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A Functional Central Limit Theorem for Hilbert-Valued Martingales

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Abstract—Weak convergence of martingales with values in Hilbert space is studied in the paper. Necessary and sufficient conditions for the convergence to Gaussian martingale with continuous trajectories are obtained.

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1. INTRODUCTION

It was shown in [1] that under the assumption of uniform integrability of jumps of local martingales the weak convergence of a sequence of local martingales to a continuous Gaussian martingale holds if and only if a convergence in probability of corresponding quadratic variations takes place. It should be noted that some particular cases of this result may be found in [1] (see also [2, 3]).

Many authors have investigated the weak convergence of martingale difference arrays and for scheme of a series of Hilbert-valued random variables [4–9]. The aim of this work is to disseminate the results of [1] on the case of local martingales with values in a Hilbert space.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, $F^n = (\mathcal{F}_t^n)_{t\geq 0}, n \geq 1$ and $F = (\mathcal{F}_t)_{t\geq 0}$ be non-decreasing right-continuous family of σ-algebras of $\mathcal{F}$ such that the σ-algebras $\mathcal{F}_0^n$ and $\mathcal{F}_0$ contain the $\mathbf{P}$ zero sets from $\mathcal{F}$. We denote by $\mathcal{M}(\mathbb{H})$, $\mathcal{M}_{loc}(\mathbb{H})$, $\mathcal{M}_{loc}^c(\mathbb{H})$, $\mathcal{M}_{loc}^d(\mathbb{H})$, $\mathcal{M}_{loc}^2(\mathbb{H})$ the classes of uniformly integrable martingales, local martingales, locally continuous, purely discontinuous local martingales and locally square-integrable martingales $X = (X_t, \mathcal{F}_t; \mathbb{H}), X_0 = 0$ with values in Hilbert space $\mathbb{H}$.

Let X be Hilbert-valued process. Then for $i \geq 1$ we denote by x_i-process $(x_i)_t = (e_i, X_t)$, where $\{e_i\}$ is orthonormal basis in $\mathbb{H}$. For $X \in \mathcal{M}_{loc}^2(\mathbb{H})$ we have the set of real predictable processes of locally integrable variation $(\langle x_i, x_j\rangle)_{i,j\geq 1}$ such that $x_i x_j - \langle x_i, x_j\rangle$ is local martingale ($\langle x_i\rangle \equiv \langle x_i, x_i\rangle$). In addition, there is a real predictable increasing process $\langle M\rangle$ such that $||M||^2 - \langle M\rangle$ is local martingale and $\langle M\rangle = \sum_{i=1}^{\infty} < m_i, m_i >$. For $X \in \mathcal{M}_{loc}(\mathbb{H})$ we denote by $[X]_t \equiv \langle X^c\rangle_t + \sum_{0<s\leq t} ||\Delta X_s||^2$, $[x_i, x_j]_t \equiv \langle x_i^c, x_j^c\rangle_t + \sum_{s\leq t} \Delta(x_i)_s \Delta(x_j)_s$.

Suppose further that M is continuous Gaussian martingale with values in Hilbert space. Then $(\langle m_i, m_j\rangle)_{i,j\geq 1}$ are continuous and deterministic functions, and quadratic variation is $\langle M\rangle_t = E||M_t||^2$.

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2. THE RESULTS

Theorem 1. *Let* $X^n \in \mathcal{M}_{loc}(\mathbb{H}), n \geq 1$ *satisfies the condition* (R): *for any* $t > 0$ *the set of random variables* $(\sup_{0<s\leq t} ||\Delta X^n_s||)$ *is uniformly integrable. Then the conditions*

$$[x^n_i, x^n_j]_t \xrightarrow{P} \langle m_i, m_j \rangle_t, \quad [X^n]_t \xrightarrow{P} \langle M \rangle_t, \quad i, j \geq 1, \quad t > 0, \tag{1}$$

holds if and only if $X^n \xrightarrow{D} M$.

Corollary 1. *Let* $X^n \in \mathcal{M}^c_{loc}(\mathbb{H}), n \geq 1$, *then the conditions*

$$\langle x^n_i, x^n_j \rangle_t \xrightarrow{P} \langle m_i, m_j \rangle_t, \quad \langle X^n \rangle_t \xrightarrow{P} \langle M \rangle_t, \quad i, j \geq 1, \quad t > 0, \tag{2}$$

holds if and only if $X^n \xrightarrow{D} M$.

The semimartingale $X^n = (X^n_t, \mathcal{F}_t; \mathbb{H})$ has a canonical representation [10]:

$$X^n_t = X_0 + B^n_t + M^{nc}_t + \int_0^1 \int_{||x|| \leq 1} x d(\mu^n - \nu^n) + \int_0^t \int_{||x||>1} x \, \mu^n (ds, dx),$$

where $B^n = (B^n_t, \mathcal{F}_t; \mathbb{H})$ is predictable process of class $\mathcal{A}_{loc}(\mathbb{H})$ (processes with locally integrable variation), $M^{nc} \in \mathcal{M}^c_{loc}(\mathbb{H})$, $\mu^n = \mu^n(ds, dx)$ is an integer-valued random measure associated to jumps of X^n and $\nu^n = \nu^n(ds, dx)$ its compensator. Then from the theorem 2 in [11] it follows that for $X^n \in \mathcal{M}_{loc}(\mathbb{H})$

$$B^n_t = -\int_0^t \int_{||x||>1} x d\nu^n.$$

Furthermore, $B^n = B^{nc} + B^{nd}$, where

$$B^{nd}_t = -\sum_{0<s\leq t} \int_{||x||>1} x \nu^n(\{s\}, dx).$$

Consequently, the variation $V_t(B^{nd})$ of the function B^{nd} on the interval $[0, t]$ is defined by the formula:

$$V_t(B^{nd}) = \sum_{0<s\leq t} \left\| \int_{||x||>1} x \nu^n(\{s\}, dx) \right\|. \tag{3}$$

Theorem 2. *Let* $X^n \in \mathcal{M}_{loc}(\mathbb{H}), n \geq 1$ *satisfies the condition*

$$\sup_{0<s\leq t} ||B^{nc}_s|| + V_t(B^{nd}) \xrightarrow{P} 0, \quad t > 0. \tag{4}$$

Then the conditions (1) *holds if and only if* $X^n \xrightarrow{D} M$.

Corollary 2. *Let* $X^n \in \mathcal{M}_{loc}(\mathbb{H}), n \geq 1$ *quasi-continuous from the left (for any predictable stopping time* $\tau \Delta X^n_\tau = 0$) *and satisfies the condition*

$$\sup_{0<s\leq t} ||B^n_s|| \xrightarrow{P} 0, \quad t > 0. \tag{5}$$

Then the conditions (1) *holds if and only if* $X^n \xrightarrow{D} M$.

Following [12], we introduce conditions for any $t > 0$ and $n \to \infty$:

$$[m^{n1}_i, m^{n1}_j]_t \xrightarrow{P} \langle m_i, m_j \rangle_t, \quad [M^{n1}]_t \xrightarrow{P} \langle M \rangle_t, \quad i, j \geq 1; \tag{6}$$

$$\langle m^{n\varepsilon}_i, m^{n\varepsilon}_j \rangle_t \xrightarrow{P} \langle m_i, m_j \rangle_t, \langle M^{n\varepsilon} \rangle_t \xrightarrow{P} \langle M \rangle_t, \varepsilon > 0, i, j \geq 1. \tag{7}$$

Corollary 3. *Let $X^n \in \mathcal{M}^2_{loc}(\mathbb{H}), n \geq 1$, satisfies Lindeberg condition:*

$$\int_0^t \int_{||x||>\varepsilon} ||x||^2 \nu^n(ds, dx) \xrightarrow{P} 0, \quad \varepsilon > 0. \tag{8}$$

Then the following statements are equivalent:

$$(1) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (2) \Leftrightarrow (X^n \xrightarrow{D} M).$$

3. THE PROOFS

3.1. The Proof Theorem 1 and Corollary 1

Following [12], we introduce conditions for any $t > 0$ and $n \to \infty$:

$$\int_0^t \int_{||x||>\varepsilon} \nu^n(ds, dx) \xrightarrow{P} 0, \quad \varepsilon \in (0, 1], \tag{9}$$

$$\sup_{0<s\leq t} ||B^n_s|| \xrightarrow{P} 0, \tag{10}$$

Following the scheme proposed in [1] for the one-dimensional case, in accordance with the [12] it suffices to establish the following implications hold:

$$(\mathrm{R}, 1) \Rightarrow^{1)} (\mathrm{R}, 1, 9) \Rightarrow^{2)} (\mathrm{R}, 1, 9, 10) \Rightarrow^{3)} (\mathrm{R}, 9, 10, 6) \Rightarrow^{4)} (\mathrm{R}, X^n \xrightarrow{D} M) \Rightarrow^{5)} (\mathrm{R}, X^n \xrightarrow{D} M, 9)$$
$$\Rightarrow^{6)} (\mathrm{R}, X^n \xrightarrow{D} M, 9, 10) \Rightarrow^{7)} (\mathrm{R}, 9, 10, 6) \Rightarrow^{8)} (\mathrm{R}, 1).$$

To this end, we prove that (1) ⇒ (9), (9, R) ⇒ (10), and verify that under conditions (9) and (R) condition (1) is equivalent to (7).

Since $\langle M \rangle$ is non-decreasing continuous function, then by Lemma 1, in [13] from the condition (1) we obtain

$$\sup_{s\leq t} |[X^n, X^n]_s - \langle M \rangle_s| \xrightarrow{P} 0, \quad t > 0.$$

Since $[M]_t = \langle M^c \rangle_t + \sum_{s\leq t} ||\Delta M_s||^2$ then $\sup_{s\leq t} ||\Delta M_s||^2 \xrightarrow{P} 0$ and from the condition (1) it follows that $\sup_{0<s\leq t} ||\Delta X_s|| \xrightarrow{P} 0, \quad t > 0$, which is equivalent to (9) according to Theorem 2 in [12].

In order to check the relation (9, R) ⇒ (10), we note that the condition (9) by the Corollary from Lenglart inequality [14] implies that

$$Z^n_t = \int_0^t \int_{||x||>1} ||x|| d\mu^n \xrightarrow{P} 0.$$

Variation of the function B^n has the property (3) and

$$V_t(B^n) \leq \int_0^t \int_{||x||>1} ||x|| d\nu^n = Z^n_t.$$

Then, since $||\Delta Z^n_s|| \leq ||\Delta X^n_s||$, then from the condition (R) and the corollary of Lenglart inequality [14] implies that $V_t(B^n) \xrightarrow{P} 0, t > 0$.

The inequality $\sup_{0<s\leq t} ||B^n_s|| \leq V_t(B^n)$ implies the validity of (9, R) ⇒ (10).

To prove the equivalence (1) and (7) (under the conditions of (9) and (R)) we denote $h = e_i, e_i + e_j$; $i, j \geq 1$,

$$J_t^n(h) = [(X^n, h)]_t - [(M^n, h)]_t = \sum_{0<s\leq t} (\Delta X_s^n, h)^2 I(||\Delta X_s^n|| > 1)$$

$$+ 2 \sum_{0<s\leq t} (\Delta X_s^n, h) I(||\Delta X_s^n|| \leq 1) \int_{||x||\leq 1} (x, h)\nu^n(\{s\}, dx)$$

$$- \sum_{0<s\leq t} \left(\int_{||x||\leq 1} (x, h)\nu^n(\{s\}, dx) \right)^2 .$$

$$J_t^n = [X^n]_t - [M^n]_t = \sum_{0<s\leq t} ||\Delta X_s^n||^2 I(||\Delta X_s^n|| > 1)$$

$$+ 2 \sum_{0<s\leq t} \left(\Delta X_s^n I(||\Delta X_s^n|| \leq 1), \int_{||x||\leq 1} x\nu^n(\{s\}, dx) \right)$$

$$- \sum_{0<s\leq t} \left|\left| \int_{||x||\leq 1} x\nu^n(\{s\}, dx) \right|\right|^2 .$$

Thus, it suffices to show that

$$(9, R) \Rightarrow J_t^n(h) \xrightarrow{P} 0, \quad J_t^n \xrightarrow{P} 0, \quad t > 0. \tag{11}$$

In the proof of Theorem 2 in [11] it has been shown that for $s > 0$ $\int_{H\setminus\{0\}} x\nu^n(\{s\}, dx) = 0$ a.s. From this and $V_t(B^n) \xrightarrow{P} 0$, $t > 0$ we have

$$|J_t^n(h)| \leq ||h||^2 I_t^n, \quad |J_t^n| \leq I_t^n,$$

$$I_t^n = \sum_{0<s\leq t} ||\Delta X_s^n||^2 I(||\Delta X_s^n|| > 1) + 2V_t(B^{nd}) + V_t^2(B^{nd}). \tag{12}$$

From the condition (9) and the corollary of Lenglart inequality [14] imply that

$$\sum_{0<s\leq t} ||\Delta X_s^n||^2 I(||\Delta X_s^n|| > 1) \xrightarrow{P} 0. \tag{13}$$

We have already proved the implication (9, R) $\Rightarrow V_t(B^n) \xrightarrow{P} 0$. Thus from $V_t(B^{nd}) \leq V_t(B^n)$, (13) and (12) we get (11).

So, the implications 1), 2), 3), 6) and 8) are proved and implications 4), 5) and 7) follow from [12]. This completes the proof of the theorem 1.

Corollary 1 is the evident consequence of Theorem 1.

3.2. The Proof of Theorem 2 and Corollaries

In the proof of Theorem 1, it was shown that $(R, 1) \Rightarrow V_t(B^n) \xrightarrow{P} 0$, $t > 0$. Hence the condition (4) is satisfied. Further (4) $\Rightarrow$ (10). Then from (9) and (4) implies the equivalence of (1) $\Leftrightarrow$ (7) therefore $(4, 1) \Rightarrow (9, 10, 6) \Rightarrow X^n \xrightarrow{D} M$.

On the other hand $(4, X^n \xrightarrow{D} M) \Rightarrow (4, 9, 6) \Rightarrow (1)$. Since the function B^n is continuous a.s. for quasi-left continuous process X^n [15] then corollary 2 follows from theorem 2.

The proof corollary 3. For $X^n \in \mathcal{M}^2_{loc}(\mathbb{H})$ we have

$$X^n_t = X^{nc}_t + \int_0^t \int_{||x||\le\varepsilon} x(\mu^n - \nu^n)(ds,dx) + \int_0^t \int_{||x||>\varepsilon} x\mu^n(ds,dx) + B^{n\varepsilon}_t,$$

where $B^{n\varepsilon}_t = -\int_0^t \int_{||x||>\varepsilon} x\nu^n(ds,dx)$ and

$$\sup_{s\le t} ||B^{n\varepsilon}_s|| \le \int_0^t \int_{||x||>\varepsilon} x\nu^n(ds,dx) \le \frac{1}{\varepsilon} \int_0^t \int_{||x||>\varepsilon} ||x||^2 \nu^n(ds,dx) \xrightarrow{P} 0.$$

Note that $s > 0$, $\int_{H\setminus\{0\}} x\nu^n(\{s\},dx) = 0$ a.s. (see proof theorem 2 in [11]), therefore

$$\langle (X^n,h)\rangle_t = \langle (X^{nc},h)\rangle_t + \int_0^t \int_{H\setminus\{0\}} (x,h)^2 \nu^n(ds,dx),$$

$$[(X^n,h)]_t = \langle (X^{nc},h)\rangle_t + \int_0^t \int_{H\setminus\{0\}} (x,h)^2 \mu^n(ds,dx),$$

the same way

$$\langle X^n\rangle_t = \langle X^{nc}\rangle_t + \int_0^t \int_{H\setminus\{0\}} ||x||^2 \nu^n(ds,dx),$$

$$[X^n]_t = \langle X^{nc}\rangle_t + \int_0^t \int_{H\setminus\{0\}} ||x||^2 \mu^n(ds,dx).$$

Condition (8) and the corollary of Lenglart inequality [14] imply that $\int_0^t \int_{||x||>\varepsilon} ||x||^2 \mu^n(ds,dx) \xrightarrow{P} 0$. With these remarks Corollary follows easily from [12] and conditions (8).

4. THE COROLLARY FOR SUMS OF RANDOM VARIABLES

Let $X^n_t = \sum_{k=0}^{[nt]} \xi_{nk}, \mathcal{F}^n_t = \sigma\{X^n_s, s\le t\}, 0 \le t \le 1, \xi_{n0} = 0, (\xi_{nk})$ is the arrays of random variables with values in Hilbert space, $0 \le k \le n$, $n \ge 1$. Later on we will be denote by $W = (W_t, \mathcal{F}_t; \mathbb{H})$ Hilbert-valued Wiener process with covariance operator $\mathbb{S}$ ($\mathbb{S}$-operator), i.e. a continuous process with independent increments that for any $u < t$, $h \in \mathbb{H}$ the real random variable $(W_t - W_u, h)$ has a Gaussian distribution with zero mean and variance $(t-u)(\mathbb{S}h,h)$ [16].

Corollary 4. *Let for any $n \ge 1$ a sequence $\xi^n = (\xi_{nk}, \mathcal{F}^n_k; \mathbb{H})$ is a martingale difference (i.e. $E||\xi_{nk}|| < \infty, E(\xi_{nk}|\mathcal{F}^n_{k-1}) = 0$, $1 \le k \le n$) and the set of random variables $(\sup_{0\le k\le n} ||\xi_{nk}||)_{n\ge1}$ is uniformly integrable. Then the conditions*

$$\sum_{k=0}^{[nt]} (\xi_{nk}, e_i)(\xi_{nk}, e_j) \xrightarrow{P} t(\mathbb{S}e_i, e_j), \quad i,j \ge 1, \tag{14}$$

$$\sum_{k=0}^{[nt]} ||\xi_{nk}||^2 \xrightarrow{P} tTr\mathbb{S} \tag{15}$$

are necessary and sufficient for the convergence of $X^n \xrightarrow{D} W$.

Proof. This corollary is a simple restatement of Theorem 1 for this special case.

Corollary 5. *Let for each* $n \geq 1$ *a sequence* $\xi^n = (\xi_{nk}, \mathcal{F}_k^n; \mathbb{H})$ *is a square-integrable martingale difference* (i.e. $E||\xi_{nk}||^2 < \infty$, $E(\xi_{nk}|\mathcal{F}_{k-1}^n) = 0$, $1 \leq k \leq n$ *and* $\xi_{n0} = 0$ *for any* $n \geq 1$).

1) *Let for all* $t \in [0,1]$ *the following conditions are satisfied:*

$$\sum_{k=1}^{n} E(||\xi_{nk}||^2 I(||\xi_{nk}|| > \varepsilon)|\mathcal{F}_{k-1}^n) \xrightarrow{P} 0, \quad \varepsilon > 0, \tag{16}$$

$$\sum_{k=1}^{[nt]} E((\xi_{nk}, e_i)(\xi_{nk}, e_j)|\mathcal{F}_{k-1}^n) \xrightarrow{P} t(\mathbb{S}e_i, e_j), \quad i,j \geq 1, \tag{17}$$

$$\sum_{k=1}^{n} E(||\xi_{nk}||^2|\mathcal{F}_{k-1}^n) \xrightarrow{P} tTr\mathbb{S}, \tag{18}$$

then $X^n \xrightarrow{D} W$.

2) *The conditions* (14) *and*

$$E\left|\sum_{k=0}^{[nt]} ||\xi_{nk}||^2 - tTr\mathbb{S}\right| \to 0 \tag{19}$$

are also sufficient for the convergence of $X^n \xrightarrow{D} W$.

3) *Let for any* $y \in \mathbb{H}$

$$\limsup_n \sum_{k=1}^{n} E(\xi_{nk}, y)^2) \leq (\mathbb{S}y, y), \quad i,j \geq 1, \tag{20}$$

then conditions (14) *and* (16)–(19) *are also necessary for the convergence of* $X^n \xrightarrow{D} W$.

Proof. The sufficiency of (16)–(18) follows from Corollary 3.

It follows from (19) (see the proof of Corollary 6 in [2]) that

$$E\sum_{k=1}^{n} ||\xi_{nk}||^2 I(||\xi_{nk}|| > \varepsilon) \to 0.$$

Hence, from the condition of (14), (19), applying Corollary 3 we obtain the convergence of $X^n \xrightarrow{D} W$.

To prove the necessity we set $y \in \mathbb{H}$ and we note that the convergence of $X^n \xrightarrow{D} W$ implies the convergence of

$$\frac{(X^n, y)}{\sqrt{(\mathbb{S}y, y)}} \xrightarrow{D} w = \frac{(W, y)}{\sqrt{(\mathbb{S}y, y)}}, \tag{21}$$

where w is standard real Wiener process.

By Theorem 2 in [17] from (20) and (21) it follows that

$$E\left|\sum_{k=1}^{[nt]} E((\xi_{nk}, y)^2 \middle| \mathcal{F}_{k-1}^n) - t(\mathbb{S}y, y)\right| \to 0, \tag{22}$$

$$E\left|\sum_{k=1}^{[nt]} (\xi_{nk}, y)^2 - t(\mathbb{S}y, y)\right| \to 0, \tag{23}$$

i.e. conditions (17) and (14) are fulfilled. Further, we note that for any $N > 1$

$$E\left|\sum_{k=0}^{[nt]}||\xi_{nk}||^2 - tTr\mathbb{S}\right| \leq \sum_{i=N}^{\infty}\sum_{k=1}^{n}E(\xi_{nk}, e_i)^2$$

$$+\sum_{i=1}^{N-1}E\left|\sum_{k=1}^{[nt]}(\xi_{nk}, e_i)^2 - t(\mathbb{S}e_i, e_i)\right| + \sum_{i=N}^{\infty}(\mathbb{S}e_i, e_i).$$

Hence, (20) and (23) imply condition (19) and, as shown above, the condition (16).

The necessity of (18) follows from (20), (22) and the following relationship:

$$E\left|\sum_{k=1}^{[nt]}E(||\xi_{nk}||^2|\mathcal{F}_{k-1}^n) - tTr\mathbb{S}\right| \leq \sum_{i=N}^{\infty}\sum_{k=1}^{n}E(\xi_{nk}, e_i)^2$$

$$+\sum_{i=1}^{N-1}E\left|\sum_{k=1}^{[nt]}E((\xi_{nk}, e_i)^2|\mathcal{F}_{k-1}^n) - t(\mathbb{S}e_i, e_i)\right| + \sum_{i=N}^{\infty}(\mathbb{S}e_i, e_i).$$

Remark 1. For the scheme of series of independent (for each n) random variables (ξ_{nk}), $1 \leq k \leq n$, in Corollary 4 and 5 conditional expectations are replaced by unconditional. For a fixed $t = 1$ Corollary 5 implies the following central limit theorem for series such random variables.

Corollary 6. *Suppose for a scheme of series of independent square integrable random variables with values in Hilbert space* $(\xi_{nk}), 0 \leq k \leq n, n \geq 1$, *the following conditions:*

$$\sum_{k=1}^{n}E(\xi_{nk}, e_i)(\xi_{nk}, e_j) \to (\mathbb{S}e_i, e_j), \quad i, j \geq 1, \quad \sum_{k=1}^{n}E||\xi_{nk}||^2 \to Tr\mathbb{S}. \tag{24}$$

If, moreover,

$$\sum_{k=1}^{n}E||\xi_{nk}||^2 I(||\xi_{nk}|| > \varepsilon) \to 0 \tag{25}$$

then $X^n = \sum_{k=1}^{n}\xi_{nk} \xrightarrow{d} \mathcal{N}(0, \mathbb{S})$.

Remark 2. Let $\mathbb{S}^n is \mathbb{S}$-operator defined by $(\mathbb{S}^n y, y) = E\sum_{k=1}^{n}(\xi_{nk}, y)^2$, $y \in \mathbb{H}$. Conditions (24) can be rewritten as follows:

$$(\mathbb{S}^n e_i, e_j) \to (\mathbb{S}e_i, e_j), i, j \geq 1, \tag{26}$$

$$Tr\mathbb{S}^n \to Tr\mathbb{S}. \tag{27}$$

One can easily verify that under the assumption (26) condition (27) is equivalent to compactness of the family $\mathbb{S}$-operator $\{\mathbb{S}^n\}$, i.e.

$$\sup_n\sum_{i=1}^{\infty}(\mathbb{S}^n e_i, e_i) < \infty, \quad \sup_n\sum_{i=r}^{\infty}(\mathbb{S}^n e_i, e_i) \to 0, r \to \infty. \tag{28}$$

Thus, the conditions (24) are equivalent to (26), (28), which have been used in the works [4, 5] in the proof of the corresponding central limit theorem, i.e. Corollary 6. It has been shown in [5] that under the additional assumption

$$\max_{1\leq k\leq n}P(||\xi_{nk}|| > \varepsilon) \to 0, \quad \varepsilon > 0,$$

condition (25) is also necessary.

Remark 3. For $t = 1$ the first part of Corollary 5 is proved in [9]. The functional central limit theorem for continuous processes built by partial sums of square-integrable martingale differences with a few more stringent assumptions than (16)–(18), has been obtained in [8].

The first assertion of Corollary 5 implies the corresponding results of [4, 5, 8, 9].

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