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Combining Reliability Functions of a Weibull Distribution

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Abstract—In this article, a large sample pooling procedure is considered for the reliability function of a Weibull distribution. Asymptotic properties of shrinkage estimation procedures based on the preliminary test are developed. It is shown that the proposed estimator has substantially smaller asymptotic mean squared error (AMSE) than the usual maximum likelihood (ML) estimator in most of the parameter space. Analytic AMSE expressions of the proposed estimators are obtained and the dominance picture of the estimators is presented by comparing them. It is shown that the suggested estimators yield a wider dominance range over the ML estimator than the usual pretest estimator and give a meaningful size of the pretest. To appraise the small sample performance of the estimators, detailed Monte-Carlo simulation studies are also carried out.

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1. INTRODUCTION AND PRELIMINARIES

The reliability function of a product, denoted as $R(t)$, indicates the probability of the product's life exceeding t time periods. When the life time of a particular product is modeled by a two-parameter Weibull distribution with shape parameter $\beta > 0$ and scale parameter $\theta > 0$ having probability density function

$$f_X(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\theta}\right)^{\beta}\right], \quad x \geq 0,$$

then the product reliability function is $R(t) = P\left(X > t\right) = \exp\left[-\left(t/\theta\right)^{\beta}\right]$. For one sample case, Baklizi and Ahmed [1] discussed improved estimation of $R(t)$ in the presence of uncertain prior information wherein $R(t)$ equals a specific value $R_0(t)$. In the present investigation, we discuss two sample version of the problem.

Suppose a reliability engineer wishes to estimate $R(t)$ of a product based on sample data of two different manufacturing plants. If we assume that the samples from both plants follow a Weibull distribution with parameters β_i and θ_i, where $i = 1, 2$, then the reliability function of the product at i-th plant is $R_i(t) = \exp[-(t/\theta_i)^{\beta_i}]$. Furthermore, suppose the reliability engineer suspects that $R_1(t) = R_2(t)$ and wishes to estimate $R_1(t)$ based on the two samples. For ease of notations, $R_i(t)$ will be denoted by γ_i.

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Let $X_{i,1}, X_{i,2} \cdots, X_{i,n_i}$ be the pair of independent random samples available from the above Weibull distribution. Then, the usual unrestricted estimator (UE) of γ_i based on n_i observations is

$$\hat{\gamma}_i = \exp\left[-\left(t/\hat{\theta}_i\right)^{\hat{\beta}_i}\right], \tag{1}$$

where $(\hat{\beta}_i, \hat{\theta}_i)$ are the ML estimators of (β_i, θ_i) and are numerically obtained by optimizing two score equations.

Our primary interest is in the estimation of γ_1 when it is suspected that both reliability functions may be equal and can thus be written in the following null hypothesis:

$$H_0 : \gamma_1 = \gamma_2. \tag{2}$$

Under H_0, the combined or restricted estimator (RE) of γ_1 is

$$\hat{\gamma}_1^{RE} = (n_1\hat{\gamma}_1 + n_2\hat{\gamma}_2)/(n_1 + n_2).$$

This estimator gives smaller quadratic risk as compared to $\hat{\gamma}_1$ when the null hypothesis is true or nearly true, while its performance is poorer in the rest of the parameter space. In order to have increased precision, one can use the preliminary test (PT) estimator suggested by Bancroft [2] defined as $\hat{\gamma}_1^{PT} = \hat{\gamma}_1^{RE} I(T_n < c_{n,\alpha}) + \hat{\gamma}_1 I(T_n \geq c_{n,\alpha})$, where $I(\cdot)$ is an indicator function, T_n is an appropriate test statistic for testing H_0 in (2), and $c_{n,\alpha}$ is the cut-point of the null distribution of T_n.

It has been observed that $\hat{\gamma}_1^{PT}$ is not the best estimator in the entire parameter space and it has a larger size of the pretest. Following Ahmed and Khan [3], an improved estimator of $\hat{\gamma}_1^{PT}$ based on the linear shrinkage methodology has been suggested here for the estimation of γ_1. This estimator is termed as shrinkage pretest (SP) and it is denoted as $\hat{\gamma}_1^{SP}$. Interestingly, $\hat{\gamma}_1^{SP}$ dominates the $\hat{\gamma}_1$ over a large portion of the parameter space and it provides a reasonable size for the preliminary test. A detailed discussion of these estimation procedures for various settings can be found in Saleh [4] and Ahmed [5].

Our primary goal in this article is to investigate the large sample properties of $\hat{\gamma}_1^{SP}$ over $\hat{\gamma}_1^{PT}$ and other estimators. The properties of small samples are also explored with the help of Monte-Carlo simulations. Mean squared error (MSE) criterion is selected to appraise the performance of the estimators under the following quadratic loss function $\ell(\hat{\gamma}_1^\star, \gamma_1) = (\hat{\gamma}_1^\star - \gamma_1)^2$, where $\hat{\gamma}_1^\star$ is a suitable estimator of γ_1. Then, the MSE of $\hat{\gamma}_1^\star$ is given by $MSE(\hat{\gamma}_1^\star, \gamma_1) = E(\hat{\gamma}_1^\star - \gamma_1)^2$. Further, $\hat{\gamma}_1^\star$ will be termed as an admissible estimator of γ_1 if there exists an alternative estimator $\hat{\gamma}_1^\diamond$ such that

$$MSE(\hat{\gamma}_1^\diamond) \leq MSE(\hat{\gamma}_1^\star) \quad \text{for all} \quad \gamma_1, \tag{3}$$

with strict inequality for some γ_1. If instead of (3) holding for every n we use the AMSE then we require

$$\lim_{n\to\infty} MSE(\hat{\gamma}_1^\diamond) \leq \lim_{n\to\infty} MSE(\hat{\gamma}_1^\star) \quad \text{for all} \quad \gamma_1,$$

with strict inequality holding for some γ_1 and $\hat{\gamma}_1^\star$ is termed as an asymptotically inadmissible estimator of γ_1.

This article is organized as follows. The proposed estimators are presented in section 2. Expressions for the asymptotic bias (AB) and AMSE of the suggested estimators under local alternatives are given in section 3 while the AMSE analysis is provided in section 4. Section 5 summarizes the findings of the simulation studies while section 6 concludes the article.

2. PROPOSED ESTIMATORS

Firstly, a linear shrinkage (LS) estimator of γ_1 is defined as $\hat{\gamma}_1^{LS} = \pi\hat{\gamma}_1^{RE} + (1-\pi)\hat{\gamma}_1$, where $\pi \in [0, 1]$ is termed as the shrinkage coefficient reflecting the degree of confidence in the available information. If the reliability engineer wishes to rely on the data completely and strongly believes that H_0 is true, then s/he should select $\pi = 1$. In the reviewed literature, it has been found that $\hat{\gamma}_1^{LS}$ provides a wider range than $\hat{\gamma}_1^{RE}$ in which it dominates $\hat{\gamma}_1$. This encouraged us to replace $\hat{\gamma}_1^{RE}$ by $\hat{\gamma}_1^{LS}$ in $\hat{\gamma}_1^{PT}$. Thus, the SP estimator of γ_1 is defined by $\hat{\gamma}_1^{SP} = \hat{\gamma}_1^{LS} I(T_n < c_{n,\alpha}) + \hat{\gamma}_1 I(T_n \geq c_{n,\alpha})$.

The following lemma, due to Bain and Engelhardt [8], will be used to obtain the test statistic T_n.

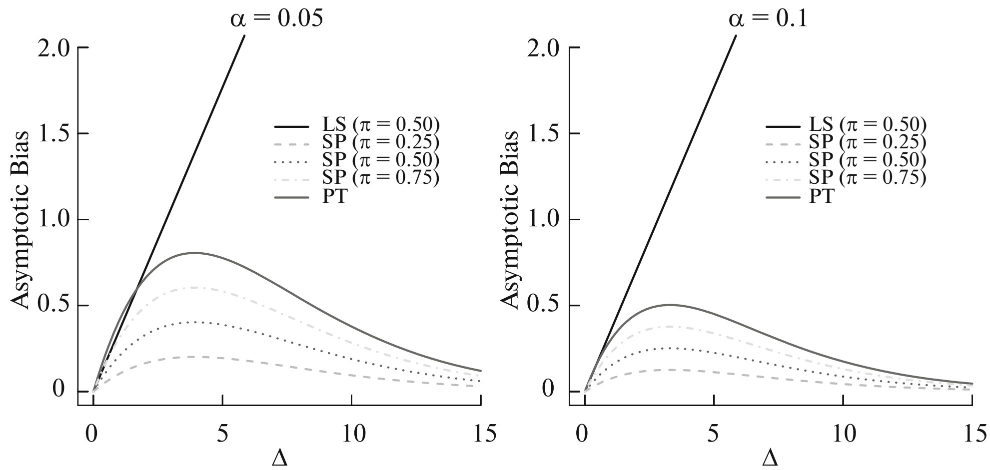

Fig. 1. Asymptotic Bias of the estimators for $\alpha = 0.05, 0.10$.

Lemma 1. *For a random sample of size n_i drawn from the Weibull distribution defined above, the following asymptotic property holds $\sqrt{n_i}(\hat{\gamma}_i - \gamma_i) \xrightarrow{d} \mathcal{N}(0, \tau_i^2)$, where $\tau_i^2 = (\gamma_i \ln \gamma_i)^2[1.109 - 0.514 \ln(-\ln \gamma_i) + 0.608\{\ln(-\ln \gamma_i)\}^2]$ and $\xrightarrow{d}$ means convergence in distribution.*

Now, for the preliminary test on H_0 in (2), we consider the following generalized squared distance

$$T_n = \frac{(\hat{\gamma}_2 - \hat{\gamma}_1)^2}{\hat{\tau}^2 \, (1/n_1 + 1/n_2)}, \tag{4}$$

where $\hat{\tau}^2 = (\hat{\gamma}_1^{RE} \ln \hat{\gamma}_1^{RE})^2[1.109 - 0.514 \ln(-\ln \hat{\gamma}_1^{RE}) + 0.608\{\ln(-\ln \hat{\gamma}_1^{RE})\}^2]$. The distribution of T_n approaches the central chi-square distribution with 1 degree of freedom asymptotically. Thus, for a given level of significance α $(0 < \alpha < 1)$ and using the null distribution of T_n one can obtain $c_{n,\alpha}$ as the upper $100\alpha\%$ critical value.

Since the test-statistic T_n is consistent against any fixed alternative $\gamma_1 = \gamma_2$, we should specify a reasonable sequence of local alternatives accordingly to avoid asymptotic degeneracy. Therefore, in order to study the asymptotic properties of the estimators, we consider the following sequence $\{\mathcal{H}_n\}$ of local alternatives $\mathcal{H}_n : \gamma_2 = \gamma_1 + \zeta/\sqrt{n}$, where ζ is a fixed real number and $n = n_1 + n_2$. With the help of the following theorem, we have studied the asymptotic properties of the estimators under $\{\mathcal{H}_n\}$

Theorem 1. *Under local alternatives, as $n \to \infty$ in such a way that $n_1/n \to \omega \in (0, 1)$ and $n_2/n \to (1 - \omega)$, the following results hold*

(1) $\sqrt{n}(\hat{\gamma}_2 - \hat{\gamma}_1) \xrightarrow{d} \mathcal{N}\left(\zeta, \frac{\tau^2}{\omega(1-\omega)}\right)$,

(2) *T_n asymptotically follows a non-central chi-squared distribution with 1 degree of freedom and non-centrality parameter $\Delta = \omega(1 - \omega)\zeta^2/\tau^2$.*

3. ASYMPTOTIC RESULTS

Following Ahmed [6, 7] and by direct computations, the AB and AMSE expressions of the proposed estimators are given in theorem 2 and 3, respectively.

Theorem 2. *Under local alternatives and using theorem 1, the asymptotic bias of the listed estimators is given as*

$$AB(\hat{\gamma}_1) = \text{asymptotic bias of } \{\sqrt{\omega n}(\hat{\gamma}_1 - \gamma_1)\} = 0,$$

$$AB(\hat{\gamma}_1^{RE}) = \text{asymptotic bias of } \{\sqrt{\omega n}(\hat{\gamma}_1^{RE} - \gamma_1)\} = \tau\zeta\sqrt{(1 - \omega)},$$

$$AB(\hat{\gamma}_1^{LS}) = \text{asymptotic bias of } \{\sqrt{\omega n}(\hat{\gamma}_1^{LS} - \gamma_1)\} = \pi\tau\zeta\sqrt{(1 - \omega)},$$

$$AB(\hat{\gamma}_1^{PT}) = \text{asymptotic bias of } \{\sqrt{\omega n}(\hat{\gamma}_1^{PT} - \gamma_1)\} = \tau\zeta\sqrt{(1 - \omega)}G_3(\chi^2_{1,\alpha}; \Delta),$$

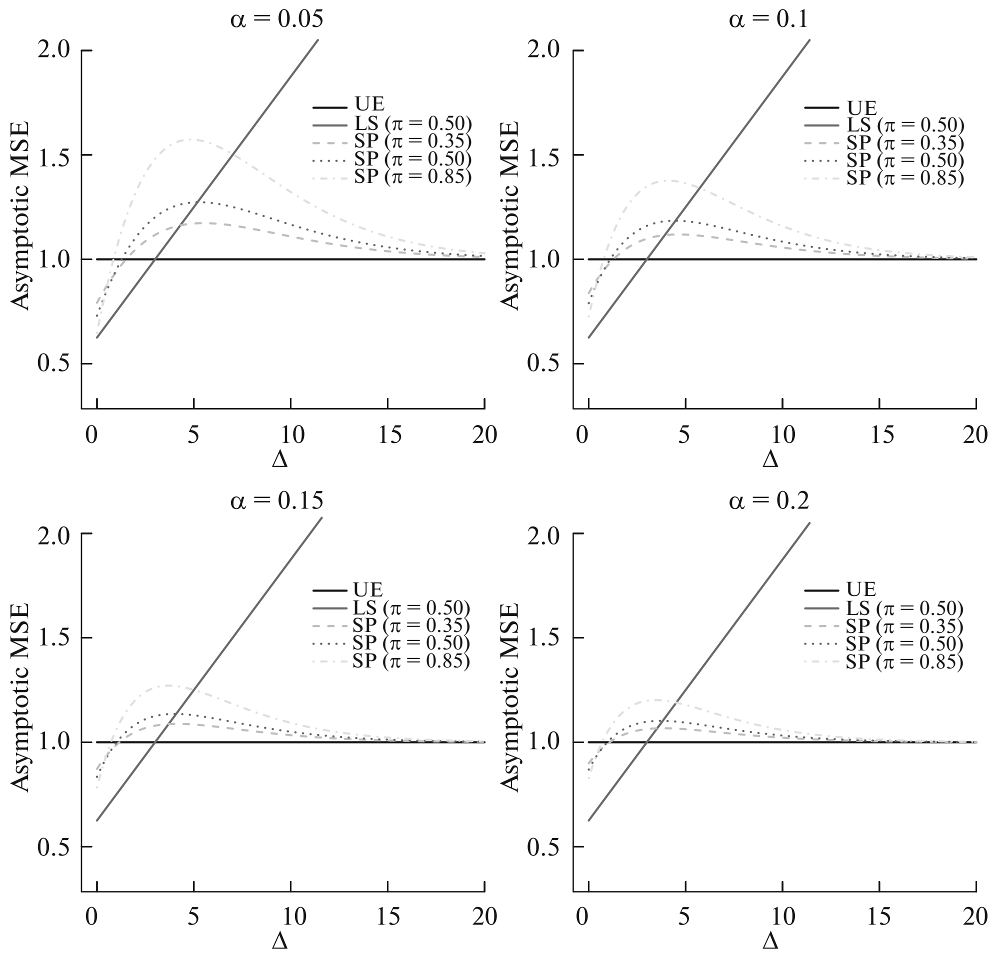

Fig. 2. Asymptotic MSE performance of the listed estimators.

$$AB(\hat{\gamma}_1^{SP}) = \text{asymptotic bias of } \{\sqrt{\omega n}(\hat{\gamma}_1^{SP} - \gamma_1)\} = \pi\tau\zeta\sqrt{(1-\omega)}G_3(\chi^2_{1,\alpha};\Delta),$$

where $G_\nu(\cdot;\Delta)$ *is the cumulative distribution function of a non-central chi-squared distribution with* ν *degrees of freedom and non-centrality parameter* Δ.

Since $AB(\hat{\gamma}_1^{LS}) = \pi AB(\hat{\gamma}_1^{RE})$ and $AB(\hat{\gamma}_1^{SP}) = \pi AB(\hat{\gamma}_1^{PT})$, therefore $AB(\hat{\gamma}_1^{LS}) < AB(\hat{\gamma}_1^{RE})$ and $AB(\hat{\gamma}_1^{SP}) < AB(\hat{\gamma}_1^{PT})$ for $\pi \in (0,1)$. Thus, the shrinkage technique worked well for the estimators and it reduced the asymptotic bias. For various values of π and α with $\omega = 0.5$, we have plotted AB of the estimators in Figure 1 against different Δ.

Theoretically speaking, bias is a component of the MSE. Thus, by controlling MSE one can control both the bias and variance of the estimators. Therefore, from this point onwards we shall only compare the AMSE. Direct computations provide expressions for the AMSE of the estimators, which are presented in theorem 3.

Theorem 3. *Using Theorem 1 and under local alternatives, expressions of the* AMSE *of the estimators are given as*

$$AMSE(\hat{\gamma}_1) = \text{asymptotic MSE of } \{\sqrt{\omega n}(\hat{\gamma}_1 - \gamma_1)\} = \tau^2,$$

$$AMSE(\hat{\gamma}_1^{RE}) = \text{asymptotic MSE of } \{\sqrt{\omega n}(\hat{\gamma}_1^{RE} - \gamma_1)\} = \tau^2 + (1-\omega)\tau^2\Delta - (1-\omega)\tau^2,$$

$$AMSE(\hat{\gamma}_1^{LS}) = \text{asymptotic MSE of } \{\sqrt{\omega n}(\hat{\gamma}_1^{LS} - \gamma_1)\} = \tau^2 + \pi^2(1-\omega)\tau^2\Delta - \pi(2-\pi)(1-\omega)\tau^2,$$

$$AMSE(\hat{\gamma}_1^{PT}) = \text{asymptotic MSE of } \{\sqrt{\omega n}(\hat{\gamma}_1^{PT} - \gamma_1)\}$$
$$= \tau^2 + (1-\omega)\tau^2\Delta\{2G_3(\chi^2_{1,\alpha};\Delta) - G_5(\chi^2_{1,\alpha},\Delta)\} - (1-\omega)\tau^2 G_3(\chi^2_{1,\alpha};\Delta);$$

$$AMSE(\hat{\gamma}_1^{SP})\& = \text{asymptotic MSE of } \{\sqrt{\omega n}(\hat{\gamma}_1^{SP} - \gamma_1)\}$$
$$= \tau^2 + (1-\omega)\tau^2\Delta\{2\pi G_3(\chi^2_{1,\alpha};\Delta) - \pi(2-\pi)G_5(\chi^2_{1,\alpha},\Delta)\} - \pi(2-\pi)(1-\omega)\tau^2 G_3(\chi^2_{1,\alpha};\Delta).$$

Asymptotic analysis of the MSE of the suggested estimators is studied in the following section.

4. ASYMPTOTIC ANALYSIS

We begin our analysis by noting that the AMSE of $\hat{\gamma}_1$ is a constant line, while the AMSE of $\hat{\gamma}_1^{RE}$ is a straight line. If the restriction $\gamma_1 = \gamma_2$ holds, then $\hat{\gamma}_1^{RE}$ has smaller AMSE than $\hat{\gamma}_1$. It can be seen that $AMSE(\hat{\gamma}_1^{RE}) \le AMSE(\hat{\gamma}_1)$ if $0 \le \Delta \le 1$ i.e., $\hat{\gamma}_1^{RE}$ performs better than $\hat{\gamma}_1$ whenever $\Delta \in [0,1]$. Beyond this interval the AMSE of $\hat{\gamma}_1^{RE}$ increases and becomes bounded. The AMSE function of $\hat{\gamma}_1^{LS}$ has similar characteristics to that of $\hat{\gamma}_1^{RE}$. It is worth noting that $\hat{\gamma}_1^{LS}$ dominates $\hat{\gamma}_1$ when $0 \le \Delta \le (2-\pi)/\pi$. Thus, the range in which $AMSE(\hat{\gamma}_1^{LS}) \le AMSE(\hat{\gamma}_1)$ is wider than the range in which $AMSE(\hat{\gamma}_1^{RE}) \le AMSE(\hat{\gamma}_1)$.

In order to compare the AMSE of the shrinkage preliminary test estimator with other estimators, the following identity will be helpful. For $\Delta \ge 0$ and $\alpha \in (0,1)$

$$G_5(\chi^2_{1,\alpha};\Delta) \le G_3(\chi^2_{1,\alpha};\Delta) \le G_3(\chi^2_{1,\alpha};0) = 1-\alpha. \tag{5}$$

The left hand side of (5) approaches 0 as Δ goes to infinity. The AMSE comparison of SP and UE yields that $\hat{\gamma}_1^{SP}$ dominates $\hat{\gamma}_1$ whenever

$$\Delta \le \frac{(2-\pi)G_3(\chi^2_{1,\alpha};\Delta)}{2G_3(\chi^2_{1,\alpha};\Delta) - (2-\pi)G_5(\chi^2_{1,\alpha};\Delta)}. \tag{6}$$

Moreover, as α approaches one, the AMSE of SP converges to the AMSE of $\hat{\gamma}_1$. For large values of Δ, $AMSE(\hat{\gamma}_1^{SP})$ increases, reaches its maximum point after crossing the AMSE of $\hat{\gamma}_1$ and then monotonically decreases and approaches $AMSE(\hat{\gamma}_1)$. Therefore, there are points in the parameter space where $\hat{\gamma}_1^{SP}$ has a larger AMSE than UE and a sufficient condition for this result to occur is $\Delta > \frac{(2-\pi)G_3(\chi^2_{1,\alpha};\Delta)}{2G_3(\chi^2_{1,\alpha};\Delta)-(2-\pi)G_5(\chi^2_{1,\alpha};\Delta)}$. Interestingly, when $\alpha \to 0$ the interval in (6) becomes

$$0 \le \Delta \le (2-\pi)\pi^{-1}. \tag{7}$$

For $\pi = 1$, it is evident that the $AMSE(\hat{\gamma}_1^{PT}) \le AMSE(\hat{\gamma}_1)$ as long as $0 \le \Delta \le 1$, whereas $AMSE(\hat{\gamma}_1^{SP}) \le AMSE(\hat{\gamma}_1)$ as long as (7) holds. Thus, the range for which SP dominates UE is greater than the range for which PT dominates UE. Clearly, this indicates the superiority of $\hat{\gamma}_1^{SP}$ over $\hat{\gamma}_1^{PT}$.

Next, we compare the AMSE of the restricted estimator and SP. Under the null hypothesis and for $\alpha \in (0,1)$ we note that

$$AMSE(\hat{\gamma}_1^{SP}) - AMSE(\hat{\gamma}_1^{RE}) = \tau^2(1-\omega)[1 - \{1-(1-\pi)^2\}G_3(\chi^2_{1,\alpha};0)] > 0,$$

which shows that $\hat{\gamma}_1^{RE}$ dominates $\hat{\gamma}_1^{SP}$. Alternatively, as Δ moves away from zero the AMSE of RE becomes unbounded whereas the AMSE of SP remains bounded. Thus, the departure from the null hypothesis is fatal to $\hat{\gamma}_1^{RE}$ whereas $\hat{\gamma}_1^{SP}$ has good AMSE control.

Now, we determine the dominance region where $\hat{\gamma}_1^{SP}$ performs better than $\hat{\gamma}_1^{LS}$ by comparing their AMSE. In the first place, under $H_0 : \gamma_1 = \gamma_2$ and for $\alpha \in (0,1)$ the AMSE of $\hat{\gamma}_1^{LS}$ is $\tau^2[1 - \{1-(1-\pi)^2\}(1-\omega)]$ and $AMSE(\hat{\gamma}_1^{SP}) - AMSE(\hat{\gamma}_1^{LS}) > 0$. Thus, at $\Delta = 0$ we conclude that $\hat{\gamma}_1^{LS}$ performs better than $\hat{\gamma}_1^{SP}$. Furthermore, we find that the AMSE of $\hat{\gamma}_1^{LS}$ is smaller than the AMSE of $\hat{\gamma}_1^{SP}$ when

$$\Delta < \frac{1 - G_3(\chi^2_{1,\alpha};\Delta)}{2(2-\pi)^{-1}[1 - G_3(\chi^2_{1,\alpha};\Delta)] - [1 - G_5(\chi^2_{1,\alpha};\Delta)]}$$

SREs of the estimators relative to $\hat{\gamma}_1$ for $\pi = 0.50$

n_i	$\Delta^\star$	$\hat{\gamma}_1^{RE}$	$\hat{\gamma}_1^{LS}$	$\hat{\gamma}_1^{PT}$		$\hat{\gamma}_1^{SP}$	
				$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$
20	0.00	1.9637	1.5825	1.6278	1.4593	1.4070	1.3090
	0.01	1.0073	1.3300	0.9065	0.8872	1.1268	1.0589
	0.02	0.6452	1.1262	0.6271	0.6582	0.9235	0.8937
	0.03	0.4606	0.9610	0.4940	0.5584	0.7903	0.8006
	0.04	0.3405	0.8085	0.4219	0.5126	0.7034	0.7490
	0.05	0.2732	0.7096	0.4109	0.5352	0.6756	0.7548
	0.10	0.1231	0.4022	0.7390	0.9017	0.8748	0.9555
	0.20	0.0598	0.2201	1.0000	1.0000	1.0000	1.0000
30	0.00	2.0128	1.6061	1.7046	1.5183	1.4493	1.3441
	0.01	0.7720	1.2155	0.7236	0.7354	1.0136	0.9675
	0.02	0.4608	0.9594	0.4938	0.5573	0.7935	0.8017
	0.03	0.3145	0.7768	0.4207	0.5244	0.6961	0.7540
	0.04	0.2296	0.6302	0.4054	0.5425	0.6596	0.7545
	0.05	0.1815	0.5414	0.4553	0.6312	0.6930	0.8108
	0.10	0.0812	0.2838	0.9687	0.9944	0.9864	0.9975
	0.20	0.0387	0.1480	1.0000	1.0000	1.0000	1.0000

and for

$$\Delta \geq \frac{1 - G_3(\chi^2_{1,\alpha}; \Delta)}{2(2-\pi)^{-1}[1 - G_3(\chi^2_{1,\alpha}; \Delta)] - [1 - G_5(\chi^2_{1,\alpha}; \Delta)]}$$

the opposite conclusion holds. Hence, neither $\hat{\gamma}_1^{SP}$ nor $\hat{\gamma}_1^{LS}$ asymptotically dominates the other estimator under local alternatives.

Lastly, we compare the AMSE performance of SP and PT and determine the dominance conditions. It may be noticed that

$$\begin{aligned} AMSE(\hat{\gamma}_1^{PT}) - AMSE(\hat{\gamma}_1^{SP}) &= \tau^2(1-\omega)\Delta\{2(1-\pi)G_3(\chi^2_{1,\alpha}; \Delta) \\ &- (1-\omega)G_5(\chi^2_{1,\alpha}; \Delta)\} - \tau^2(1-\omega)(1-\pi)^2 G_3(\chi^2_{1,\alpha}; \Delta). \end{aligned} \tag{8}$$

It is clear from (8) that the AMSE of $\hat{\gamma}_1^{PT}$ will be smaller than $\hat{\gamma}_1^{SP}$ in the neighborhood of $\Delta = 0$. However, for large values of π, this AMSE difference may be negligible. On the other hand, as Δ increases, the AMSE difference in (8) becomes positive and $\hat{\gamma}_1^{SP}$ starts dominating $\hat{\gamma}_1^{PT}$ uniformly in the rest of the parameter space. To be specific, let Δ_π be a point in the parameter space at which the AMSE of SP and PT intersect for a given π. Then, for $\Delta \in (0, \Delta_\pi]$, PT performs better than SP, while for $\Delta \in (\Delta_\pi, \infty]$, its SP that dominates PT uniformly. Further, for large values of π (close to 1), the interval $\Delta \in (0, \Delta_\pi]$ may be negligible. Nevertheless, PT and SP share a common asymptotic property in that as $\Delta \to \infty$, their AMSE converge to a common limit, i.e., to the AMSE of $\hat{\gamma}_1$.

We have plotted the AMSE of $\hat{\gamma}_1$, $\hat{\gamma}_1^{LS}$, and $\hat{\gamma}_1^{SP}$ against Δ for $\pi = 0.35, 0.50, 0.85$, $\omega = 0.5$, $\tau^2 = 1$, and for selected values of α. Figure 2 exhibits the aforementioned properties of the estimators.

From the above figure, we see that for smaller values of α and fixed π, the variation in the AMSE is greater. The larger α values are used in figure 2 to observe the variation in the AMSE of the selected estimators. Furthermore, the larger value of π results in greater variation in the AMSE of $\hat{\gamma}_1^{SP}$.

Finally, we conclude that none of the five estimators is superior to the others under the local alternatives. However, at $\Delta = 0$ and the nominal choice of α, the AMSE of the estimators may be ordered as follows:

$$\hat{\gamma}_1^{RE} \succ \hat{\gamma}_1^{LS} \succ \hat{\gamma}_1^{PT} \succ \hat{\gamma}_1^{SP} \succ \hat{\gamma}_1$$

for a range of π, where $\succ$ denotes the dominance.

5. MONTE-CARLO SIMULATIONS

In this section, we investigate the properties of the proposed estimators in small samples as the analytical treatment of the MSE is difficult in such cases. However, using Monte-Carlo simulations, we can study the MSE efficiency of the estimators relative to $\hat{\gamma}_1$. Following Bain and Engelhardt [8], we first note that (1) can be written as:

$$\hat{\gamma} = \exp\left[\left(\frac{\ln \gamma_0}{(\hat{\theta}/\theta_0)^{\beta}}\right)^{\hat{\beta}/\beta_0}\right],$$

where θ_0, β_0, and γ_0 are some specific values.

The indices of our simulation design are the sample size (n_i), the significance level (α), the shrinkage coefficient (π), the true value of reliability function (γ_0) and a parameter $\Delta^\star = \sum_{i=1}^{2}(\gamma_i - \gamma_0)^2$, which is essentially a measure of how far away we go from the hypothesized value γ_0. The hypothesized value of reliability was chosen to be $\gamma_0 = 0.6$. For the sake of simplicity, we considered an equal number of random samples from these populations and the chosen sample sizes were $n_i = 20, 25, 30$, and 50. The significance level was fixed at 5% and 10%, while different values of π were selected, i.e., $\pi = 0.25, 0.50, 0.75, 0.90$. In order to save space, the results are reported only for $\pi = 0.50$.

To facilitate the computational process, we have generated $N = 5000$ samples from a Weibull distribution with $\theta_i = \beta_i = 1$ by writing a computer program in `R` language [9]. A special `R` package named **`fitdistrplus`** by Delignette–Muller and Dutang [10] was used to obtain the MLEs of (θ_i, β_i) by fitting a Weibull distribution to the simulated data. Based on the numerical estimates of $(\hat{\theta}_i, \hat{\beta}_i)$ and true value of $\gamma_0 = 0.6$, we have computed the estimators and their simulated relative efficiencies (SRE) relative to $\hat{\gamma}_1$ by the following formula

$$SRE(\hat{\gamma}_1^* : \hat{\gamma}_1) = \text{Simulated}MSE(\hat{\gamma}_1)/\text{Simulated}MSE(\hat{\gamma}_1^*),$$

where the simulated MSE of the estimator $\hat{\gamma}_1^*$ is the average MSE of $N = 5000$ replications defined as $\frac{1}{5000}\sum_{b=1}^{5000}\sum_{i=1}^{2}(\hat{\gamma}_{i(b)}^* - \gamma_0)^2$. A value of SRE greater than one indicates the superiority of $\hat{\gamma}_1^*$ over $\hat{\gamma}_1$. Obviously, the SRE of the unrestricted estimator $\hat{\gamma}_1$ will be 1.

In order to assess the behavior of the estimators when the null hypothesis (2) is false, further samples were drawn from a Weibull distribution with a shift to the right side under $\gamma_1 \neq \gamma_2$ such that $\Delta^\star$ takes a specific value greater than zero. The results of the simulation studies are reported in table 1 and can be summarized as follows:

At $\Delta^\star = 0$, when the null hypothesis holds, $\hat{\gamma}_1^{RE}$ has maximum SRE as compared to the other estimators. But the SRE drops to zero as $\Delta^\star$, increases showing the inferiority of the restricted estimator. Similarly, the performance of $\hat{\gamma}_1^{PT}$ is much better than $\hat{\gamma}_1^{SP}$ when the null hypothesis holds. However, as we move away from the hypothesized value, the SRE performance of $\hat{\gamma}_1^{SP}$ is comparable to $\hat{\gamma}_1^{PT}$ and $\hat{\gamma}_1^{RE}$.

The simulated relative efficiency of the estimators based on the preliminary test depends upon the size of the pretest (α) i.e., a smaller size yields larger SRE and vice-versa. The performance of $\hat{\gamma}_1^{LS}$ depends on π and has the same characteristics as that of $\hat{\gamma}_1^{RE}$. Graphical representation of table 1 is given in Figure 3 and 4 for $n_i = 20$ and $n_i = 30$, respectively. Overall, the results of the simulation study are in agreement with the asymptotic results mentioned in Section 4.

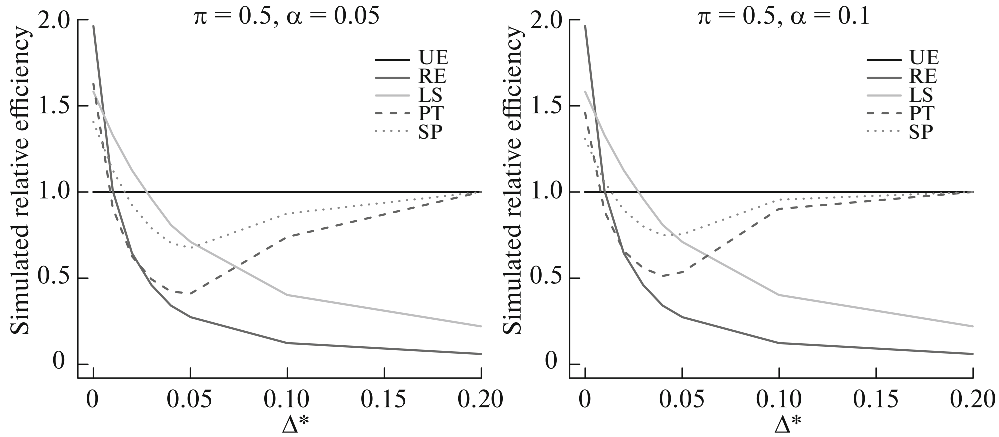

Fig. 3. SRE of the estimators for $n_i = 20$.

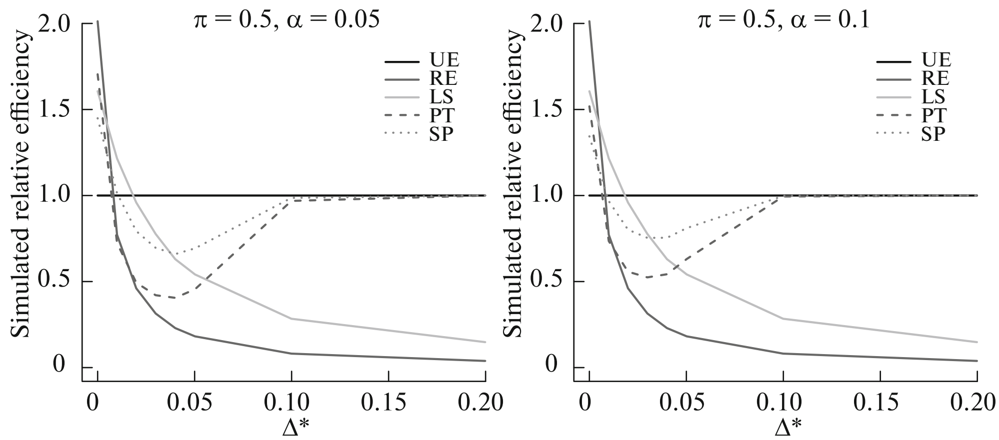

Fig. 4. SRE of the estimators for $n_i = 30$.

6. CONCLUDING REMARKS

In this article, we compared the performance of various estimators based on the linear shrinkage, preliminary test, and shrinkage pretest principles with the unrestricted and restricted estimation strategies using the AMSE criterion. Monte-Carlo simulations were also designed to assess the robustness of the estimators in small samples. It was concluded that the restricted estimator should be used when the assumption of homogeneity of reliability functions holds, i.e., $\gamma_1 = \gamma_2$. When the assumption of homogeneity is rather dubious, the use of $\hat{\gamma}_1^{SP}$ is suggested as it has the smallest AMSE in most of the parametric space as compared to the other estimators.

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