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# Design of Optimal Control for Motions of Elastic Bodies: Variational Approaches

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**Abstract**—For the forced motion of elastic bodies, we provide the variational and projection statements of initial-boundary problems. In the framework of the spatial linear model, we investigate the optimal control problem for an elastic rectilinear beam with a rectangular cross-section. Using the proposed generalized formulations, we develop a design algorithm for optimal displacements of elastic beams. Results of the numerical simulation and the analysis of the dynamics are provided.

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## INTRODUCTION

Usually, systems with distributed parameters are described by systems of partial differential equations; in special cases, integral or integrodifferential relations might be included. Also, functionals of the unknown variables might be included. Such a functional achieves its stationary value at an admissible set of functions corresponding to the stationary point, i. e., the sought solution of the problem. Usually, this is caused by the problem formulation based on the corresponding variational principle with a certain physical interpretation. In special cases, the solution might correspond to an extremum of the functional. Variational principles are especially important because the main equations describing the medium behavior directly follow from the corresponding principles: the said equations are the stationary conditions for the corresponding functional. Also, variational problem statements have a number of advantages compared with partial differential problem statements.

First, the variational technique is suitable to transform a problem originally posed as a partial differential one to an equivalent problem such that it is frequently easier to solve it than the original one. If we have a variational statement with additional constraints, then the problem is usually transformed by means of the method of Lagrange multipliers; this procedure is very efficient and regular. Thus, one can obtain pairwise equivalent families of variational principles.

Secondly, if it is not possible to find a precise solution of the problem, then the variational method frequently provided various finite-dimensional formulations to find approximate solutions.

Finally, the application of variational principles guarantees that the numerical algorithms are stable and the approximate solutions are optimal. Usually, the resulting system of equations is symmetric and positive definite. Disadvantages of the variational approach exist as well. For example, variational principle formulations are possible not for all problems of mathematical physics. Frequently, it is rather hard to construct reliable quality estimates for the solution. If we use, e. g., the Hu—Washizu elasticity principle to find approximate solutions of a variational problem posed by means of Lagrange multipliers, then the positive definiteness and symmetry of the problem are lost.

In the present paper, we discuss the method of integrodifferential relations (see [1]). This approach has a number of advantages and takes into account disadvantages of variational methods, projection

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methods, and the method of least squares. The sense of this approach is as follows. Several control equations are satisfied precisely, while other relations (a priori selected from physical considerations) are taken into account in their integral form. To find an approximate solution of the integrodifferential problem, we minimize a quadratic functional under differential constraints (such as equilibrium equations, kinematic relations, and boundary-value conditions). Such a statement is completely coordinated with the ideas of the method of least squares; however, it is a variational principle as well.

For various variational statements implied by the method of integrodifferential relations, we propose two-side energy estimates of the quality of the approximate solution. We construct finite-element algorithms both for the verification of the error of the mathematical model and for the adaptive refinement of finite-element nets to improve the solution quality.

Following the ideas of the method of integrodifferential relations, we construct a projection approach as a modification of the Petrov–Galerkin method. Using semidiscrete polynomial approximations and the projection technique, one can obtain high-precision solutions of three-dimensional static and dynamic elasticity problems.

Approaches discussed in the present work (on the example of the motion optimization for elastic bodies) are applied to static and spectral elasticity problems (see [2]) and to direct and inverse initial-boundary value problems of mechanics of deformable solid bodies (see [3]), hydrodynamics, and thermodynamics (see [4, 5]).

## 1. THE MOTION PROBLEM FOR ELASTIC BODIES: A VARIATIONAL STATEMENT

Consider a three-dimensional elastic body occupying a bounded domain $V$ with a piecewise-smooth boundary $\Gamma$. Introduce dynamic variables $\sigma(t, \mathbf{x})$ and $\mathbf{p}(t, \mathbf{x})$ and kinematic variables $\mathbf{w}(t, \mathbf{x})$ and $\varepsilon(t, \mathbf{x})$; they characterize the behavior of the elastic system and depend on the time $t \in [0, T]$ and the vector $\mathbf{x} = (x_1, x_2, x_3) \in V$ of the spatial coordinates. Here the vector functions $\mathbf{p}$ and $\mathbf{w}$ are the momentum density and displacement respectively, while $\sigma$ and $\varepsilon$ are second-rank tensors determining the spatio-temporal distribution of elastic stresses and strains. Also, define the spatio-temporal domain $\Omega = (0, T) \times V$.

In the linear elasticity theory, the local medium state equations linking the velocities $\mathbf{w}_t$ of points of the system with the function $\mathbf{p}$ of the momentum density and the strains $\varepsilon$ with the stresses $\sigma$ can be written as follows (see [6]):

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{0}, \quad \xi(t, \mathbf{x}) = \mathbf{0}, \quad (t, \mathbf{x}) \in \overline{\Omega}. \tag{1}$$

Here

$$\mathbf{v} = \mathbf{w}_t - \rho^{-1}(\mathbf{x})\mathbf{p} \tag{2}$$

is the residual vector-function with respect to velocities, while

$$\xi = \varepsilon - \mathsf{C}^{-1}(\mathbf{x}) : \sigma. \tag{3}$$

is the residual tensor-function with respect to strains. The strain tensor $\varepsilon$ linearly depends on the movement vector:

$$\varepsilon = \frac{1}{2}(\nabla \mathbf{w} + \nabla \mathbf{w}^{\mathrm{T}}). \tag{4}$$

The solid density $\rho$ of the body and the tensor $\mathsf{C}$ of elasticity modules are given functions of the coordinates $\mathbf{x}$. The fourth-rank tensor $\mathsf{C}$ is such that its components possess the following symmetry properties: $C_{ijkl} = C_{ijlk} = C_{klij}$. The sign ":" between the tensors (in the notation) denotes their scalar product (the double convolution with respect to indices). In equation (4), we use the gradient operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ in the space of $\mathbf{x}$-coordinates.

Using relations (1) and (4), one can express the variation law for the momentum density of an elastic body via the stress tensor $\sigma$ and the momentum-density vector $\mathbf{p}$:

$$\mathbf{p}_t(t, \mathbf{x}) = \nabla \cdot \sigma(t, \mathbf{x}), \quad (t, \mathbf{x}) \in \Omega. \tag{5}$$

It is assumed that there are no external volume forces. The operation between the vector and the tensor at the right-hand part of (5) is the convolution with respect to one index.

Consider the case where the boundary-value conditions can be represented via components of the displacement vector $\mathbf{w}$ and the external-load vector $\mathbf{q} = \sigma \cdot \mathbf{n}$ as follows:

$$\mathbf{w}(t,\mathbf{x}) = \mathbf{w}_0(t,\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \qquad (6)$$
$$\mathbf{q}(t,\mathbf{x}) = \mathbf{q}_0(t,\mathbf{x}), \quad \mathbf{x} \in \Gamma_2.$$

Here $\mathbf{n}$ is the unit vector of the outward normal to the boundary $\Gamma$, $\mathbf{w}_0$ and $\mathbf{q}_0$ are given boundary vectors of movements and stresses, and $\Gamma_1$ and $\Gamma_2$ are disjoint parts of the boundary, i. e., $\Gamma_1 \bigcap \Gamma_2 = \emptyset$ and $\overline{\Gamma}_1 \bigcup \overline{\Gamma}_2 = \Gamma$.

To describe the motion of an elastic body completely, it is necessary to define its initial state (without loss of generality, one can assign $t = 0$ for that state). To do that, one can define initial distributions of elastic displacements $\mathbf{w}^0$ and the momentum density $\mathbf{p}^0$ as sufficiently smooth functions of $\mathbf{x}$-coordinates:

$$\mathbf{w}(0,\mathbf{x}) = \mathbf{w}^0(\mathbf{x}), \quad \mathbf{p}(0,\mathbf{x}) = \mathbf{p}^0(\mathbf{x}), \quad \mathbf{x} \in V. \qquad (7)$$

The proposed approach changes equations (1), which are local state equations of an elastic body, for an integral relation linking the vector $\mathbf{p}$ of the momentum density and the vector $\mathbf{w}_t$ of the velocity as well as the stress tensor $\sigma$ and the deformation tensor $\varepsilon$.

In [3], the following integrodifferential posing of (1)–(7), which is an initial-boundary motion problem for an elastic body, is proposed: to find a displacement field $\mathbf{w}^*(t,x)$, stress field $\sigma^*(t,x)$, and momentum density $\mathbf{p}^*(t,x)$ such that the integral relation

$$\Phi[\mathbf{w},\sigma,\mathbf{p}] = \int_\Omega \varphi(t,\mathbf{x})\, \mathrm{d}\Omega = 0, \quad \varphi = \frac{1}{2}\left(\rho(\mathbf{x})\mathbf{v}\cdot\mathbf{v} + \xi : \mathsf{C}(\mathbf{x}) : \xi\right), \qquad (8)$$

is satisfied provided that the kinematic equation (4), the momentum variation equation (5), and the boundary-value and initial-value conditions (6)-(7) are satisfied.

The integrand function $\varphi$ in (8) has the energy-density dimension and is nonnegative. The latter property implies the nonnegativity of the integral $\Phi$ for arbitrary functions $\mathbf{w}$, $\sigma$ and $\mathbf{p}$; thus, the integrodifferential problem (4)–(7), (8) can be reduced to the following minimization statement: to find admissible functions $\mathbf{w}^*$, $\sigma^*$, and $\mathbf{p}^*$ providing the least (zero) value of the functional

$$\Phi[\mathbf{w}^*,\sigma^*,\mathbf{p}^*] = \min_{\mathbf{w},\sigma,\mathbf{p}} \Phi[\mathbf{w},\sigma,\mathbf{p}] = 0 \qquad (9)$$

under constraints (4)–(7).

Denote actual and arbitrary selected admissible displacements, stresses, and momenta by $\mathbf{w}^*$, $\sigma^*$, and $\mathbf{p}^*$ and $\mathbf{w}$, $\sigma$, and $\mathbf{p}$ respectively. Assign

$$\mathbf{w} = \mathbf{w}^* + \delta\mathbf{w}, \quad \sigma = \sigma^* + \delta\sigma, \text{ and } \mathbf{p} = \mathbf{p}^* + \delta\mathbf{p}.$$

Then, taking into account (9), we have the relation

$$\Phi[\mathbf{w},\sigma,\mathbf{p}] = \Phi[\mathbf{w}^*,\sigma^*,\mathbf{p}^*] + \delta_{\mathbf{w}}\Phi + \delta_\sigma\Phi + \delta_{\mathbf{p}}\Phi + \delta^2\Phi = \Phi[\delta\mathbf{w},\delta\sigma,\delta\mathbf{p}],$$

where $\delta_{\mathbf{w}}\Phi$, $\delta_\sigma\Phi$, and $\delta_{\mathbf{p}}\Phi$ are the first variations of the functional $\Phi$ with respect to the unknowns $\mathbf{w}$, $\sigma$, and $\mathbf{p}$, while $\delta^2\Phi$ is its second variation; note that $\delta^2\Phi \geq 0$.

Using the functional $\Phi$ defined by (8) for arbitrary admissible fields of displacements $\mathbf{w}$, stresses $\sigma$, and momentum density $\mathbf{p}$ satisfying constraints (5)–(7), one can propose various criteria of the closeness to the precise solution. The integral quality of approximate functions $\mathbf{w}$, $\sigma$, and $\mathbf{p}$ can be estimated by the value of the dimensionless relation

$$\Delta = \Phi\Psi^{-1} < \delta \ll 1. \qquad (10)$$

Here $\delta$ is a selected positive number, while the integral $\Psi$ of the full mechanical energy with respect to time is given as follows:

$$\Psi = \frac{1}{2}\int_\Omega \left(\rho^{-1}(\mathbf{x})\mathbf{p}\cdot\mathbf{p} + \varepsilon : \mathsf{C}(\mathbf{x}) : \varepsilon\right) \mathrm{d}\Omega. \qquad (11)$$

For any admissible functions $\mathbf{w}$, $\sigma$, and $\mathbf{p}$, the spatio-temporal distribution of its precision is characterized by the function $\varphi(\mathbf{w},\sigma,\mathbf{p})$ defined by (8).

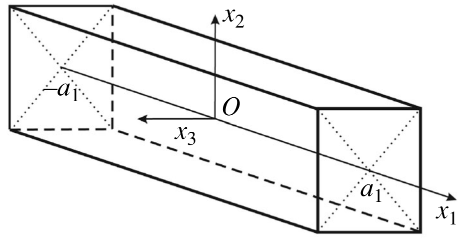


**Fig. 1.** The domain $V$ occupied by an elastic body.

## 2. THE MOTION PROBLEM FOR ELASTIC BODIES: A PROJECTION STATEMENT

Various projection approaches related to the method of integrodifferential relations are also applicable to obtain authentic numerical models of the considered mechanical processes. Those approaches use integral projections of defining relations to a special function space selected to construct a consistent system of equations. In [4], a modification of the method of integrodifferential relations is developed to find the profiles of the temperature and heat flow for one-dimensional heat-exchange problems; the said modification is based on the projection technique and polynomial representations of the unknown functions.

Below, we explain a variant of the Petrov—Galerkin method (see [2]) using integral projections of the velocity residual vector $\mathbf{v}$ and strain residual vector $\xi$ introduced in (1). In that case, the linear elasticity problem is posed as follows: to find admissible fields of the displacements $\mathbf{w}$, stresses $\sigma$, and momentum density $\mathbf{p}$ satisfying the momentum variation law expressed by (5), boundary-value conditions (6), and initial-value conditions (7) and such that the following integral relations are satisfied:

$$\int\limits_{\Omega} \rho(x)\mathbf{v}_t(t,\mathbf{x}) \cdot \mathbf{r}(t,\mathbf{x})\, \mathrm{d}\Omega = 0 \quad \forall\, \mathbf{r} \in L^2(\Omega) \tag{12}$$

and

$$\int\limits_{\Omega} \xi(t,\mathbf{x}) : \tau(t,\mathbf{x})\, \mathrm{d}\Omega = 0 \quad \forall\, \tau \in L^2(\Omega). \tag{13}$$

Here $\mathbf{r}$ is the vector of virtual displacements, $\tau$ is the tensor of virtual stresses, and the vector $\mathbf{v}$ and tensor $\xi$ are defined by (2) and (3) respectively.

## 3. OPTIMAL CONTROL PROBLEM FOR DISTRIBUTED SYSTEMS

Consider an elastic body (beam) shaped as a rectangular parallelepiped of length $2a_1$ and cross-section of sizes $2a_2 \times 2a_3$ such that $a_1 \gg a_2 + a_3$ (see fig. 1). Introduce the Cartesian coordinate system $Ox_1x_2x_3$ such that its origin is located at the middle of the body and its axes $Ox_k$ are parallel to the sides of lengthes $2a_k$, $k = 1, 2, 3$. The (three-dimensional) domain of the problem is defined as follows:

$$V = \{\mathbf{x} : |x_i| < a_i, i = 1, 2, 3\}.$$

Consider the case where the long sides of the beam are load-free:

$$\sigma(t,\mathbf{x}) \cdot \mathbf{e}_n = 0, \quad x_n = \pm a_n, \quad n = 2, 3. \tag{14}$$

Assume that one front cross-section is load-free:

$$\sigma(t,\mathbf{x}) \cdot \mathbf{e}_1 = \mathbf{0}, \quad x_1 = a_1. \tag{15}$$

Also, we assume that another cross-section is not deformed and moves according to the given control law $u(t)$:

$$\mathbf{w}(t,\mathbf{x}) = (0, y_1(t), 0), \quad \ddot{y}_1(t) = u(t), \quad x_1 = -a_1. \tag{16}$$

Here $\mathbf{e}_k = (\delta_{1k}, \delta_{2k}, \delta_{3k})$, $k = 1, 2, 3$, are unit vectors of the coordinate system $Ox_1x_2x_3$ normal to various parts of the boundary of the body $V$ and $\delta_{jk}$ is the Kronecker delta. The displacement of the end section $y_1(t)$ along the axis $Ox_2$ satisfies the following initial-value conditions:

$$\dot{y}_1(0) = y_1(0) = 0. \tag{17}$$

We have to find an optimal control (see [8]), i.e., an acceleration $u^*(t)$ transferring the beam from the initial state of rest

$$\mathbf{w}(0, \mathbf{x}) = \mathbf{0}, \quad \mathbf{p}(0, \mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in V, \tag{18}$$

to the terminal state

$$\mathbf{w}(T, \mathbf{x}) = (0, y_T, 0), \quad \mathbf{p}(T, \mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in V, \tag{19}$$

and minimizing the quality functional

$$J[u^*] = \min_{u \in L^2(0,T)} J[u]. \tag{20}$$

Consider the quadratic functional

$$J = \frac{1}{2} \int_0^T u^2(t)\, \mathrm{d}t + \gamma \Psi, \quad \gamma \geq 0, \tag{21}$$

where $\gamma$ is a given weight coefficient and $\Psi$ is the integral of the energy with respect to time defined by (11).

## 4. A DISCRETIZATION ALGORITHM

In [1], we propose algorithms based on polynomial and piecewise-polynomial approximations of the unknown functions with respect to coordinates and time. Such a complete discretization reduces the generalized initial-boundary problem to a finite-dimensional linear system of algebraic equations with respect to the unknown functions. Under the given requirements to the precision of the solution and the class of admissible solutions, this approach has several constraints for the choice parameters of the problem. For numerical models of motions of elastic bodies, it is frequently more convenient to reduce the original problem to an approximate finite-dimensional system of ordinary differential equations by means of the semidiscretization method (see [2]).

### *4.1. Discretization with respect to spatial variables*

To apply this approach, we exclude the vector-function of the momentum density from our consideration, integrating equation (5) with respect to $t$ and taking into account conditions (18):

$$\mathbf{p}(t, \mathbf{x}) = \int_0^t \nabla \cdot \sigma(\widetilde{t}, \mathbf{x})\, \mathrm{d}\widetilde{t}. \tag{22}$$

Substituting expression (22) in the expression for the vector $\mathbf{v}$ introduced by (2), we obtain the relation

$$\mathbf{v} = \mathbf{w}_t - \rho^{-1} \int_0^t \nabla \cdot \sigma(\widetilde{t}, \mathbf{x})\, \mathrm{d}\widetilde{t}. \tag{23}$$

Once we substitute the Cauchy tensor (4), the residual tensor with respect to deformation, defined by (3), takes the form

$$\xi = \frac{1}{2}\left(\nabla \mathbf{w} + \nabla \mathbf{w}^{\mathrm{T}}\right) - \mathsf{C}^{-1} : \sigma. \tag{24}$$

Define the approximations of the unknown displacement fields and stress fields as follows:

$$w_1(t, \mathbf{x}) = \sum_{k+l=0}^{N} w_1^{(kl)}(t, x_1)\widetilde{x}_2^k\, \widetilde{x}_3^l, \quad w_3(t, \mathbf{x}) = \sum_{k+l=0}^{N-1} w_3^{(kl)}(t, x_1)\widetilde{x}_2^k\, \widetilde{x}_3^l,$$

$$w_2(t,\mathbf{x}) = y_1(t) + \sum_{k+l=0}^{N-1} w_2^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,$$

$$\sigma_{11}(t,\mathbf{x}) = \sum_{k+l=0}^{N} \sigma_{11}^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l, \quad \sigma_{nn}(t,\mathbf{x}) = g_n \sum_{k+l=0}^{N} \sigma_{nn}^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,$$

$$\sigma_{1n}(t,\mathbf{x}) = g_n \sum_{k+l=0}^{N-1} \sigma_{1n}^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\ \sigma_{23}(t,\mathbf{x}) = g_2 g_3 \sum_{k+l=0}^{N-2} \sigma_{23}^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l, \tag{25}$$

$$g_n = 1 - \widetilde{x}_n^2, \quad \widetilde{x}_n = a_n^{-1} x_n, \quad n = 2,3.$$

Here $N$ is a given positive integer defining the degrees of the polynomial expansions of the unknown functions with respect to the dimensionless coordinates $\widetilde{x}_2$ and $\widetilde{x}_3$. Approximations selected this way satisfy conditions (14), which are homogeneous boundary-value conditions on the lateral side of a prism.

Approximations (25) and projection relations (12)-(13) allow us to construct a system of partial differential-algebraic equations with respect to $t$ and $x_1$.

First, we form a group of equations containing partial derivatives with respect to $x_1$. To do that, we compute the projections

$$\int_\Omega \mathbf{v}_t(t,\mathbf{x}) \cdot \mathbf{r}(t,\mathbf{x})\, \mathrm{d}\Omega = 0 \quad \forall\, \mathbf{r} \tag{26}$$

and

$$\int_\Omega \mathbf{e}_1 \cdot \xi(t,\mathbf{x}) \cdot \mathbf{s}(t,\mathbf{x})\, \mathrm{d}\Omega = 0 \quad \forall\, \mathbf{s}, \tag{27}$$

where $\mathbf{r}$ is the vector of virtual momenta and $\mathbf{s} = \tau \cdot \mathbf{e}_1$ is the vector of virtual stresses defined as follows:

$$r_1(t,\mathbf{x}) = \sum_{k+l=0}^{N} r_1^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l, \quad s_1(t,\mathbf{x}) = \sum_{k+l=0}^{N} s_1^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,$$

$$r_n(t,\mathbf{x}) = \sum_{k+l=0}^{N-1} r_n^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l, \quad s_n(t,\mathbf{x}) = \sum_{k+l=0}^{N-1} s_n^{(kl)}(t,x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l, \quad n = 2,3. \tag{28}$$

Equations (26) and (27) can be explicitly resolved with respect to the first derivatives of the functions $\partial w_m^{(kl)}/\partial x_1$ and $\partial \sigma_{1m}^{(kl)}/\partial x_1$, $m = 1,2,3$. Due to (25) and (28), the full number of those functions is equal to the number of virtual functions $r_m^{(kl)}$ and $s_m^{(kl)}$, i. e., is equal to $2N_d$, where $N_d = (N+1)(3N+2)/2$. If we substitute the expressions for those derivatives with respect to the coordinate $x_1$ in the functional $\Phi$ from (8), then the algebraic relations needed to find the stress functions $\sigma_{22}^{(kl)}$, $\sigma_{33}^{(kl)}$, and $\sigma_{23}^{(kl)}$ (their full number is equal to $N_a = \dfrac{3}{2}N^2 + \dfrac{5}{2}N + 2$) can be obtained from the condition

$$\delta_{\sigma_{22}}\Phi + \delta_{\sigma_{23}}\Phi + \delta_{\sigma_{33}}\Phi = 0 \tag{29}$$

(the first variation should be equal to zero) provided that the values of other stress and displacement functions are fixed.

If the test functions $r_m^{(kl)}$ and $s_m^{(kl)}$ and variations $\delta\sigma_{22}^{(kl)}$, $\delta\sigma_{33}^{(kl)}$, and $\delta\sigma_{23}^{(kl)}$ are chosen arbitrarily, then system (26), (27), (29) of integral equations is equivalent to a system of $2N_d + N_a$ linear equations with respect to the variables $w_m^{(kl)}$ and $\sigma_{mn}^{(kl)}$. The differential order of the system is equal to $2N_d$ (both with respect to $x_1$ and $t$).

Condition (15) imply $N_d$ stress boundary-value conditions:

$$\sigma_{11}^{(ij)}(t,a_1) = 0, \quad i+j \le N; \quad \sigma_{12}^{(kl)}(t,a_1) = \sigma_{13}^{(kl)}(t,a_1) = 0, \quad k+l \le N-1. \tag{30}$$

Condition (16) and the form of approximations (25) yield $N_d$ homogeneous displacement conditions:

$$w_1^{(ij)}(t,-a_1)=0,\quad i+j\le N;\quad w_2^{(kl)}(t,-a_1)=w_3^{(kl)}(t,-a_1)=0,\quad k+l\le N-1. \tag{31}$$

The initial-value conditions for the displacement functions directly follow from (18):

$$\begin{aligned}&w_1^{(ij)}(0,x_1)=0,\quad i+j\le N;\quad w_2^{(kl)}(0,x_1)=w_3^{(kl)}(0,x_1)=0,\quad k+l\le N-1;\\&w_{1,t}^{(ij)}(0,x_1)=0,\quad i+j\le N;\quad w_{2,t}^{(kl)}(0,x_1)=w_{3,t}^{(kl)}(0,x_1)=0,\quad k+l\le N-1.\end{aligned} \tag{32}$$

To obtain a consistent system, the differential equation with respect to the displacement function $y_1(t)$ for the beam section (see (16)) and the initial-value conditions (17) have to be added to relations (26), (27), and (29)–(32).

### *4.2. The eigenvalue problem*

To solve the finite-dimensional problem posed in partial derivatives, we represent the sought functions (25) as follows (see [7]):

$$\begin{aligned}&w_1(t,\mathbf{x})=\sum_{i=1}^{M-1}\widetilde{w}_{1,i}(\mathbf{x})y_{i+1}(t),\quad w_3(t,\mathbf{x})=\sum_{i=1}^{M-1}\widetilde{w}_{3,i}(\mathbf{x})y_{i+1}(t),\\&w_2(t,\mathbf{x})=y_1(t)+\sum_{i=1}^{M-1}\widetilde{w}_{2,i}(\mathbf{x})y_{i+1}(t),\quad \sigma=\sum_{i=1}^{M-1}\widetilde{\sigma}_i(\mathbf{x})y_{i+1}(t),\end{aligned} \tag{33}$$

where $M$ is a given positive number.

Apply substitution (33) and change the variables as follows: $y_j(t)=\exp(i\omega t)$, $j=1,\dots,M$. This reduces problem (26), (27), (29)–(32), which is a partial derivative problem, to a system of ordinary differential equations with respect to $x_1$ with homogeneous boundary-value conditions posed on the beam end-sides (this is the problem to find eigenfrequencies $\omega$).

Once the eigenvalue problem is solved, one can take into account the notation introduced by (25) and represent the solution as follows (see [2]):

$$\begin{aligned}&\widetilde{w}_1=\sum_{k+l=0}^{N}\widetilde{w}_1^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\quad \widetilde{w}_n=\sum_{k+l=0}^{N-1}\widetilde{w}_n^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\\&\widetilde{\sigma}_{11}=\sum_{k+l=0}^{N}\widetilde{\sigma}_{11}^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\quad \widetilde{\sigma}_{nn}=g_n\sum_{k+l=0}^{N}\widetilde{\sigma}_{nn}^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\\&\widetilde{\sigma}_{1n}=g_n\sum_{k+l=0}^{N-1}\widetilde{\sigma}_{1n}^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l,\quad \widetilde{\sigma}_{23}=g_2g_3\sum_{k+l=0}^{N-2}\widetilde{\sigma}_{23}^{(kl)}(x_1)\widetilde{x}_2^k\,\widetilde{x}_3^l.\end{aligned} \tag{34}$$

Here $\widetilde{w}_i^{(kl)}(x_1)$ and $\widetilde{\sigma}_{ij}^{(kl)}(x_1)$ are the components of the corresponding eigenvectors.

In the sequel, we investigate the special case of oscillations of a homogeneous isotropic beam such that its cross-section is a square ($a_2=a_3$). Introducing the characteristic length $\widetilde{x}=a_2$ and time $\widetilde{t}=a_2\sqrt{\rho/E}$, where $E$ is the Young module, one can reduce all the considered linear elasticity equations to a dimensionless form. The only two parameters left in the system are the beam relative length $a=a_1/a_2$ and the Poisson coefficient $\nu$. The unknown is the dimensionless frequency $\widetilde{\omega}=\widetilde{t}\omega$. For simplicity, the symbol "~" is omitted. In the sequel, the following values are used for the computation: $\nu=0.3$ and $a=20$.

To increase the efficiency of the numerical algorithm, one can take into account the symmetry properties of the problem. Since the cross-section of the beam has two symmetry axes, it follows that the system of equations (26), (27), and (29) can be divided into four independent subsystems according to the evennes of the polynomial part of base functions from (34): the said subsystems approximately

**Table 1.** The symmetry property of base functions $\sim x_2^{N_2} x_3^{N_3}$

| | Extension along $Ox_1$ | | Bend around $Ox_2$ | | Bend around $Ox_3$ | | Torsion around $Ox_1$ | |
|---|---|---|---|---|---|---|---|---|
| | $N_2$ | $N_3$ | $N_2$ | $N_3$ | $N_2$ | $N_3$ | $N_2$ | $N_3$ |
| $w_1, \sigma_{jj}$ | $2n$ | $2n$ | $2n$ | $2n+1$ | $2n+1$ | $2n$ | $2n+1$ | $2n+1$ |
| $w_2, \sigma_{12}$ | $2n-1$ | $2n-2$ | $2n-1$ | $2n-1$ | $2n$ | $2n$ | $2n$ | $2n+1$ |
| $w_3, \sigma_{13}$ | $2n-2$ | $2n-1$ | $2n$ | $2n$ | $2n-1$ | $2n-1$ | $2n+1$ | $2n$ |
| $\sigma_{23}$ | $2n-3$ | $2n-3$ | $2n-1$ | $2n-2$ | $2n-2$ | $2n-1$ | $2n$ | $2n$ |

describe the extension—contraction, bends with respect to the axis $Ox_2$, bends with respect to the axis $Ox_3$, and the torsion of the beam (see [2]).

For various eigenmotions, Table 1 provides the greatest degrees $N_2(n)$ and $N_3(n)$ of the variables $x_2$ and $x_3$ contained in the approximations of displacement and stresses (34). Here $j = 1, 2, 3$, while the positive integer $n$ determines the differential order of the corresponding boundary-value problem. The evenness (oddness) of the numbers $N_2$ and $N_3$ characterizes symmetry (antisymmetry) properties of the displacement and stress functions with respect to the coordinate planes $Ox_2$ and $Ox_3$ respectively. If at least one of those numbers is less than zero, then there are no corresponding functions.

The order of the system of differential equations is equal to $(n+1)(3n+2)$ for the extension—contraction problem, to $(n+1)(3n+4)$ for the bending problems, and to $(n+1)(3n+6)$ for the torsion problem. The least possible dimensions of the approximations from representation (34) are equal to 2, 4, and 6 respectively.

Since we consider the case where the end of beam for $x_1 = -a_1$ moves along the axis $Ox_2$ according to (16), it follows that only bending oscillations with respect to the axis $Ox_3$ arise in the system.

For example, consider the eigenfrequency problem for bending oscillations of a beam, assuming that its cross-section is a square and $n = 0$. For this case, the functions from (34) have the form

$$\widetilde{w}_1 = \widetilde{w}_1^{(10)}(x_1)x_2, \quad \widetilde{w}_2 = \widetilde{w}_2^{(00)}(x_1),$$
$$\widetilde{\sigma}_{11} = \widetilde{\sigma}_{11}^{(10)}(x_1)x_2, \quad \widetilde{\sigma}_{12} = \widetilde{\sigma}_{12}^{(00)}(x_1)(1-x_2^2),$$
$$\widetilde{\sigma}_{22} = \widetilde{\sigma}_{22}^{(10)}(x_1)x_2(1-x_2^2), \quad \widetilde{w}_3 = \widetilde{\sigma}_{13} = \widetilde{\sigma}_{23} = \widetilde{\sigma}_{33} = 0. \tag{35}$$

Then the resulting system of the differential-algebraic equations (26), (27), (29) can be represented as the system

$$\frac{4}{3}\frac{\mathrm{d}\widetilde{\sigma}_{11}^{(10)}}{\mathrm{d}x_1} - \frac{8}{3}\widetilde{\sigma}_{12}^{(00)} + \frac{4}{3}\omega^2\widetilde{w}_1^{(10)} = 0, \quad \frac{8}{3}\frac{\mathrm{d}\widetilde{\sigma}_{12}^{(00)}}{\mathrm{d}x_1} + 4\omega^2\widetilde{w}_2^{(00)} = 0,$$
$$\frac{16}{45}\frac{\mathrm{d}\widetilde{\sigma}_{12}^{(00)}}{\mathrm{d}x_1} + \frac{16}{15}\widetilde{\sigma}_{22}^{(10)} = 0,$$
$$\frac{4}{3}\frac{\mathrm{d}\widetilde{w}_1^{(10)}}{\mathrm{d}x_1} - \frac{4}{3}\widetilde{\sigma}_{11}^{(10)} + \frac{4}{25}\widetilde{\sigma}_{22}^{(10)} = 0, \quad \frac{4}{3}\frac{\mathrm{d}\widetilde{w}_2^{(00)}}{\mathrm{d}x_1} + \frac{4}{3}\widetilde{w}_1^{(10)} - \frac{208}{75}\widetilde{\sigma}_{12}^{(00)} = 0 \tag{36}$$

with the boundary-value conditions

$$\widetilde{w}_1^{(10)}(-a_1) = \widetilde{w}_2^{(00)}(-a_1) = \widetilde{\sigma}_{11}^{(10)}(a_1) = \widetilde{\sigma}_{12}^{(00)}(a_1) = 0. \tag{37}$$

Eigenvalues $\omega$ are found from system (36): they are roots of the characteristic equation

$$\frac{4}{3}\,\lambda^4 + \frac{406}{75}\,\omega^2\lambda^2 - 4\omega^2 + \frac{104}{25}\,\omega^4 = 0, \tag{38}$$

where $\lambda$ is the corresponding wave number.

**Table 2.** Eigenfrequencies $\omega_i$ of a cantilever beam

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\omega_i^c$ | $1.269 \times 10^{-3}$ | $7.951 \times 10^{-3}$ | $2.226 \times 10^{-2}$ | $4.363 \times 10^{-2}$ |
| $n = 0, \omega_i^0$ | $1.266 \times 10^{-3}$ | $7.843 \times 10^{-3}$ | $2.157 \times 10^{-2}$ | $4.123 \times 10^{-2}$ |
| $n = 1, \omega_i^1$ | $1.269 \times 10^{-3}$ | $7.861 \times 10^{-3}$ | $2.162 \times 10^{-2}$ | $4.135 \times 10^{-2}$ |
| $(\omega_i^c - \omega_i^0)/\omega_i^0$ | 0.2% | 1.4% | 3.2% | 5.8% |
| $(\omega_i^1 - \omega_i^0)/\omega_i^0$ | 0.21% | 0.22% | 0.25% | 0.29% |
| $\Delta_i^1$ | 0.30% | 0.31% | 0.32% | 0.33% |

To compare, consider the characteristic equation for the Timoshenko beam with the same parameters (see [9]):

$$\frac{4}{3}\,\lambda^4 + \frac{412}{75}\,\omega^2\lambda^2 - 4\omega^2 + \frac{104}{25}\,\omega^4 = 0. \tag{39}$$

We see that the only difference between equations (38) and (39) is the coefficient at $\lambda^2$ and the value of that difference is less than 2%.

To compute eigenfrequencies and oscillation shapes of beams more precisely, one has to apply higher powers of polynomial approximations ($n > 0$). The initial three rows of Table 2 provide the values of the four lowest eigenfrequencies for the oscillations of a cantilever beam with a square cross-section for the Euler—Bernoulli model and for the model proposed in the present paper with $n = 0$ and $n = 1$ (respectively). The third and fourth lines show the difference of between frequencies obtained in the framework of those approximations. The difference between the obtained frequencies and the ones computed according to the Euler—Bernoulli model is substantial: it achieves 5.8% already for the fourth mode. The last row shows the relative error $\Delta_i^1$ of the computation of the $i$th eigenshape of the oscillations (for $n = 1$) according to the integral criterion (10).

### *4.3. The system of ordinary differential equations with respect to time*

To construct the system of ordinary differential equations with respect to $t$, take the approximations from (33) and substitute the eigenshapes of oscillations from (34), corresponding to the frequencies $\omega_i$, $i = 1, \ldots, M-1$, for the displacement functions $\widetilde{\mathbf{w}}_i(x)$ and stress functions $\widetilde{\sigma}_i(x)$. The obtained approximations are substituted in the integral equations from (26), (27), and (29). In relations (26) and (27), the vector $\mathbf{v}_t$ and the tensor $\xi$ are projected to the base vector-functions $\mathbf{r}_i(\mathbf{x})$ and $\mathbf{s}_i(\mathbf{x})$ with the components

$$\widetilde{r}_{1,i} = \sum_{k+l=0}^{N} \widetilde{r}_{1,i}^{(kl)}(x_1)\widetilde{x}_2^k\widetilde{x}_3^l, \quad \widetilde{r}_{n,i} = \sum_{k+l=0}^{N+1} \widetilde{r}_{n,i}^{(kl)}(x_1)\widetilde{x}_2^k\widetilde{x}_3^l,$$
$$\widetilde{s}_{1,i} = \sum_{k+l=0}^{N} \widetilde{s}_{1,i}^{(kl)}(x_1)\widetilde{x}_2^k\widetilde{x}_3^l, \quad \widetilde{s}_{n,i} = \sum_{k+l=0}^{N-1} \widetilde{s}_{n,i}^{(kl)}(x_1)\widetilde{x}_2^k\widetilde{x}_3^l. \tag{40}$$

To find the unknown functions $\widetilde{r}_{j,i}^{(kl)}(x_1)$ and $\widetilde{s}_{j,i}^{(kl)}(x_1)$, $j = 1, 2, 3$, we solve the adjoint boundary-value eigenvalue problem (see [10]).

The above choice of such projections reduces the integral equations from (26), (27), and (29) to the diagonal form

$$\ddot{y}_j = -\omega_{j-1}^2 y_j + b_{M+j}u(t), \quad j = 2, \ldots, M, \tag{41}$$

where $\omega_i$, $i = 1, \ldots, M-1$, are the approximate eigenfrequencies obtained for the selected approximation degree $n$, while $b_k$, $k = M+1, \ldots, 2M$, are the control coefficients.

## 5. THE FINITE-DIMENSIONAL CONTROL PROBLEM

Taking into account the boundary-value conditions (17), (32), transform system (41), joining the equation with respect to $y_1(t)$ from (16), to the form

$$\begin{gathered}\dot{y}_j = y_{M+j}, \quad j = 1, \ldots, M,\\ \dot{y}_{M+j} = -\omega_{j-1}^2 y_j + b_{M+j} u(t), \quad j = 1, \ldots, M,\\ y_k(0) = 0, \quad k = 1, \ldots, 2M. \end{gathered} \tag{42}$$

The new variables $y_{M+j}(t)$ are derivatives of $y_j(t)$ with respect to time. Also, we assume that the frequency $\omega_0$ is equal to zero and the control coefficient $b_{M+1}$ is equal to one.

System (42) can be represent in the vector form

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}, u) = \mathbf{A}\mathbf{y}(t) + \mathbf{b}u(t) \tag{43}$$

with the homogeneous initial-value conditions

$$\mathbf{y}(0) = \mathbf{0} \tag{44}$$

and the terminal relations

$$\mathbf{y}(T) = (y_T, 0, \ldots, 0)^{\mathrm{T}} \tag{45}$$

implied by (19).

Here $\mathbf{y} = (y_1(t), \ldots, y_{2M}(t))^{\mathrm{T}}$ is the vector of phase variables, $\mathbf{A}$ from $\mathbb{R}^{2M \times 2M}$ is a constant matrix, and $\boldsymbol{b} \in \mathbb{R}^{2M}$ is a constant vector.

Thus, (43) is a finite-dimensional dynamical system approximately determining lateral motions of the considered elastic body. For this system, we pose the optimal control problem corresponding to problem (18)–(20): to find a control function $u^*(t)$ moving the linear system (43) from the zero state (44) to the terminal state (45) of rest, where $T$ is fixed, and minimizing the quality functional:

$$\widetilde{J}[u^*] = \min_{u \in L^2(0,T)} \widetilde{J}[u]. \tag{46}$$

Here the quadratic integral

$$\widetilde{J} = \int_0^T f_0(t)\, \mathrm{d}t, \quad f_0 = \frac{1}{2}\, u^2(t) + \frac{\gamma}{2}\, \mathbf{y}(t)^{\mathrm{T}} \mathbf{W} \mathbf{y}(t), \tag{47}$$

is obtained by means of the discretization of the functional $J$ introduced in (21).

Introducing the vector $\mathbf{z}(t) \in \mathbb{R}^{2M}$ of adjoint variables, one can define the Hamiltonian of the system according to the Pontrjagin maximum principle (see [11]):

$$\mathcal{H}[\mathbf{y}, \mathbf{z}, u] = -f_0 + \mathbf{f}^{\mathrm{T}} \mathbf{z}. \tag{48}$$

Using (48), represent the adjoint system of equations as follows:

$$\dot{\mathbf{z}}(t) = -\frac{\partial \mathcal{H}}{\partial \mathbf{y}} = \gamma \mathbf{W} \mathbf{y}(t) - \mathbf{A}^{\mathrm{T}} \mathbf{z}(t). \tag{49}$$

Then the optimal control treated as a linear function of adjoint variables is

$$u = \mathbf{b}^{\mathrm{T}} \mathbf{z}(t). \tag{50}$$

Substitute this function in equation (43). Then the problem to find optimal motions is reduced to the boundary-value problem (43)–(45), (49), (50).

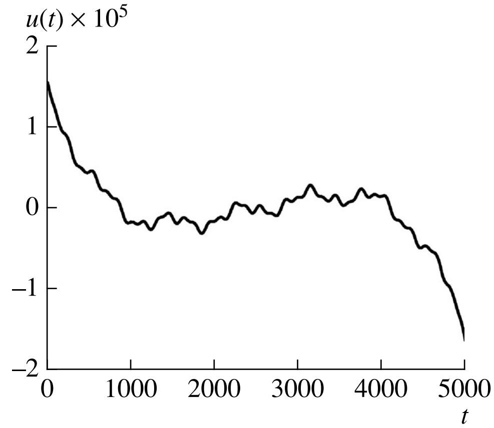


**Fig. 2.** The optimal control law $u(t)$.

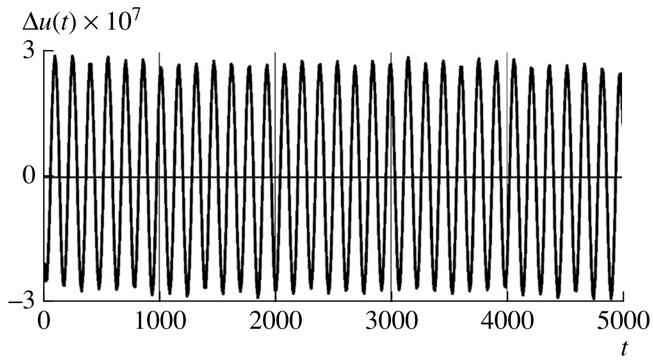


**Fig. 3.** The variation of the control $\Delta u(t)$ under the growth of the dimension of the provlem.

## 6. NUMERICAL RESULTS

To illustrate the work of the proposed algorithm to find controlled motions of a prism-shaped elastic body, we select the approximation dimensions $n = 1$, $M = 4$, and 5 and the control parameters $y_T = 20$, $T = 5000$, and $\gamma = 5 \cdot 10^{-7}$. The control time $T$ exceeds the oscillation period of the beam with respect to the first mode $T_1 \simeq 4952$, while the displacement $y_T$ is equal to a half of the dimensionless length of the beam $a$.

Figure 2 provides the optimal control law $u(t)$ obtained for the approximation parameter $M = 5$ (only the four initial oscillation modes are taken into account). This function slightly differs from the control obtained for $M = 4$, which is demonstrated by Fig. 3 providing the difference $\Delta u(t)$ between those two laws. From both figures, it follows that, taking into account the additional oscillation mode in the model, we add a negligible high-frequency component to the control.

Figure 4 provides the displacements $w_2(t, \mathbf{x}_\pm)$ of the two points of an elastic body; their coordinates in the phase plane are $\mathbf{x}_\pm = (\pm a, 0, 0)$. The dashed curve shows the location and velocity of the point lying at the middle of the boundary cross-section of the beam ($x_1 = -a$) moving as a rigid body. The dense curve denotes the phase trajectory of the point $x_1 = a$ placed at the middle of the boundary cross-section free from external loads. The difference between those curves shows that the optimal control law generates substantial elastic deformations for the selected set of parameters.

The energy distribution with respect to oscillation modes is shown at Fig. 4. The solid curve $E_1(t)$ is the variation of the kinetic energy corresponding to the translational movement of the beam. The said energy is proportional to the second power of the variable $\dot{y}_1(t)$. The dashed curve $E_2(t)$ corresponds to the kinetic energy of the first oscillation mode, while the dash-dotted curve $E_3(t)$ corresponds to the second one. The values of those energies are proportional to $\dot{y}_2^2(t)$ and $\dot{y}_3^2(t)$ respectively. The graph does

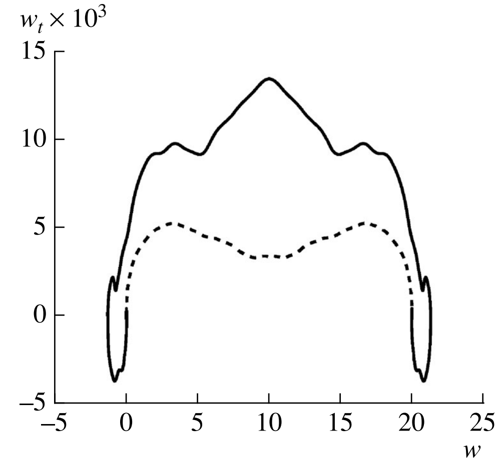


**Fig. 4.** The displacements (in the phase plane) of points $\mathbf{x}_\pm$ of boundary cross-sections.

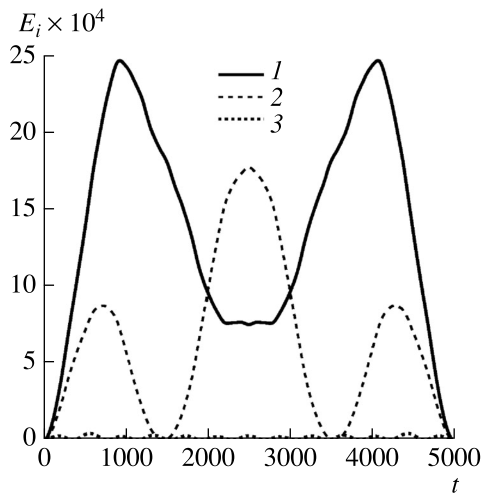


**Fig. 5.** The variation of the kinetic energy $E_i(t)$ of lower modes.

not show other energies because their values are negligible. The main elastic motions refer to the first mode and are comparable (with respect to the energy) with the motion of the whole beam as a rigid body.

To obtain more precise optimal control law $u^*(t)$ for the motion of an elastic body and to find the corresponding fields of displacements, stresses, and momenta, it is necessary to increase (in a coordinated way) both the order $n$ of the polynomial expansion and the number $M$ of modes taken into account (provided that it is affordable due to computational resources). Note that high-frequency oscillations of the control function are caused by the growth of the model dimension; this might substantially complicate the implementation of the obtained laws in technical applications.

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