm-*admissible sets is an* Ihm-*admissible quasiordered set; d) The class of* Ihm-*admissible quasiordered sets is closed under ultraproducts.*

Corollary 7 [17]. *Any quasiorder can be embedded in some* Ihm-*admissible quasiorder. Let us formulate some results on relations among lattices* $\mathrm{Alg}_n\mathfrak{A}$, $\mathrm{Sub}\mathfrak{A}$, *the semigroup* $\mathrm{Ihm}\mathfrak{A}$, *and other derived structures of universal algebras.*

Proposition 1 [18]. *a) There exist universal algebras* $\mathfrak{A}_1 = \langle A; \sigma_1 \rangle$ *and* $\mathfrak{A}_2 = \langle A; \sigma_2 \rangle$ *with the same basic set* A *such that the quasiorders* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_1}$ *and* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_2}$ *coincide but the lattices* $\mathrm{Alg}_1\mathfrak{A}_1$ *and* $\mathrm{Alg}_1\mathfrak{A}_2$ *are different.*

b) There exist universal algebras $\mathfrak{A}_1 = \langle A; \sigma_1 \rangle$ *and* $\mathfrak{A}_2 = \langle A; \sigma_2 \rangle$ *with the same basic set* A *such that the quasiorders* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_1}$ *and* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_2}$ *coincide but the* lattices $\mathrm{Sub}\mathfrak{A}_1$ *and* $\mathrm{Sub}\mathfrak{A}_2$ *are different.*

c) There exist universal algebras $\mathfrak{A}_1 = \langle A; \sigma_1 \rangle$ *and* $\mathfrak{A}_2 = \langle A; \sigma_2 \rangle$ *with the same basic set* A *such that the quasiorders* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_1}$ *and* $\leqslant_{\mathrm{Ihm}\mathfrak{A}_2}$ *coincide, the lattices* $\mathrm{Sub}\mathfrak{A}_1$ *and* $\mathrm{Sub}\mathfrak{A}_2$ *coincide but the semigroups* $\mathrm{Ihm}\mathfrak{A}_1$ *and* $\mathrm{Ihm}\mathfrak{A}_2$ *are different.*

I am grateful to the referee for his numerous comments on the text of my work.

ACKNOWLEDGMENTS

Supported by the Russian Ministry of Education and Science (contract 2014/138, project no. 1052).

REFERENCES

1. B. Plotkin, "Varieties of algebras and algebraic varieties. Categories of algebraic varieties," Mat. Tr. **7** (2), 64–97 (1997).
2. B. Plotkin,"Seven lectures in universal algebraic geometry," arXiv: math.RA/0502212-2002.
3. B. Plotkin, "Some results and problems related to universal algebraic geometry," Int. J. Algebra Comput. **17**, 1133–1164 (2007).
4. B. Plotkin and E. Plotkin, "Multi-sorted logic and logical geometry: some problems," Demonstr. Math. **48**, 578–619 (2015).
5. G. Baumslag, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over groups I. Algebraic sets and ideal theory," J. Algebra **219**, 16–79 (1999).
6. E. Yu. Danijarova, A. G. Myasnikov, and V. N. Remeslennikov, "Algebraic geometry over algebraic structures II. Foundations," Fund. Appl. Math. **17**, 65–106 (2011–2012).
7. E. Yu. Danijarova, A. G. Myasnikov, and V. N. Remeslennikov, "Algebraic geometry over algebraic structures. V. The case of arbitrary signature," Algebra Logic **51**, 28–40 (2012).
8. I. Shafarevich, *The Foundations of Algebraic Geometry* (Nauka, Moskow, 1972) [in Russian].
9. A. G. Pinus, "On the lattices of algebraic subsets of universal algebras," in *Algebra and Model Theory 8, Collection of Papers* (Novosib. Gos. Technol. Univ., Novosibirsk, 2011), pp. 67–72 [in Russian].
10. A. G. Pinus, "The dimensions of functional clons and the metric on the family of this clons," Sib. Elektron. Mat. Izv. **13**, 355–374 (2016).
11. A. G. Pinus, "On the universal algebras with identical derived objects (congruences, algebraic sets)," Sib. Elektron. Mat. Izv. **11**, 752–758 (2014).
12. A. G. Pinus, "Inner homomorphisms and positive-conditional terms," Algebra Logic **40**, 87–95 (2001).
13. A. G. Pinus, "Algebras with identical algebraic sets," Algebra Logic **54**, 316–322 (2015).
14. A. G. Pinus, "Algebraically equivalent clones," Algebra Logic **55**, 501–506 (2017).
15. A. G. Pinus, "On the quasiorder induced by inner homomorphisms and operator of algebraic closure," Sib. Math. J. **56**, 499–504 (2015).
16. A. G. Pinus, "n-algebraically complete algebras, pseudodirect product and operator of algebraic closure on subsets of universal algebras," Sib. Zh. Chist. Prikl. Mat. **16** (4), 97–102 (2016).
17. A. G. Pinus, "On Ihm-admissible and Ihm-forbidden quasiorders on sets," Sib. Math. J. **57**, 1109–1113 (2016).
18. A. G. Pinus, "Ihm-quasiorders and derived structures of universal algebras," Tr. Irkutsk Univ., Ser. Mat. **12**, 72–78 (2015).