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On Positive Definiteness of Some Radial Functions

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Abstract—We consider the functions $h_{\mu,\nu}$ introduced by Zastavnyi in 2002. The family of these functions is a subfamily of Buhmann's functions and contains the families of functions introduced by Trigub in 1987 and Wendland in 1995. We investigate the problems of positive definiteness and smoothness at zero for the linear combinations $\beta_2^{\varepsilon} h_{\mu,\nu}(x/\beta_2) - \beta_1^{\varepsilon} h_{\mu,\nu}(x/\beta_1)$.

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1. INTRODUCTION. FORMULATION OF THE RESULTS

A complex-valued function $f : \mathbb{R}^m \to \mathbb{C}$ is said to be positive definite on $\mathbb{R}^m$ ($f \in \Phi(\mathbb{R}^m)$) if the inequality $\sum\limits_{k,j=1}^{n} c_k \overline{c_j} f(x_k - x_j) \geqslant 0$ is satisfied for any finite systems of complex numbers $\{c_k\}_{k=1}^n \subset \mathbb{C}$ and points $\{x_k\}_{k=1}^n \subset \mathbb{R}^m$. A real-valued and continuous function φ on $[0, +\infty)$ is called a radial positive definite function on $\mathbb{R}^m$, if $\varphi(||x||)$ is positive definite on $\mathbb{R}^m$ (we denote this class by Φ_m). We stick to the standard notation for the inner product $(u, v) = u_1 v_1 + \ldots + u_m v_m$ of two vectors $u = (u_1, \ldots, u_m)$ and $u = (v_1, \ldots, v_m)$ in $\mathbb{R}^m$, and $||u|| = \sqrt{(u, v)}$ for the Euclidean norm of u.

In this paper, we study the problems of positive definiteness and smoothness at zero for the linear combinations of functions $h_{\mu,\nu}$ being introduced by Zastavnyi (2002) [34, 33] and defined as follows: $h_{\mu,\nu}(x) := 0$ for $|x| \geq 1$ and

$$h_{\mu,\nu}(x) := \int\limits_{|x|}^{1} (2u - |x|) g_{\mu,\nu}(u) g_{\mu,\nu}(u - |x|) du, \quad |x| < 1, \tag{1}$$

where $g_{\mu,\nu}(u) := u^{\mu-1}(1 - u^2)^{\nu-1}$, $u \in (0, 1)$, $\mu, \nu \in \mathbb{C}_+$, and $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The family of these functions is a subfamily of Buhmann's functions (2000) and contains the families of functions introduced by Trigub (1987) and Wendland (1995) (for more details see section 2). Functions of the form (1) arise in the study of exponential type entire functions without zeros in the lower half-plane [36, Proposition 5.1].

Problem 1. *Let $\mu > 0$, $\nu > 1/2$, $\mu + \nu > 1$. Then $h_{\mu,\nu} \in C(\mathbb{R})$ (see [34, 33]). Let $\varepsilon > 0$ and*

$$f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}(x) := \beta_2^{\varepsilon} h_{\mu,\nu}\left(\frac{x}{\beta_2}\right) - \beta_1^{\varepsilon} h_{\mu,\nu}\left(\frac{x}{\beta_1}\right), \quad x \in \mathbb{R}. \tag{2}$$

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Let $m \in \mathbb{N}$. Show the conditions on (ε, μ, ν) such that, for any $\beta_2 > \beta_1 > 0$, we have

$$f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in \Phi_m. \tag{3}$$

A similar problem for the function $h_{1,1}(x) = (1 - |x|)_+$ was considered by Ramachandran [17] and Gneiting [11, p. 362].

The main results of our paper are as follows.

Theorem 1. *The following assertions are true:*

1. *If (3) is true, then $\varepsilon \geq 2\nu - 1$.*

2. *If $m \in \mathbb{N}$, $\nu > \frac{1}{2}$, $\varepsilon \geq 2\nu - 1$ and $\mu \geq (m-1)/2 + \nu + 3$, then condition (3) is true. If, in addition, $\varepsilon = 2\nu - 1$, then (3) is true if and only if $\mu \geq (m-1)/2 + \nu + 3$.*

3. *Suppose that for some $n = 1, 2, 3$, we have $\varepsilon \geq 2^{1-n}(m + (2\nu - 1)(2^{n-1} + 1))$, $(m-1)/2 + \nu + 1 - n > 0$ and*

$$\mu - n \geq \min\left\{m - 1 + 2\nu + 2 - 2n; \max\{1; \frac{m-1}{2} + \nu + 1 - n\}\right\}.$$

Then, condition (3) is true.

Theorem 2. *Let $\nu \in \mathbb{N}$, $\mu > 0$, $\varepsilon \in \mathbb{R}$, $\beta_2, \beta_1 > 0$, and $\beta_2 \neq \beta_1$. Let $q := \min(\beta_1, \beta_2)$. Then,*

1. *If $\varepsilon \neq 2\nu - 1$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^{2\nu-2}(-q,q)$, and $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \notin C^{2\nu-1}(-q,q)$.*

2. *If $\varepsilon = 2\nu - 1$, $\mu \notin \{1, 2\}$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^{2\nu}(-q,q)$, and $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \notin C^{2\nu+1}(-q,q)$.*

3. *If $\varepsilon = 2\nu - 1$, $\mu = 1$ or $\mu = 2$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}$ is a even polynomial of degree at most $\mu + 2\nu - 2$ on $[-q,q]$, and therefore $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^{\infty}(-q,q)$.*

The section 5 is devoted to a discussion on the applications of positive definite functions to spatial statistics.

2. BUHMANN FUNCTIONS. AUXILIARY FACTS AND ASSERTIONS

We denote $C(\mathbb{R}^m)$ the set of continuous functions on $\mathbb{R}^m$, for $m = 1, 2, \ldots$. Let $\delta, \mu, \nu \in \mathbb{C}_+$ and $\alpha \in \mathbb{C}$. Zastavnyi (2006) [35] proposed the following even functions given on $\mathbb{R}$:

$$\varphi_{\delta,\mu,\nu,\alpha}(x) := \begin{cases} \int\limits_{|x|}^{1} (s^2 - x^2)^{\nu-1}(1 - s^{\delta})^{\mu-1} s^{\alpha-2\nu+1} \mathrm{d}s, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{4}$$

If $\delta, \mu, \nu \in \mathbb{C}_+$, then $\varphi_{\delta,\mu,\nu,\alpha} \in C(-1,1) \iff \alpha \in \mathbb{C}_+$ (see [35, Proposition 1]) and $\varphi_{\delta,\mu,\nu,\alpha} \in C(\mathbb{R}) \iff \alpha, \mu + \nu - 1 \in \mathbb{C}_+$ (see [35, Theorem 1]). If $\delta, \mu, \nu, \alpha \in \mathbb{C}_+$, then $\varphi_{\delta,\mu,\nu,\alpha}(0) = B(\alpha/\delta, \mu)/\delta$.

The functions $\varphi_{\delta,\mu,\nu,\alpha}$ coincide (modulo some positive factors) with the functions

$$\phi_{\delta,\varrho,\lambda,\alpha}(x) \equiv 2\varphi_{2\delta,\varrho+1,\lambda+1,2\alpha+2}(x), \quad x \in \mathbb{R},$$

introduced by Martin Buhmann (2000) [5]. We thus term them Buhmann functions throughout. The class $\varphi_{\delta,\mu,\nu,\alpha}$ includes a wealth of interesting special cases. For instance, $\mu\delta\varphi_{\delta,\mu,1,\delta}(x) = (1 - |x|^{\delta})_+^{\mu}$ and

$$\varphi_{1,\mu,\nu,2\nu-1}(x) \equiv h_{\mu,\nu}(x) \equiv \frac{2^{\nu-1}\Gamma(\nu)}{\mu}\psi_{\mu,\nu-1}(x), \quad x \in \mathbb{R}, \tag{5}$$

with the functions $h_{\mu,\nu}$ (see (1)) being introduced by Zastavnyi (2002) [34, 33]. The functions $\psi_{\mu,\nu-1}$, with $\mu > 0$, $\nu \in \mathbb{N}$, have been introduced by Wendland (1995) [29]: for $\mu > 0, k \in \mathbb{Z}_+$, we have

$$\psi_{\mu,0}(x) := \psi_{\mu}(x) := (1 - |x|)_+^{\mu}, \quad \psi_{\mu,k} := I^k \psi_{\mu} \quad (k \in \mathbb{N}),$$

where $I(f)(x) := \int_{|x|}^{+\infty} s f(s) \mathrm{d}s$ is the Matheron's [13] Montée operator (provided the integral is well defined), and where I^k is the k-fold application of the operator I. Arguments in [29] and subsequently [12] show that $I\varphi$ belongs to the class Φ_{m-2} whenever $\varphi \in \Phi_m$, for $m \geq 3$. For $k < 2m$, the k-fold application of the Montée operators shows that $I^k\varphi \in \Phi_{m-2k}$, $k \in \mathbb{N}$.

Gneiting (2002) [12, Equation (17)] has proposed a generalizion of Wendland functions on the basis of the fractional Montée operator, which coincide with the normalized Buhmann functions $\varphi_{1,\mu+1,\nu,2\nu}(x)/\varphi_{1,\mu+1,\nu,2\nu}(0)$, $\mu,\nu>0$, and coincide with the functions $h_{\mu,\nu+1}(x)/h_{\mu,\nu+1}(0)\equiv\psi_{\mu,\nu}(x)/\psi_{\mu,\nu}(0)$ (see Equation (7)). Arguments in [5] show that Wu functions [31] and consequently the spherical model are a special case of the Buhmann class.

For $r\in\mathbb{Z}_+$ and $k\in\mathbb{N}$, we have $h_{r+k,r+1}(x)\equiv B(r+k,2r+1)A_{r,2k-1}(x)$, with the splines $A_{r,2k-1}$ introduced by Trigub (1987), and we refer to [24], [25, § 6.2.13, 6.2.16, 6.3.12], [26, § 6] for their analytical expression which is not reported here. Equation above in turn highlights the explicit connection between Trigub splines and Wendland functions: $A_{r,2k-1}(x)\equiv\psi_{r+k,r}(x)/\psi_{r+k,r}(0)$, for $r\in\mathbb{Z}_+$ and $k\in\mathbb{N}$.

For a proof of the identities above, the reader is referred to Zastavnyi and Trigub [34, Remarks 10 and 11], to [33, Theorems 12 and 13], [35] and [37, § 4.7].

Arguments in Proposition 4 of [35] show that, for $\delta,\mu,\nu\in\mathbb{C}_+$ and $x\in\mathbb{R}$,

$$\varphi_{2,\frac{\mu}{2},\frac{\mu}{2}+\nu,2\nu-1}(x)\equiv\frac{2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu}{2}+\nu\right)}{\Gamma(\mu)}\varphi_{1,\mu,\nu,2\nu-1}(x),$$
$$2\nu\varphi_{\delta,\mu+1,\nu,2\nu}(x)\equiv\delta\mu\varphi_{\delta,\mu,\nu+1,2\nu+\delta}(x),\tag{6}$$

and, for $\mu,\nu\in\mathbb{C}_+$ and $x\in\mathbb{R}$, we also have the obvious identities:

$$\varphi_{1,\mu+1,\nu,2\nu}(x)\equiv\frac{\mu}{2\nu}\varphi_{1,\mu,\nu+1,2\nu+1}(x)\equiv\frac{\mu}{2\nu}h_{\mu,\nu+1}(x)\equiv2^{\nu-1}\Gamma(\nu)\psi_{\mu,\nu}(x).\tag{7}$$

For a function h defined on $(0,\infty)$ and $m\in\mathbb{C}$, we define the Hankel transform $\mathfrak{F}_m$ as follows:

$$\mathfrak{F}_m(h)(t):=t^{1-\frac{m}{2}}\int_0^\infty h(u)u^{\frac{m}{2}}J_{\frac{m}{2}-1}(tu)\mathrm{d}u=\int_0^\infty h(u)u^{m-1}j_{\frac{m}{2}-1}(tu)\mathrm{d}u,\quad t>0,\tag{8}$$

where J_λ is the Bessel function of the first kind (see [27, Sec. 3.1]) and

$$j_\lambda(x):=\frac{J_\lambda(x)}{x^\lambda}=\frac{1}{2^\lambda}\sum_{k=0}^\infty\frac{1}{\Gamma(k+\lambda+1)}\cdot\frac{\left(-\frac{x^2}{4}\right)^k}{k!},\quad x\in\mathbb{C},\quad\lambda\in\mathbb{C}.\tag{9}$$

Remark 1. For functions f defined on $\mathbb{R}^m$, $m\in\mathbb{N}$, the Fourier transform is given by the formula

$$F_m(f)(x):=\int_{\mathbb{R}^m}f(u)e^{-i(u,x)}du,\quad x\in\mathbb{R}^m.$$

For $m\in\mathbb{N}$ the transform $\mathfrak{F}_m$ is connected with the Fourier transforms F_m of radial functions through the identity

$$F_m(h(||\cdot||))(x)=(2\pi)^{\frac{m}{2}}\mathfrak{F}_m(h)(||x||),\quad x\in\mathbb{R}^m.$$

These facts and Bochner–Khintchine theorem (see, for example [18, 20, 25]) imply that, *if h is a continuous functions on $[0,\infty)$ and $\int_0^\infty t^{m-1}|h(t)|dt<\infty$, then $h\in\Phi_m$ if and only if $\mathfrak{F}_m(h)$ is nonnegative on the positive real line.*

For $\delta,\mu,\alpha+1\in\mathbb{C}_+$ and $\nu\in\mathbb{C}$, we define the function $I_{\delta,\mu,\nu,\alpha}:\mathbb{R}_+\to\mathbb{C}$ through

$$I_{\delta,\mu,\nu,\alpha}(t):=t^{-\alpha-1-\delta(\mu-1)}\int_0^t(t^\delta-u^\delta)^{\mu-1}u^{\alpha-\nu+\frac{1}{2}}J_{\nu-\frac{1}{2}}(u)\mathrm{d}u$$
$$=\int_0^1(1-x^\delta)^{\mu-1}x^\alpha j_{\nu-\frac{1}{2}}(tx)\mathrm{d}x,\quad t>0.\tag{10}$$

The following result reports succinctly a collection of useful results from [35] (Theorems 2, 3, Proposition 4 (Assertions 1, 3) and Proposition 6).

Theorem 3 (Zastavnyi [35]). *Let the functions I and $\mathfrak{F}_m$ as being defined through Equations (10) and (8), respectively. Denote with I' the first derivative of I. Then, the following assertions are true:*

1. *Let $\delta, \mu, \nu, m, \alpha + m \in \mathbb{C}_+$. Then $\mathfrak{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(t) = 2^{\nu-1}\Gamma(\nu) I_{\delta,\mu,\frac{m-1}{2}+\nu,m-1+\alpha}(t)$. Moreover, if $n, m - n + 2\nu \in \mathbb{C}_+$, then*

$$\mathfrak{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(t) = \frac{2^{\frac{n-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-n}{2}+\nu)}\mathfrak{F}_n(\varphi_{\delta,\mu,\frac{m-n}{2}+\nu,m-n+\alpha})(t), \quad t \geq 0.$$

2. *Let $\delta, \mu, \alpha + 1 \in \mathbb{C}_+$ and $\nu \in \mathbb{C}$. Then*

$$I'_{\delta,\mu,\nu,\alpha}(t) = -tI_{\delta,\mu,\nu+1,\alpha+2}(t), \quad t > 0. \tag{11}$$

3. *If $\mu, \nu \in \mathbb{C}_+$, $t > 0$, then*

$$I_{1,\mu,\nu,2\nu-1}(t) = \frac{2^{\frac{1}{2}-\nu}\Gamma(\mu)\Gamma(2\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\mu+2\nu)}\,{}_1F_2\left(\nu; \frac{\mu+2\nu}{2}, \frac{\mu+2\nu+1}{2}; -\frac{t^2}{4}\right). \tag{12}$$

4. *Let $\delta, \mu, \nu \in \mathbb{C}_+$ and $\alpha \in \mathbb{C}$. Then*

$$\varphi_{\delta,\mu,\nu+1,\alpha+2}(t) = 2\nu\int_{|t|}^{\infty} u\varphi_{\delta,\mu,\nu,\alpha}(u)\mathrm{d}u, \quad t \neq 0. \tag{13}$$

5. *Let $\delta, \mu, \nu + \frac{1}{2}, \alpha + 1 > 0$ and let $L \in \mathbb{Z}_+$. Then the following equality holds as $t \to +\infty$:*

$$\frac{\delta}{2^{\frac{1}{2}-\nu}\Gamma(\mu)} \cdot I_{\delta,\mu,\nu,\alpha}(t) = \frac{2^\nu\delta^\mu}{\sqrt{\pi}t^{\mu+\nu}}\left\{\cos\left(t - \frac{\pi}{2}(\mu+\nu)\right) + O\left(\frac{1}{t}\right)\right\}$$
$$+\sum_{l=0}^{L}\frac{\delta(-1)^l}{2}\frac{\Gamma((\alpha+1+\delta l)/2)}{\Gamma(\nu-(\alpha+\delta l)/2)\Gamma(\mu-l)\Gamma(l+1)} \cdot \left(\frac{2}{t}\right)^{\alpha+1+\delta l} + o\left(\frac{1}{t^{\alpha+1+\delta L}}\right).$$

Another remarkable consequence of Theorem 3 is that, for $\mu, \nu, m, 2\nu - 1 + m \in \mathbb{C}_+$,

$$\mathfrak{F}_m(h_{\mu,\nu})(t) = \mathfrak{F}_m(\varphi_{1,\mu,\nu,2\nu-1})(t) = \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)}\mathfrak{F}_1(h_{\mu,\frac{m-1}{2}+\nu})(t), \quad t \geq 0, \tag{14}$$

which in turn shows, in concert with [34, Lemma 12], that in some cases the Hankel transforms above can be written in closed form. Specifically, we have

$$\mathfrak{F}_m(h_{\mu,\nu})(t) = \mathfrak{F}_m(\varphi_{1,\mu,\nu,2\nu-1})(t) = 2^{\nu-1}\Gamma(\nu)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t)$$
$$= D(m,\mu,\nu)\cdot {}_1F_2\left(\frac{m-1}{2}+\nu; \frac{m-1}{2}+\nu+\frac{\mu}{2}, \frac{m-1}{2}+\nu+\frac{\mu+1}{2}; -\frac{t^2}{4}\right), \tag{15}$$

with

$$D(m,\mu,\nu) := \frac{2^{-\frac{m}{2}}\Gamma(\nu)\Gamma(\mu)\Gamma(m-1+2\nu)}{\Gamma\left(\frac{m}{2}+\nu\right)\Gamma(\mu+m-1+2\nu)}, \quad \mu,\nu,m,2\nu-1+m \in \mathbb{C}_+.$$

For the one-dimensional Fourier transform of the function $h_{\mu,\nu}$, we use the notation $\widehat{h}_{\mu,\nu}$. We also denote with L the Laplace transform operator. For $\mu, \nu \in \mathbb{C}_+$, arguments in Zastavnyi and Trigub [34, Equation (44)] show that

$$L\left(t^{2\nu+\mu-1}\widehat{h}_{\mu,\nu}(t)\right)(x) := \int_0^\infty e^{-tx}t^{2\nu+\mu-1}\widehat{h}_{\mu,\nu}(t)\mathrm{d}t = \frac{\Gamma^2(\nu)\Gamma(\mu)2^{2\nu-1}}{x^\mu(1+x^2)^\nu}, \quad x > 0. \tag{16}$$

Thus, for $\mu, \nu, m, 2\nu - 1 + m \in \mathbb{C}_+$ and $x > 0$, we have

$$L\left(t^{m-1+2\nu+\mu-1}\mathfrak{F}_m(h_{\mu,\nu})(t)\right)(x) = \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)} L\left(t^{m-1+2\nu+\mu-1}\mathfrak{F}_1(h_{\mu,\frac{m-1}{2}+\nu})(t)\right)(x)$$

$$= \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)} \cdot \frac{\Gamma^2(\frac{m-1}{2}+\nu)\Gamma(\mu)2^{m-1+2\nu-1}}{(2\pi)^{\frac{1}{2}}} \cdot \frac{1}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu}} = \frac{C(m,\mu,\nu)}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu}}, \qquad (17)$$

with

$$C(m,\mu,\nu) := \frac{2^{-\frac{m}{2}}\Gamma(\nu)\Gamma(\mu)\Gamma(m-1+2\nu)}{\Gamma\left(\frac{m}{2}+\nu\right)} = D(m,\mu,\nu)\Gamma(\mu+m-1+2\nu).$$

Remark 2. Equation (17) is the crux of the proof of the main part of Theorem 11 in [33]:

(i) If $\nu > 1/2$ and $\mu \geq \max\{\nu, 1\}$, then $h_{\mu,\nu} \in \Phi_1$. If, additionally, $(\mu;\nu) \neq (1;1)$, then there exist constants $c_i > 0$, $i = 1, 2$, depending on μ and ν only, such that $c_1 \leq (1+t^2)^\nu \cdot \widehat{h}_{\mu,\nu}(t) \leq c_2$, $t \in \mathbb{R}$

(ii) If $\nu \geq 1$, then $h_{\mu,\nu} \in \Phi_1 \iff \mu \geq \nu$.

(iii) If $m \geq 2$, then $h_{\mu,\nu} \in \Phi_m \iff \nu > 1/2$ and $\mu \geq \frac{m-1}{2} + \nu$. In this case, there exist two constants $c_i > 0$, $i = 1, 2$, depending on μ, ν and m, and such that

$$c_1 \leq (1+t^2)^{\frac{m-1}{2}+\nu} \cdot \mathfrak{F}_m(h_{\mu,\nu})(t) \leq c_2, \ t \geq 0.$$

This theorem is related to the positiveness of the function $I_{1,\mu,\nu,2\nu-1}(t)$ for all $t > 0$. Theorems on positiveness of the functions $I_{\delta,\mu,\nu,\alpha}(t)$ are obtained in [35, Theorems 4, 5, 6] (the well-known cases given before Theorem 4 from [35]).

3. PROOF OF THEOREM 1

A function $f : (0,\infty) \to \mathbb{R}$ is called completely monotone if it is infinitely often differentiable and $(-1)^n f^{(n)}(x) \geq 0$, for all $n \in \mathbb{Z}_+$ and for all $x > 0$. The set of completely monotone functions on $(0,\infty)$ is denoted $\mathcal{CM}$.

Theorem 4 (Hausdorff–Bernstein–Widder. See, for example, [8, 18, 20, 30]). *$f \in \mathcal{CM}$ if and only if*

$$f(x) = \int_0^{+\infty} e^{-xs}\, d\mu(s), \quad x > 0, \qquad (18)$$

where μ is a nonnegative Borel measure on $[0,+\infty)$ such that the integral (18) converges for all $x > 0$. The measure μ is finite on $[0,+\infty)$ if and only if $f(+0) < +\infty$.

Remark 3. It follows from Hausdorff–Bernstein–Widder theorem that if $g \in C[0,+\infty)$ and its Laplace transform $Lg(x) := \int_0^{+\infty} e^{-xs}g(s)\, ds$ converges for all $x > 0$, then $g(s) \geq 0$ for $s \geq 0$ if and only if $Lg \in \mathcal{CM}$.

Let us start with a general assertion regarding the structure of Problem 1.

Proposition 1. *The following conditions are equivalent:*

1. *Condition (3) is satisfied.*

2. *For any $\beta_2 > \beta_1 > 0$, the function $t \mapsto \beta_2^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(\beta_2 t) - \beta_1^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(\beta_1 t)$ is nonnegative in interval $(0,\infty)$.*

3. *The function $t^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(t) = 2^{\nu-1}\Gamma(\nu)t^{\varepsilon+m}I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t)$ increases in the interval $(0,\infty)$.*

4. *The following inequality is true:*

$$(\varepsilon+m)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) + tI'_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) =$$

$$(\varepsilon+m)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) - t^2 I_{1,\mu,\frac{m-1}{2}+\nu+1,m-1+2\nu+1}(t) \geq 0, \quad \forall t > 0$$

5. *Let* $n := \frac{m-1}{2} + \nu$. *Then,* $(\varepsilon + m)\mathfrak{F}_1(h_{\mu,n})(t) - \frac{t^2}{2n}\mathfrak{F}_1(h_{\mu,n+1})(t) \geq 0$, $\forall t > 0$.

6. *We have*

$$L\left(t^{2n+\mu-1}\Big((\varepsilon+m)\widehat{h}_{\mu,n}(t) - \frac{t^2}{2n}\widehat{h}_{\mu,n+1}(t)\Big)\right)(x)$$
$$= \Gamma^2(n)\Gamma(\mu)2^{2n-1}\left(\frac{\varepsilon+m}{x^\mu(1+x^2)^n} - \frac{2n}{x^\mu(1+x^2)^{n+1}}\right) \in \mathcal{CM}.$$

7.

$$\frac{\varepsilon - 2\nu + 1 + (\varepsilon+m)x^2}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu+1}} \in \mathcal{CM} \tag{19}$$

The proof is an easy consequence of Remark 1 in concert with the Hausdorff–Bernstein–Widder theorem (see Remark 3), Theorem 3 (statements 1 and 2), and equalities (14) and (16).

Note that, if $\mu, \nu > 0$, then $x^{-\mu}(1+x^2)^{-\nu} \in \mathcal{CM}$ if and only if $\widehat{h}_{\mu,\nu}(t) \geq 0$ for all $t > 0$, if and only if $I_{1,\mu,\nu,2\nu-1}(t) \geq 0$ for all $t > 0$ (see Equations (16) and (15)).

Proposition 1 will now be combined with the following facts:

1. If $\nu \geq 1$, then $x^{-\mu}(1+x^2)^{-\nu} \in \mathcal{CM}$ if and only if $\mu \geq \nu$. The sufficiency of this result can be found in [9], and the necessity has been proved in [14], [32, Lemma 8].

2. If $0 < \nu < 1$, $\mu \geq 1$, then $x^{-\mu}(1+x^2)^{-\nu} \in \mathcal{CM}$ [14] and [36, Example 5.4], [37, § 4.7, Example 4.7.7].

3. If $\nu > 0$, $\mu \geq 2\nu$, then $x^{-\mu}(1+x^2)^{-\nu} \in \mathcal{CM}$ [2].

4. If $n = 1, 2, 3$, then $(a + x^2)/(x^n(1+x^2)^n) \in \mathcal{CM}$ if and only if $a \geq 1/(2^{n-1}+1)$ [34, § 2].

We can now combine the first three sufficient conditions above to obtain the following assertion: if $\nu > 0$ and $\mu \geq \min\{2\nu; \max\{1,\nu\}\}$, then $x^{-\mu}(1+x^2)^{-\nu} \in \mathcal{CM}$.

The combination of these facts with Proposition 1 has just offered the proof of Theorem 1. It is only necessary to take into account the following equations:

$$\frac{\varepsilon - 2\nu + 1 + (\varepsilon+m)x^2}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu+1}} = \frac{\varepsilon - 2\nu + 1}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu+1}} + \frac{\varepsilon+m}{x^{\mu-2}(1+x^2)^{\frac{m-1}{2}+\nu+1}},$$
$$\frac{\varepsilon - 2\nu + 1 + (\varepsilon+m)x^2}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu+1}} = \frac{\frac{\varepsilon-2\nu+1}{\varepsilon+m} + x^2}{x^n(1+x^2)^n} \cdot \frac{\varepsilon+m}{x^{\mu-n}(1+x^2)^{\frac{m-1}{2}+\nu+1-n}}.$$

4. PROOF OF THEOREM 2

If $\mu, \nu > 0$, then the arguments in [34, Equality (40)] show that

$$h_{\mu,\nu}(x) = (1-x)^{\mu+\nu-1}\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(1-t+(1+t)x)^{\nu-1}\mathrm{d}t, \quad x \in (0,1). \tag{20}$$

Let $\nu \in \mathbb{N}$. Then from Proposition 1 in [35] we have that $h_{\mu,\nu} \in C^{2\nu-2}(-1,1)$ and $h_{\mu,\nu} \notin C^{2\nu-1}(-1,1)$. Thus, from (20) it follows that

$$h_{\mu,\nu}(x) = \sum_{k=0}^{\infty} a_k(\mu,\nu)x^{2k} + |x|^{2\nu-1}\sum_{k=0}^{\infty} b_k(\mu,\nu)x^{2k}, \quad |x| < 1,$$
$$b_0(\mu,\nu) \neq 0, \quad b_1(\mu,\nu) = \frac{h_{\mu,\nu}^{(2\nu+1)}(+0)}{(2\nu+1)!}. \tag{21}$$

From (13) we have that $h'_{\mu,\nu}(x) = -2(\nu-1)xh_{\mu,\nu-1}(x)$ for $\nu \geq 2$, $x > 0$, and $h_{\mu,\nu}^{(k+1)}(x) = -2(\nu-1)(xh_{\mu,\nu-1}^{(k)}(x) + kh_{\mu,\nu-1}^{(k-1)}(x))$ for $k \geq 1$, $0 < x < 1$. From the last equation it follows that $h_{\mu,\nu}^{(k+1)}(+0) =$

$-2(\nu-1)kh_{\mu,\nu-1}^{(k-1)}(+0)$ for $\nu \geq 2,\ k \geq 1$, and for $\nu \geq 2,\ k \geq 2\nu-3$, we have (for convenience, we consider that $0!! := 1$ and $(-1)!! := 1$)

$$h_{\mu,\nu}^{(k+1)}(+0) = (-2)^{\nu-1}(\nu-1)! \cdot \frac{k!!}{(k-2\nu+2)!!} h_{\mu,1}^{(k-2\nu+3)}(+0). \tag{22}$$

The equality (22) is true for $\nu = 1$, $k \geq -1$. It's obvious that $h_{\mu,1}(x) = (1-|x|)_+^{\mu}/\mu$ and $h_{\mu,1}^{(p)}(+0) = (-1)^p \Gamma(\mu)/\Gamma(\mu-p+1)$, $p \in \mathbb{Z}_+$. Therefore, for $k \geq \nu-1$, with $\nu \in \mathbb{N}$, we have

$$h_{\mu,\nu}^{(2k+1)}(+0) = (-4)^{\nu-1}(\nu-1)! \cdot \frac{k!}{(k-\nu+1)!} \cdot h_{\mu,1}^{(2k-2\nu+3)}(+0)$$
$$= -(-4)^{\nu-1}(\nu-1)! \cdot \frac{k!}{(k-\nu+1)!} \cdot \frac{\Gamma(\mu)}{\Gamma(\mu-2k+2\nu-2)};$$
$$h_{\mu,\nu}^{(2\nu+1)}(+0) = -(-4)^{\nu-1}(\nu-1)!\nu!(\mu-1)(\mu-2).$$

On the other hand, (21) shows that, for $|x| < q = \min(\beta_1, \beta_2)$,

$$f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}(x) = \sum_{k=0}^{\infty} a_k(\mu,\nu)\left(\beta_2^{\varepsilon-2k} - \beta_1^{\varepsilon-2k}\right)x^{2k}$$
$$+ |x|^{2\nu-1} \sum_{k=0}^{\infty} b_k(\mu,\nu)\left(\beta_2^{\varepsilon-2\nu+1-2k} - \beta_1^{\varepsilon-2\nu+1-2k}\right)x^{2k}.$$

Assertions 1. and 2. are thus proved.

If, in addition, $\mu \in \mathbb{N}$, then $h_{\mu,\nu}$ is a polynomial of degree $\mu + 2\nu - 2$ on the compact interval $[0,1]$. If $\mu = 1$ or $\mu = 2$, then in the second sum of (21), the terms with $k \geq 1$ vanish. Therefore, if $\varepsilon = 2\nu - 1$, $\mu = 1$ or $\mu = 2$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}$ on the interval $[-q,q]$ is an even polynomial of degree $\leq \mu + 2\nu - 2$. Assertion 3. is proved. The proof is completed.

5. COVARIANCE FUNCTIONS

Let $\{Z(\mathbf{x}), \mathbf{x} \in D\}$ be a stationary and isotropic Gaussian field observed over a bounded set D of $\mathbb{R}^m$. The assumption of Gaussianity implies that we only need focus on the first and second order moments in order to specify the probabilistic properties of the field. If $\varphi \in \Phi_m$ with $\varphi(0) = 1$, then $\varphi(||\cdot||)$ is the correlation function of some Gaussian field Z as previously defined.

Covariance functions are fundamental to geostatistics for both modeling, inference, and best linear unbiased prediction (called kriging in the geostatistical literature). Let us define $\Phi_\infty := \bigcap_m \Phi_m$. Arguments in Bernstein–Widder in concert with Schoenberg [21] show that a mapping f belongs to the class Φ_∞ if and only if $f(\sqrt{\cdot})$ is completely monotonic on the positive real line. This is the case for the Matérn function (see [23], with the references therein), defined through

$$\mathcal{M}_\nu(r) = r^\nu \mathcal{K}_\nu(r), \quad r \geq 0, \tag{23}$$

where $\nu > 0$ is a parameter that allows to govern the differentiability at the origin, and thus the mean square differentiability of the associated Gaussian field. Here, $\mathcal{K}_\nu$ is a modified Bessel function of order ν. We have $\mathcal{M}_{1/2}(r) = \exp(-r)$ and $\mathcal{M}_\infty = \exp(-r^2)$. When $\nu = k + 1/2$, with k positive integer, the Matérn simplifies into the product of the exponential function with polynomial of degree k, and exact smoothness can be determined in this case. Another function in the class Φ_∞ is the Dagum function [15, 3], admitting equation

$$\mathcal{D}_{\alpha,\beta}(r) = 1 - \frac{r^{\alpha\beta}}{(1+r^\alpha)^\beta}, \quad r \geq 0, \tag{24}$$

where the conditions on the positive parameters are shown in Berg, Porcu and Mateu [3]. This function has been used in a number of papers devoted to statistical mechanics, and we refer the reader to [22] with the references therein.

The use of compactly supported covariance functions has been advocated in a number of papers, and we refer the reader to [12], with the references therein, and to [7] for a recent effort under the framework on multivariate Gaussian fields. Covariance functions with compact support represent the building block for the construction of methods allowing to overcome the big data problem ([10]). The recent work of [4] brought even more attention on the role of some classes of compactly supported covariances for asymptotically optimal prediction on a bounded set of $\mathbb{R}^d$. The Buhmann functions as defined in Zastavnyi [35] (see also [5]) includes as special case many other classes of compactly supported covariance functions, such as Askey [1] and Wendland [29] functions, as well as the Zastavnyi [33–35] and Trigub [24, 25] classes. Finally, also Wu functions [31] and the celebrated spherical model [28] are included as special cases. From the cited works it has become apparent that the smoothness at the origin (intended as even extension) of a compactly supported and isotropic covariance function plays a crucial role for both estimation and prediction. Wendland functions [29] have been especially popular, being compactly supported over balls of $\mathbb{R}^d$ with arbitrary radii, and additionally allowing for a continuous parameterization of differentiability at the origin, in a similar way to the Matérn family ([23]).

It should be noted that many applications of positive definite radial functions arise in applications like Engineering and Physics, because scientists use shifted translates of them as trial functions for PDE solving. An overview of such applications is given e.g. in Schaback and Wendland [19].

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