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Hermitian Finite Element Complementing the Bogner–Fox–Schmit Rectangle Near Curvilinear Boundary

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Abstract—We discuss the Hermitian finite elements of high-order accuracy for solving boundary value problems for partial differential equations in domains with curvilinear boundaries. New elements are constructed in such a way that they can be used in conjunction with the Bogner–Fox–Schmit rectangular elements.

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1. INTRODUCTION

The finite elements with inter-elemental continuous differentiability are more complicated than those providing only continuity. Such two-dimensional elements are mostly developed for triangles: the Argyris triangle [1], the Bell reduced triangle [2], the family of Morgan–Scott triangles [3], the Hsieh–Clough–Tocher macrotriangle [4], the reduced Hsieh–Clough–Tocher macrotriangle [5], the family of Douglas–Dupont–Percell–Scott triangles [6], the Powell–Sabin macrotriangles [7]. The Fraeijs de Veubeke–Sander quadrilateral [8] and its reduced version [9] are also composed of triangles. As for single, non-composite rectangles, the Bogner–Fox–Schmit (BFS) element [10] is the most popular and simplest one in the family of elements by Zhang [11]. All these elements are widely used in the conforming finite element method for the biharmonic equation and other fourth-order equations (see [12, 13, 17–21] and references therein) along with mixed statements of problems and a nonconforming approach [12, 17, 20].

Using the terminology of paper [12], we shall describe finite elements as the triple (e, P_e, Σ_e) consisting of a cell e, a space of functions P_e, and a set of degrees of freedom Σ_e.

Denote by P_k with positive *integer* k the space of all polynomials in two variables of full degree k: $\sum_{0\le i+j\le k} a_{i,j}x^iy^j$. And denote by Q_k the space of all polynomials in two variables of degree k for each variable: $\sum_{0\le i,j\le k} a_{i,j}x^iy^j$.

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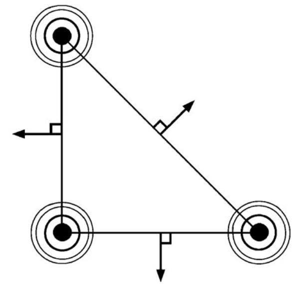

Fig. 1. The Argyris P_5-triangle. The bold point at vertex means a free value of a function in DoF, the first bold circle around it means 2 free first-order derivatives in DoF, and the next double circle around it means 3 free second-order derivatives in DoF. Arrow at the midpoint of a side means the free normal derivative in DoF.

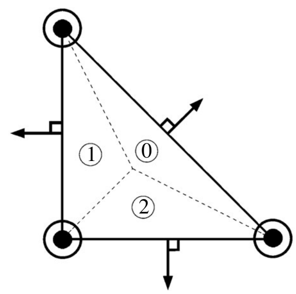

Fig. 2. The Hsieh–Clough–Tocher composite element combining 3 triangles.

2. TRIANGULAR C^1-ELEMENTS

The first C^1-element (with inter-element continuity of trial functions and their first derivatives) was the Argyris P_5-triangle (Fig. 1) with a triangle e, a space of functions P_5, and 21 degrees of freedom (DoF): 6 ones (u, $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$, $\partial^2 u/\partial y^2$) at each vertex and 3 ones ($\partial u/\partial n$) at the midpoint of each side where $\partial u/\partial n$ means the derivative in the direction of an external normal. Later some restrictions on the polynomial degree along sides were applied to reduce the number of DoF [2, 3] without loss of inter-elemental C^1-smoothness. Nevertheless, the number of DoF stays great enough.

Composite-triangular C_1-elements have a less number of DoF. For example, the Hsieh–Clough–Tocher macro-triangle consists of 3 triangles (Fig. 2) on each of them the space of functions is P_3.

For 30 coefficients of three polynomials, 18 restrictions are imposed for C_1-continuity along internal sides of constituting triangles. Therefore only 12 DoF remain: three ($u, \partial u/\partial x$, $\partial u/\partial y$) at each vertex and one ($\partial u/\partial n$) at the midpoint of each side.

3. QUADRANGULAR C^1-ELEMENTS

Now consider the composite-quadrangular C^1-elements by Fraeijs de Veubeke and Sander. It is defined with the help of 3 (piecewise) cubic functions $w = w^a + w^b + w^c$, where $w^a \in P_3$ with 10 coefficients provides 10 degrees of freedom (Fig. 3(a)). The function w^b is defined in a piecewise manner. In the lower triangle in Fig. 3(b), it is assumed to be zero; in the upper triangle, a cubic expression $w^b = a_1 y'^2 + a_2 y'^3 + a_3 x'^2$ with three coefficients satisfies the continuity of values and first-order derivatives along the diagonal (the Ox' axis) and still provides 3 DoF. Similarly (Fig. 3(c)), the third function w^c equals 0 in the lower triangle; in the upper one, we put $w^c = b_1 y''^2 + b_2 y''^3 + b_3 x'' y''^2$ that satisfies the continuity of values and first-order derivatives along another diagonal (the Ox'' axis) and provides 3 more degrees of freedom.

The most simple rectangular C^1-elements is the Bogner–Fox–Schmit one (Fig. 4). Its space of function is Q_3. 16 coefficients of polynomial generate 16 DoF: 4 at each vertex of a rectangle ($u, \partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x \partial y$). This element is the most efficient by the criteria of [24] among all those

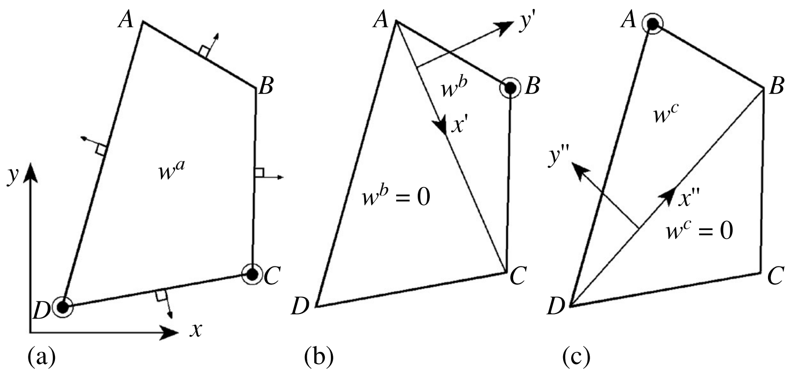

Fig. 3. The Fraeijs de Veubeke–Sander element.

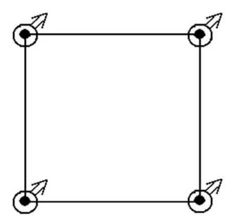

Fig. 4. The Bogner–Fox–Schmit element. The double arrow at vertex means free mixed second-order derivative.

containing the polynomial space P_3 to provide the corresponding order of approximation. Recall that the global number of DoF for an approximate solution u^h on the triangulation is not proportional to the number of DoF of one element multiplied by the number of cells. A part of DoF coincides along inter-elemental boundaries. Therefore, we suggested the number M of DoF of an element on the half-closed quadrangular cell as the local characteristic for the global number of DoF. This follows from the proportionality of the number AM of equations or unknowns in the finite element method to the number M with a factor A depending only on the size of a domain for the same meshsize h of its rectangular triangulation. For example, $M = 9$ for two Argiris triangles, $M = 6$ for two Hsieh–Clough–Tocher composite triangles, $M = 5$ for the Fraeijs de Veubeke–Sander composite element, and $M = 4$ for the Bogner–Fox–Schmit element.

However, this simple and efficient element has a significant drawback. It may not be used without application of other elements near the curvilinear boundary. For three previous elements, the isoparametric transformation may be used for the approximation of the curvilinear boundary. But for two neighboring Bogner–Fox–Schmit elements, the isoparametric transformation destroys inter-elemental continuity of first-order derivatives. It is therefore proposed to use them on a rectangular grid complementing rectangular cells by curvilinear triangles [23] or trapezoids near the boundary.

4. TRAPEZOIDAL ELEMENTS COMPLEMENTING THE BFS-ELEMENT

First, consider the 'reference' element on the trapezoidal cell ABCD with vertices A=(0,0), B=(1,0), C=(1,a), and D=(0,1) where the straight side DC (Fig. 5) is supposed to be mapped on a part of the boundary of the domain. We subdivide the trapezium by the diagonal AC into two triangles. In the triangle ABC, we define the polynomial space $Q_3' \subset Q_3$ with 15 free coefficients where the coefficient of x^3y^3 identically equals zero. And in the triangle ACD we define a polynomial space $Q_3' \cup L\{x^4\}$ with 16 free coefficients where $L\{x^4\}$ means the linear span of the set in braces. Totally two polynomials have 31 coefficients. For C^1-coalescence of this element with adjacent elements of the BFS-type, it is necessary to define four DoF $u, \partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x\partial y$ at each vertex that gives totally 16 DoF. But at the vertices A and C, the coincidence of these DoF in two triangles imposes 8 more restrictions. Two more DoF are taken as values of a function u on the 'boundary' segment CD for improved approximation of the boundary values. So we get 18 DoF and 8 restriction. Note that along the diagonal AC, the

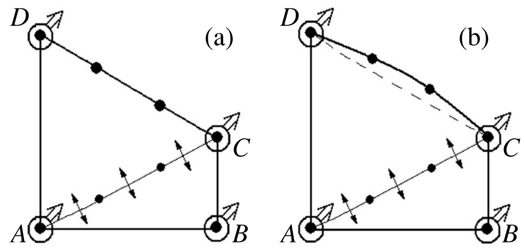

Fig. 5. Composite quadrangle with straight (a) and curved (b) side.

traces of the two-dimensional polynomials are one-dimensional polynomials of degree 5. 6 conditions are necessary to match their values. Four of them (two values and two derivatives along the diagonal at the nodes A and C) are derived from the definition of DoF for this element. Two more conditions are taken as coincidence of values at two additional points on the diagonal with uniform arrangement (Fig. 5). This ensures the continuity of the function and its first-order derivative along the diagonal. In each triangle along the diagonal, the derivative in the orthogonal direction is a fourth-degree polynomial. For its coincidence on both sides, it is necessary to impose 5 conditions. Two of them are derived from the definition of DoF at the nodes A and C; and we take three more ones as coincidence of the derivative normal to the diagonal at three additional points with uniform arrangement (Fig. 5a). As a result, we have 18 DoF and 13 restrictions which equals the amount of the coefficients of two polynomials.

In fact, to prove the unique solvability [12] of this element, one should prove the solvability of the corresponding systems of linear algebraic equations for the construction of each of 18 basis functions. Instead of this, we omit checking the solvability of these systems and write out their solutions.

For example, the first basis function (which is equal to 1 at the node A with all other DoF being vanished) has the form

$$\varphi_{1,\text{low}}(x,y) = 1 - 3x^2 - 3y^2 + 2x^3 - (3a^3 - 9a^2 - 3a + 3)xy^2/a^2 \\ + (3a^2-3)y^3/a^2 + (6a^3 - 9a^2 - 6a + 6)x^2y^2/a^2 - (6a^2-6)xy^3/a^2 \\ - (3a^3 - 3a^2 - 3a + 3)x^3y^2/a^2 + (3a^2-3)x^2y^3/a^2$$

in the lower triangle ABC and

$$\varphi_{1,\text{up}}(x,y) = 1 - 3x^2 - 3y^2 + (a^3 - 9a^2 + 3a + 5)x^3 - (3a^3 - 18a^2 + 3a + 6)x^2y/a + 2y^3 \\ + (18a^3 - 18a^2 - 6a + 6)x^3y/a - (12a^3 - 6a - 3)x^2y^2/a^2 \\ - (3a^3 - 3a^2 - 3a + 3)x^3y^2/a^2 + (3a^2-3)x^2y^3/a^2 - (6a^3 + 9a^2 - 3)x^4$$

in the upper triangle ACD.

Then the second basis function (which satisfies $\partial\varphi_2/\partial x = 1$ at the node A with all other DoF being vanished) has the form

$$\varphi_{2,\text{low}}(x,y) = y - 2x^2 + x^3 + (a-1)xy^2/a^2 - y^3/a^2 - (2a-2)x^2y^2/a^2 \\ + 2xy^3/a^2 + (a-1)x^3y^2/a^2 - x^2y^3/a^2$$

in the lower triangle ABC and

$$\varphi_{2,\text{up}}(x,y) = y - 2x^2 + (-3a^2 + a + 2)x^3 + (6a^2 - a - 2)x^2y/a - 3xy^2 \\ + (6a^3 - 6a^2 - 2a + 2)x^3y/a + (-6a^3 - 3a^2 - 2a - 1)x^2y^2/a^2 + 2xy^3 \\ + (a-1)x^3y^2/a^2 + (-2a^3 + 3a^2 - 1)x^4$$

in the upper triangle ACD.

The third basis function (which satisfies $\partial\varphi_3/\partial y = 1$ at the node A with all other DoF being vanished) has the form

$$\varphi_{3,\text{low}}(x,y) = y - 2y^2 - \frac{8a^5 + 2a^4 + 443a^3 - 222a^2 - 42a + 54}{4a^4 - 8a^3 + 4a^2}xy^2$$

$$
\begin{aligned}
&+\frac{8a^4+26a^3+275a^2-12a-54}{4a^4-8a^3+4a^2}y^3\\
&-3x^2y+2x^3y+\frac{8a^5+14a^4+419a^3-210a^2-42a+54}{2a^4-4a^3+2a^2}x^2y^2\\
&-\frac{8a^4+26a^3+275a^2-12a-54}{2a^4-4a^3+2a^2}xy^3\\
&-\frac{8a^5+18a^4+411a^3-206a^2-42a+54}{4a^4-8a^3+4a^2}x^3y^2+\frac{8a^4+26a^3+275a^2-12a-54}{4a^4-8a^3+4a^2}x^2y^3
\end{aligned}
$$

in the lower triangle ABC and

$$
\begin{aligned}
\varphi_{3,\mathrm{up}}(x,y)={}&y-2y^2-\frac{66a^4+99a^3+198a^2-66a-54}{4a^2-8a+4}x^3\\
&-\frac{4a^5-98a^4+85a^3-432a^2+90a+108}{4a^3-8a^2+4a}x^2y\\
&+y^3+\frac{20a^5-22a^4+459a^3-230a^2-38a+54}{2a^3-4a^2+2a}x^3y\\
&-\frac{14a^5+14a^4+221a^3+81a^2-60a-27}{2a^4-4a^3+2a^2}x^2y^2\\
&-\frac{8a^5+18a^4+411a^3-206a^2-42a+54}{4a^4-8a^3+4a^2}x^3y^2\\
&+\frac{8a^4+26a^3+275a^2-12a-54}{4a^4-8a^3+4a^2}x^2y^3-(3a^3-6a^2+30a+27/2)x^4
\end{aligned}
$$

in the upper triangle ACD.

The fourth basis function (which satisfies $\partial\varphi_4/\partial x\partial y=1$ at the node A with all other DoF being vanished) has the form

$$
\begin{aligned}
\varphi_{4,\mathrm{low}}(x,y)={}&xy-2x^2y+(a-1)xy^2/a-y^3/a+x^3y-(2a-2)x^2y^2/a\\
&+2xy^3/a+(a-1)x^3y^2/a-x^2y^3/a
\end{aligned}
$$

in the lower triangle ABC and

$$
\begin{aligned}
\varphi_{4,\mathrm{up}}(x,y)={}&xy+(a-a^2)x^3+(3a-4)x^2y-2xy^2+(3a^2-6a+3)x^3y\\
&-(3a^2-4a-1)x^2y^2/a+xy^3+(a-1)x^3y^2/a-x^2y^3/a-(a^3+2a^2-a)x^4
\end{aligned}
$$

in the upper triangle ACD.

The basic functions at the other three vertices of the trapezium have a similar form. In principle, for proving the unique solvability it is enough to find at least one basis function, since the coefficients of each of them are found out of the system of linear algebraic equations with the same matrix. And if this matrix is non-singular, then the coefficients of other basic functions are also founded in unique way.

Therefore, we write down only two more basic functions those differ from previous ones and related to DoF on the arc CD. For example, φ_{17} equals 1 in the node $(1/3,(2+a)/3)$ and has zeroth other DoF:

$$
\begin{aligned}
\varphi_{17,\mathrm{low}}(x,y)={}&-\frac{162a^3+243a^2-297a+117}{4a^4-8a^3+4a^2}x^2y\\
&+\frac{162a^3+162a^2-135a+36}{4a^5-8a^4+4a^3}xy^2\\
&+\frac{162a^3+243a^2-297a+117}{2a^4-4a^3+2a^2}xy^3-\frac{162a^3+162a^2-135a+36}{2a^5-4a^4+2a^3}x^3y^2\\
&-\frac{162a^3+243a^2-297a+117}{4a^4-8a^3+4a^2}x^2y^3+\frac{162a^3+162a^2-135a+36}{4a^5-8a^4+4a^3}x^2y^2
\end{aligned}
$$

in the lower triangle ABC and

$$\varphi_{17,\mathrm{up}}(x,y) = -x^2 - \frac{162a^3+59a^2+71a-67}{4a^2-8a+4}x^3 + \frac{162a^3+189a-126}{4a^3-8a^2+4a}y^3 + \frac{162a^3+189a^2-189a+63}{2a^3-4a^2+2a}x^3y - \frac{324a^3+189a^2-63}{4a^4-8a^3+4a^2}xy^3 - \frac{162a^3+189a^2-189a+63}{4a^4-8a^3+4a^2}x^2y^3 + \frac{162a^2+126a-63}{4a^4-8a^3+4a^2}x^2y^2 - \frac{63}{4}x^4$$

in the upper triangle ACD.

And finally, φ_{18} equals 1 in the node $(2/3,(1+2a)/3)$ and has zeroth other DoF:

$$\varphi_{18,\mathrm{low}}(x,y) = \frac{81a^3+486a^2-432a+126}{4a^4-8a^3+4a^2}x^2y - \frac{81a^3+324a^2-108a-36}{4a^5-8a^4+4a^3}xy^2 - \frac{81a^3+486a^2-432a+126}{2a^4-4a^3+2a^2}xy^3 + \frac{81a^3+324a^2-108a-36}{2a^5-4a^4+2a^3}x^3y^2 + \frac{81a^3+486a^2-432a+126}{4a^4-8a^3+4a^2}x^2y^3 - \frac{81a^3+324a^2-108a-36}{4a^5-8a^4+4a^3}x^2y^2$$

in the lower triangle ABC and

$$\varphi_{18,\mathrm{up}}(x,y) = -x^2 + \frac{81a^3+184a^2+172a-176}{4a^2-8a+4}x^3 - \frac{81a^3+540a-360}{4a^3-8a^2+4a}y^3 - \frac{81a^3+540a^2-540a+180}{2a^3-4a^2+2a}x^3y + \frac{81a^3+270a^2-90}{2a^4-4a^3+2a^2}xy^3 + \frac{81a^3+540a^2-540a+180}{4a^4-8a^3+4a^2}x^2y^3 - \frac{81a^2+360a-180}{4a^4-8a^3+4a^2}x^2y^2 + 45x^4$$

in the upper triangle ACD.

Now consider the case of curved side CD. In this case, the construction of the basis functions differs from the previous description only in selecting two points on the arc CD (Fig. 5b). It is clear that the basis functions are not exactly the same as listed above; so it is necessary to solve a system of linear algebraic equations to determine them. The quantitative factor for these calculations is a relatively small number of $O(h^{-1})$ of such elements (that are located along the boundary of the domain) in comparison with the number of $O(h^{-2})$ of the standard BFS-elements inside the domain. As regards the solvability of these systems, it is guaranteed at least by a slight deviation of the selected points on the arc CD from those on the straight segment due to the continuous dependence of the coefficients of systems on the coordinates of these points. A small difference is performed automatically and tends to zero for small h and a smooth section of the approximated boundary since the maximal distance from the smooth arc to a chord joining its ends is of $O(h^2)$.

Note that the proposed finite element is one of a family of three possible elements of this type. In fact, in the triangle ACD, we can take a more complex polynomial space $Q_3' \cup L\{x^4, x^4y\}$ or $Q_3' \cup L\{x^4, x^4y, x^5\}$. This does not violate the constructions presented above since a one-dimensional polynomial along the diagonal AC is still of the 5th degree. In this case, there is a possibility to introduce additional DoF on the boundary segment CD. For example, we can take one or two additional values of the normal derivative on the segment CD to improve the approximation of the boundary conditions.

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