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Subharmonic Test Functions and the Distribution of Zero Sets of Holomorphic Functions

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Abstract—Let $m, n \geq 1$ are integers and D be a domain in the complex plane $\mathbb{C}$ or in the m-dimensional real space $\mathbb{R}^m$. We build positive subharmonic functions on a part of D vanishing on the boundary ∂D of domain D. We use such (test) functions to study the distribution of zero sets of holomorphic functions f on $D \subset \mathbb{C}^n$ with restrictions on the growth of f near the boundary ∂D.

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1. MOTIVATION AND STATEMENT OF THE PROBLEM

1.1. Notations, Definitions, and Agreements

We use an information and definitions from [1–4]. As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, resp. We set

$$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R},\ \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}, \mathbb{R}_{\pm\infty} := \mathbb{R}_{-\infty} \cup \mathbb{R}_{+\infty},$$

$$\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\},\ \mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\},\ \mathbb{R}^+_{+\infty} := \mathbb{R}^+ \cup \{+\infty\},$$

where the usual order relation $\leq$ on $\mathbb{R}$ is complemented by the inequalities $-\infty \leq x \leq +\infty$ for all $x \in \mathbb{R}_{\pm\infty}$. Let $f \colon X \to Y$ be a function. For $Y \subset \mathbb{R}_{\pm\infty}$, $g \colon X_1 \to \mathbb{R}_{\pm\infty}$ and $S \subset X \cup X_1$, we write "$f = g$ *on* S" or "$f \leq g$ *on* S" if $f(x) = g(x)$ or $f(x) \leq g(x)$ for all $x \in S$ respectively.

Let $m \in \mathbb{N}$. Denote by $\mathbb{R}^m$ the *m-dimensional Euclidean real space*. Then $\mathbb{R}^m_\infty := \mathbb{R}^m \cup \{\infty\}$ is the *Alexandroff* (⇔one-point) *compactification of* $\mathbb{R}^m$. Given a subset S of $\mathbb{R}^m$ (or $\mathbb{R}^m_\infty$), the closure clos$S$, the interior int$S$ and the boundary ∂S will always be taken relative $\mathbb{R}^m_\infty$.

Let $S_0 \subset S \subset \mathbb{R}^m_\infty$. If the closure clos$S_0$ is a compact subset of S in the topology induced on S from $\mathbb{R}^m_\infty$, then we write $S_0 \Subset S$. An open connected (sub-)set of $\mathbb{R}^m$ is a *(sub-)domain* of $\mathbb{R}^m_\infty$. Given $x \in \mathbb{R}^m$ and $r \in \mathbb{R}^+_{+\infty}$, we set $B(x, r) := \{x' \in \mathbb{R}^m \colon |x' - x| < r\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$, $|\infty| := +\infty$; $B(r) := B(0, r)$. Besides, $B(\infty, r) := \{x \in \mathbb{R}^m \colon |x| > 1/r\}$, $\overline{B}(x, r) := \text{clos}B(x, r)$ for $r > 0$, but $\overline{B}(x, 0) := \{x\}$ and $\overline{B}(+\infty) := \mathbb{R}^m_\infty$.

Let A, B are sets, and $A \subset B$. The set A is a *non-trivial subset* of the set B if the subset $A \subset B$ is non-empty ($A \neq \varnothing$) and *proper* ($A \neq B$).

We understand always the "*positivity*" or "*positive*" as ≥ 0, where the symbol 0 denotes the number zero, the zero function, the zero measure, etc. So, a function $f : X \to R \subset \mathbb{R}_{\pm\infty}$ is positive on X if $f(x) \geq 0$ for all $x \in X$. In such case we write "$f \geq 0$ *on* X".

The operation of superposition of functions denoted by $\circ$.

By $\mathcal{M}^+(S)$ denote the class of all Borel positive measures on S.

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Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{R}^m_\infty$. We denote by $\mathrm{sbh}(\mathcal{O})$ the class of all subharmonic functions $u\colon \mathcal{O}\to\mathbb{R}_{-\infty}$ on $\mathcal{O}$ for $m\geq 2$, and all (local) convex functions $u\colon \mathcal{O}\to\mathbb{R}_{-\infty}$ on $\mathcal{O}$ for $m=1$. By $\star\colon \mathbb{R}^m_\infty\to\mathbb{R}^m_\infty$ denote the inversion in the unit sphere $\partial B(0,1)$:

$$\star\colon x\mapsto x^\star := \begin{cases} 0 & \text{for } x=\infty,\\ \frac{1}{|x|^2}x & \text{for } x\neq 0,\infty,\\ \infty & \text{for } x=0.\end{cases} \tag{1}$$

A function u is subharmonic on a neighborhood of $\infty\in R^m_\infty$ if its *Kelvin transform*

$$u^\star(x^\star)=|x|^{m-2}u(x),\quad x^\star\in\mathcal{O}^\star:=\{x^\star\colon x\in\mathcal{O}\}, \tag{2}$$

is subharmonic on a neighborhood of 0. The class $\mathrm{sbh}(\mathcal{O})$ contains the function $\boldsymbol{-\infty}\colon x\mapsto-\infty$, $x\in\mathcal{O}$ (identically equal to $-\infty$);

$$\mathrm{sbh}^+(\mathcal{O}):=\{u\in\mathrm{sbh}(\mathcal{O})\colon u\geq 0 \text{ on } \mathcal{O}\},\quad \mathrm{sbh}_*(\mathcal{O}):=\mathrm{sbh}(\mathcal{O})\setminus\{\boldsymbol{-\infty}\}.$$

For $u\in\mathrm{sbh}_*(\mathcal{O})$, the *Riesz measure of* u is the Borel (or Radon [2, A.3]) positive measure

$$\nu_u:=c_m\Delta u\in\mathcal{M}^+(\mathcal{O}),\quad c_m:=\frac{\Gamma(m/2)}{2\pi^{m/2}\max\{1,(m-2)\}}, \tag{3}$$

where Δ is the *Laplace operator* acting in the sense of distribution theory, and Γ is the gamma function. In particular, $\nu_u(S)<+\infty$ for each subset $S\Subset\mathcal{O}$. By definition, $\nu_{\boldsymbol{-\infty}}(S):=+\infty$ for all $S\subset\mathcal{O}$.

1.2. Test Functions

Subjects of our investigation are presented by

Definition 1. Throughout what follows $m,n\in\mathbb{N}$ and $\varnothing\neq K=\mathrm{clos}K\Subset D\subset\mathbb{R}^m_\infty$, where D is *a subdomain in* $\mathbb{R}^m_\infty$ or $\mathbb{C}^n_\infty$. A function $v\in\mathrm{sbh}^+(D\setminus K)$ is a *test function for D outside of K* if

$$\lim_{D\ni x'\to x} v(x')=0 \quad \text{for each } x\in\partial D, \tag{4}$$

$$\sup_{x\in D\setminus K} v(x)<+\infty. \tag{5}$$

The class of test functions for D outside of K is denoted by $\mathrm{sbh}^+_0(D\setminus K)$.

We give *elementary properties* and simple examples of test functions.

t1. The condition (4) can be replaced by the condition: *for each number $\varepsilon\in\mathbb{R}^+_*$ there is a subset $S_\varepsilon\Subset D$ such that* $0\leq v<\varepsilon$ *on* $D\setminus S_\varepsilon$.

t2. If a function $v\in\mathrm{sbh}^+_0(D\setminus K)$ is continued (extended) by zero as

$$v(x):=\begin{cases} v(x), & \text{for } x\in D\setminus K,\\ 0, & \text{for } x\in\mathbb{R}^m_\infty\setminus D,\end{cases} \tag{6}$$

then the extended function v is a subharmonic function on $\mathbb{R}^m_\infty\setminus K$ and $v\in\mathrm{sbh}^+_0(\mathbb{R}^m_\infty\setminus K)$.

t3. If $v\in\mathrm{sbh}^+_0(\mathbb{R}^m_\infty\setminus K)$ and $v=0$ on $\mathbb{R}^m_\infty\setminus D$, then $v\in\mathrm{sbh}^+_0(D\setminus K)$. Throughout what follows we identify a test function $v\in\mathrm{sbh}^+_0(D\setminus K)$ and its continuation (6) of the class $\mathrm{sbh}^+_0(\mathbb{R}^m_\infty\setminus K)$.

t4. The condition (5) can be replaced by the condition

$$\sup_{x\in\partial S}\ \limsup_{D\setminus S\ni x'\to x} v(x')<+\infty \text{ (the maximum Principle for } \mathrm{sbh}(\mathbb{R}^m_\infty\setminus K)).$$

t5. If $v\in\mathrm{sbh}^+_0(D\setminus K)\subset\mathrm{sbh}^+_0(\mathbb{R}^m_\infty\setminus K)$, then its Riesz measure ν_v belongs to $\mathcal{M}^+(\mathrm{clos}D\setminus K)$.

Example 1. Let $D \subset \mathbb{R}^m_\infty$ be a domain, $\widetilde{D} \subset D$ a regular (for the Dirichlet problem) subdomain of D, and $\exists x_0 \in \widetilde{D}$. Then *the extended Green's function* $g_{\widetilde{D}}(\cdot, x_0)$ *for* $\widetilde{D}$ *with pole at* x_0 is a test function from the class $\mathrm{sbh}_0^+\bigl(D \setminus \{x_0\}\bigr)$. Its Riesz measure is *the harmonic measure* $\omega_{\widetilde{D}}(x_0, \cdot)$ *for* $\widetilde{D}$ *at* x_0 such that

$$\mathrm{supp}\omega_{\widetilde{D}}(x_0, \cdot) \in \mathcal{M}^+(\partial \widetilde{D}) \subset \mathcal{M}^+(\mathrm{clos} D), \quad \omega_{\widetilde{D}}(x_0, \partial \widetilde{D}) = 1.$$

1.3. Holomorphic Functions

Let $n \in \mathbb{N}$. Denote by $\mathbb{C}^n$ the *n-dimensional Euclidean complex space*. Then $\mathbb{C}^n_\infty := \mathbb{C}^n \cup \{\infty\}$ is the *Alexandroff* ($\Leftrightarrow$one-point) *compactification of* $\mathbb{C}^n$. If it is necessary, we identify $\mathbb{C}^n$ (or $\mathbb{C}^n_\infty$) with $\mathbb{R}^{2n}$ (or $\mathbb{R}^{2n}_\infty$). Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{C}^n_\infty$. We denote by $\mathrm{Hol}(\mathcal{O})$ and $\mathrm{sbh}(\mathcal{O})$ the class of holomorphic and subharmonic functions on $\mathcal{O}$, resp. For $u \in \mathrm{sbh}_*(\mathcal{O})$, the *Riesz measure of* u is the Borel (and the Radon) positive measure

$$\nu_u := c_{2n} \Delta u \in \mathcal{M}^+(\mathcal{O}), \quad c_{2n} := \frac{(n-1)!}{2\pi^n \max\{1, 2n-2\}}. \tag{7}$$

For $k \in \{0\} \cup \mathbb{N}$, we denote by σ_k the *k-dimensional surface* ($\Leftrightarrow$Hausdorff) *measure* on $\mathbb{C}^n$ and its restrictions to subsets of $\mathbb{C}^n$. So, if $k = 0$, then $\sigma_0(S) = \sum_{z \in S} 1$ for each $S \subset \mathbb{C}^n$, i. e. $\sigma_0(S)$ is equal to the number of points in the set $S \subset \mathbb{C}^n$.

Theorema A (see [5, Corollary 1.1] for the case $n = 2$, and [6–8, Corollary 1] for $n > 1$). *Let D be a non-trivial domain in $\mathbb{C}^n_\infty$, K a compact subset of D with* $\mathrm{int} K \neq \varnothing$. *Let $M \in \mathrm{sbh}_*(D)$ be a function with the Riesz measure $\nu_M \in \mathcal{M}^+(D)$, and $v \in \mathrm{sbh}_0^+(D \setminus K)$ a test function for D outside of K. Assume that*

$$\int\limits_{D \setminus K} v \, \mathrm{d}\nu_M < +\infty. \tag{8}$$

Let $f \in \mathrm{Hol}(D)$ *and* $\mathrm{Zero}_f := \{z \in D \colon f(z) = 0\} \supset \mathsf{Z}$. *If*

$$|f| \leq e^M \text{ on } D, \qquad \int\limits_{\mathsf{Z} \setminus K} v \, \mathrm{d}\sigma_{2n-2} = +\infty, \tag{9}$$

then $f = 0$ *on* D, *i.e.* $\mathrm{Zero}_f = D$.

This Theorem A shows that each constructed test function of the class $\mathrm{sbh}_0^+(D \setminus K)$ gives a uniqueness theorem in terms of the distribution of the zero set of holomorphic functions. The main goal of our article is to give some methods for constructing of test functions in the sense of Definition 1 with applications to the distribution of the zero sets of holomorphic functions. Many such constructions have been proposed for domains in the complex plane $\mathbb{C}$ in our work [5, sections 4–5].

2. RADIAL CASE

2.1. Radial Subharmonic Functions

A subset $S \subset R^m_\infty$ is *radial,* if from the conditions $x \in S$ and $|x'| = |x|$ it follows that $x' \in S$. A function f on radial set S is *radial,* if $f(x) = f(x')$ for all $|x| = |x'|$, $x \in S$. By $\mathrm{im} f$ denote the image of f. Further

$$\mathrm{sp} S := \{|x| \colon x \in S\}, \quad \mathrm{sp}_f \colon \mathrm{sp} S \to \mathrm{im} f, \quad \mathrm{sp}_f(r) := f\bigl(|x|\bigr) \text{ for } r = |x|,$$

is *the spherical projection* of radial function f on radial set S. Let $0 \leq r_1 < r_2 \leq +\infty$ and $h \colon (r_1, r_2) \to \mathbb{R}$ be a strictly increasing function. A function $f \colon (r_1, r_2) \to \mathbb{R}$ is *convex of* h if the function $f \circ h^{-1}$ is convex on $\bigl(h(r_1), h(r_2)\bigr) \subset \mathbb{R}$. Given $t \in \mathbb{R}^+_*$, we set

$$h_m(t) := \begin{cases} t & \text{for } m = 1, \\ \log t & \text{for } m = 2, \\ -\dfrac{1}{t^{m-2}} & \text{for } m \geq 3, \end{cases} \qquad t \in \mathbb{R}^+_*;$$

$$A(r_1,r_2) := \{x \in \mathbb{R}^m \colon r_1 < |x| < r_2\}. \tag{10}$$

Proposition 1. *Let $Q\colon A(r_1,r_2) \to \mathbb{R}$ be a radial function and $q := \mathrm{sp}_Q$. The following five conditions are equivalent:*

I. *The function Q is subharmonic on $A(r_1,r_2)$, $Q \neq -\infty$;*

II. *The function q is convex of h_m on $\bigl(h_m(r_1), h_m(r_2)\bigr) \subset \mathbb{R}$;*

III. *The function q has the following properties:* i) *q is continuous;* ii) *there exist the left derivative q'_{left} and the right derivative $q'_{\text{right}}(r)$;* iii) *q'_{left} is continuous on the left, and q'_{right} is continuous on the right;* iv) *the functions $r \mapsto r^{m-1}q'_{\text{left}}(r)$, $r \mapsto r^{m-1}q'_{\text{right}}(r)$ are increasing;* v) $q'_{\text{left}} \leq q'_{\text{right}}$ on (r_1,r_2); vi) *there is a no-more-than countable set $R \subset (r_1,r_2)$ such that $q'_{\text{left}} = q'_{\text{right}}$ on $(r_1,r_2) \setminus R$;*

IV. *For any $r_0 \in (r_1,r_2)$ there is an increasing function $p_0\colon (r_1,r_2) \to \mathbb{R}$ such that*

$$q(r) = q(r_0) + \int\limits_{r_0}^{r} \frac{p_0(t)}{t^{m-1}}\,\mathrm{d}t, \quad r \in (r_1,r_2),$$

where the function p_0 can be chosen in the form

$$p_0(r) := r^{m-1}q'_{\text{left}}(r) \quad \text{or} \quad p_0(r) := r^{m-1}q'_{\text{right}}(r), \quad r \in (r_1,r_2);$$

V. *The function q is upper semicontinuous, locally integrable on (r_1,r_2), and $r \mapsto \bigl(r^{m-1}q'(r)\bigr)'$ is a positive distribution (measure).*

The proof is omitted (see [1]–[4] and [5, § 4] for $m = 2$ or $\mathbb{C}$).

2.2. Radial Test Functions

Let $0 < r_0 < R \in \mathbb{R}^+_{+\infty}$. The following statement describes all radial test functions for the domain $D = B(R) \subset \mathbb{R}^m_\infty$. Recall that $B(+\infty) = \mathbb{R}^m$.

Proposition 2. *Let $v\colon B(R) \setminus \overline{B}(r_0) \to \mathbb{R}^+$ be a radial function on $B(R) \setminus \overline{B}(r_0)$. The following three conditions are equivalent:*

1. *The function v is a test function for $B(R)$ outside of $\overline{B}(r_0)$;*
2. *There is a decreasing function $d\colon (r_0,R) \to \mathbb{R}^+$ such that*

$$v(x) = \int\limits_{|x|}^{R} \frac{d(t)}{t^{m-1}}\,\mathrm{d}t < +\infty, \quad x \in B(R) \setminus \overline{B}(r_0);$$

3. *The function $\mathrm{sp}_v \circ h_m^{-1}$ is convex on $\bigl(h_m(r_0), h(R)\bigr)$ and*

$$\lim_{h_m(R) > x \to h_m(R)} \mathrm{sp}_v(x) = 0.$$

Proof. If we apply the inversion and the Kelvin transform from (1)–(2) to the extended function (6) with $D = B(R)$ and $K = B(r_0)$, then the equivalences 1⇔2 and 1⇔3 follow from the equivalences I⇔IV and I⇔II of Proposition 1 respectively. □

We can easily add other equivalences to Proposition 2 based on Proposition 1.

2.3. Cases $D = B(R)$

The following result follows immediately from Theorem A and Proposition 1 (see [5, § 4] for $m = 2$ or $\mathbb{C}$).

Theorem 1. *Let* $0 < r_0 < R \in \mathbb{R}^+_\infty$. *Let* $M\colon B(R) \to \mathbb{R}$ *be a continuous radial function and* $q := \mathrm{sp}_M$. *Suppose that* q *is convex of* h_{2n} *on* $(0, R)$ *and there is a decreasing function* $d\colon (r_0, R) \to \mathbb{R}^+$ *such that*

$$\int\limits_{r_0}^{R} d(r) q'_{\text{right}}(r)\mathrm{d}r \overset{(8)}{<} +\infty.$$

If the function $f \in \mathrm{Hol}\bigl(B(R)\bigr)$ *with zero set* $\mathrm{Zero}_f \supset \mathtt{Z}$ *satisfies the conditions* (see 9)

$$|f| \le e^M \text{ on } B(R), \quad \int\limits_{r_0}^{R} d(r) s_{\mathtt{Z}}(r)\frac{\mathrm{d}r}{r^{2n-1}} = +\infty,$$

where $s_{\mathtt{Z}}(r) = \sigma_{2n-2}\bigl(Z \cap B(r)\bigr)$, $r \in (r_0, R)$, *then* $f = 0$ *on* D.

The proof is a direct computation of (8)–(9) for radial case with $D = B(R)$ and $K = \overline{B}(r_0)$ using the integration by parts. So, in (8),

$$\mathrm{d}\nu_M(rz){=}c_{2n}\mathrm{d}\bigl(r^{m-1}q'_{\text{right}}(r)\bigr) \otimes \mathrm{d}\sigma_{2n-1}(z), \quad z \in \partial B(1),$$

and we consider the test function v from Proposition 2 (2), in (9),

$$\int\limits_{\mathtt{Z}\setminus B(r_0)} v\mathrm{d}\sigma_{2n-2} = \int\limits_{r_0}^{R} d(r)s_{\mathtt{Z}}(r)\frac{\mathrm{d}r}{r^{2n-1}} - s_{\mathtt{Z}}(r_0)\int\limits_{r_0}^{R}\frac{d(t)}{t^{m-1}}\mathrm{d}t.$$

Radial test functions can also be considered for sets $A(r_1, r_2) \overset{(10)}{\subset} \mathbb{C}^n$. But it is not of interest for holomorphic functions on $A(r_1, r_2) \subset \mathbb{C}^n$ in view of the classical Hartogs extension phenomenon. Here we do not consider also holomorphic functions on polydiscs in $\mathbb{C}^n$, $n > 1$.

3. GREEN'S CASE

Throughout this section 3 $D \subset \mathbb{R}^m_\infty$ is a regular domain with Green's function $g_D := g_D(\cdot, x_0)$ (with the pole at $x_0 \in D$). We set

$$D_t := \{x \in D\colon g_D(x, x_0) > t\} \ni x_0, \quad 0 < t \le t_0 \in \mathbb{R}^+_*. \tag{11}$$

3.1. Superpositions of Convex Functions and Green's Functions

Proposition 3. *Let* $q\colon [0, t_0) \to \mathbb{R}^+$ *be a convex function such that* $q(0) = 0$. *Then the superposition* $q \circ g_D$ *is a test function for* D *outside of* D_{t_0}, *i.e.* $q \circ g_D \in \mathrm{sbh}^+_0(D \setminus \overline{D}_{t_0})$.

Proof. The superposition of convex function f and harmonic function $g_D(\cdot, x_0)$ is subharmonic. For $v := f \circ g_D$, the condition (4) follows from the condition $f(0) = 0$, since the Green's function $g_D(\cdot, x_0)$ vanishes on the boundary ∂D of regular domain D. □

Proposition 4. *Let* $F\colon (-\mathbb{R}^+_*) \to \mathbb{R}^+$ *be a convex increasing function,* $F(-\infty) := \lim\limits_{x\to-\infty} F(x) \in \mathbb{R}_{-\infty}$, *where* $(-\mathbb{R}^+_*) := \mathbb{R}_{-\infty} \setminus \mathbb{R}^+$. *Then the superposition* $F \circ (-g_D)$ *is subharmonic on* D.

Proof. Obviously, the function $-g_D(\cdot, x_0)$ is subharmonic on D. The superposition of convex increasing function F and subharmonic function $-g_D(\cdot, x_0)$ is subharmonic on D. □

3.2. A Uniqueness Theorem with Green's Functions

For simplicity, we assume that the boundaries ∂D and ∂D_t of D_t from (11) belong to the class C^2.

Theorem 2 [see [5, Theorem 7] for $n = 1$]. *Suppose that the functions q and F are the same as in Propositions* 3 *and* 4. *Let $q \in C^1(0, t_0)$ and $F \in C^1(-\mathbb{R}_*^+)$, and*

$$\int_0^{t_0} q'(t)F'(-t)\mathrm{d}t < +\infty. \tag{12}$$

If the function $f \in \mathrm{Hol}(D)$ with zero set $\mathrm{Zero}_f \supset \mathtt{Z}$ satisfies the conditions

$$|f| \leq \exp\bigl(F \circ (-g_D)\bigr) \text{ on } D, \quad \int_0^{t_0} q'(t)s_{\mathtt{Z},D}(t)\mathrm{d}t = +\infty, \tag{13}$$

where $s_{\mathtt{Z},D}(t) = \sigma_{2n-2}\bigl(Z \cap D_t\bigr)$, $t \in (0, t_0)$, then $f = 0$ on D.

Proof. Let ν_M be the Riesz measure of $M := F \circ (-g_D) \in \mathrm{sbh}(D)$, and $v := q \circ g_D$. We have the following equalities:

$$\nu_M(D_{t_2} \setminus \overline{D}_{t_1}) = F'(-t_2) - F'(-t_1), \quad -t_0 < -t_1 < -t_2 < 0 \quad [5, 6.2.1],$$

$$\int_{D \setminus \overline{D}_{t_0}} v\mathrm{d}\nu_M = \int_{D \setminus \overline{D}_{t_0}} (q \circ g_D)\mathrm{d}\nu_M = \int_0^{t_0} q(t)\mathrm{d}\bigl(-F'(-t)\bigr), \tag{14}$$

$$\int_{\mathtt{Z} \setminus \overline{D}(t_0)} v\mathrm{d}\sigma_{2n-2} = \int_{\mathtt{Z} \setminus \overline{D}(t_0)} (q \circ g_D)\mathrm{d}\sigma_{2n-2} = \int_0^{t_0} q(t)\mathrm{d}s_{\mathtt{Z},D}(t). \tag{15}$$

Next we apply the integration by parts to the right-hand sides of (14)–(15) and Theorem A with $K = \overline{D}_{t_0}$. □

Remark 1. The conditions to the boundaries ∂D and ∂D_t can be considerably weakened [9]. In addition, if we replace the derivatives q', F' by q'_{right}, F'_{right} in (12) and by q'_{left} in (13) respectively, we can remove the conditions $q \in C^1(0, t_0)$ and $F \in C^1(-\mathbb{R}_*^+)$.

We will provide a more general and subtle results on the test functions and their construction elsewhere.

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