

On the Equivalence Problem of Generalized Abel ODEs under the Action of the Linear Transformations Pseudogroup

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Abstract—In the present paper we establish the necessary and sufficient conditions for two generalized Abel differential equations to be locally equivalent under the action of the pseudogroup of linear transformations of the form $\{x \mapsto f(x), y \mapsto g(x) \cdot y + h(x)\}$. These conditions are formulated in terms of differential invariants.

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1. INTRODUCTION

The first kind Abel differential equation is the ODE of the form

$$y' = a(x)y^3 + b(x)y^2 + c(x)y + d(x). \quad (1)$$

It was introduced by Abel in the paper [1]. In what follows we assume that the functions a, b, c, d are of class C^∞ and that $a \neq 0$.

The pseudogroup G of point transformations of the form

$$x \mapsto f(x), \quad y \mapsto g(x) \cdot y + h(x), \quad f, g, h \in C^\infty(\mathbb{R}), \quad (2)$$

preserves the class of such ODEs. The problem of equivalence of equations (1) under the action of this pseudogroup was studied in the papers of Liouville [4] and Appell [2]. There are two basic relative invariants (i.e. functions F in coefficients of ODE (1) and their derivatives, for which the equality $F = 0$ remains invariant under the action of G):

$$s_1 = a, \quad s_3 = a'b - b'a + abc - \frac{2}{9}b^3 - 3a^2d.$$

Starting from s_3 one can construct a sequence of relative invariants by the formula

$$s_{2n+1} = a \frac{ds_{2n-1}}{dx} - (2n-1)s_{2n-1} \left(a' + ac - \frac{1}{3}b^2 \right), \quad n \geq 2.$$

Using these, one obtains a sequence of absolute invariants (i.e. functions that remain invariant under the action of G):

$$J_1 = s_5^3/s_3^5, \quad J_2 = s_5s_7/s_3^4, \quad J_3 = s_9/s_3^3, \dots \quad (3)$$

Appell proved that this sequence can also be obtained from two basic absolute invariants J_1, J_2 , by expressing J_2 as a function of J_1 and then differentiating the result with respect to J_1 . Using the

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transformations (2) one can reduce the Abel equation to the canonical form $Y' = Y^3 + R$. Moreover, the transformations

$$\tilde{X} = K^{-2}(X + h), \quad \tilde{Y} = KY, \quad \text{where } K, h \in \mathbb{R}, K \neq 0 \tag{4}$$

send one canonical form to another one.

Theorem 1 ([2, 6]). *Two Abel equations are equivalent if and only if they have the same canonical form modulo transformations (4).*

The problem of equivalence of Abel equations with non-constant invariants was also considered by Cheb-Terrab and Roche in [3]. Note that $R = 0$ if and only if $s_3 = 0$ (see, e. g., [2, 6]). It follows that all Abel equations satisfying $s_3 = 0$ are equivalent.

2. ANOTHER APPROACH TO THE EQUIVALENCE OF Abel ODEs

We formulate another theorem about the equivalence of Abel equations under the action of G . Every Abel equation \mathcal{E} is a section of the 4-dimensional bundle

$$\pi : \mathbb{R}^5 \rightarrow \mathbb{R}, \quad (x, a, b, c, d) \rightarrow x$$

and the pseudogroup G acts on these sections. The Lie algebra \mathfrak{g} corresponding to G consists of vector fields

$$X = \xi(x) \frac{\partial}{\partial x} + (\eta(x) \cdot y + \zeta(x)) \frac{\partial}{\partial y}.$$

The representation of the Lie algebra \mathfrak{g} into the Lie algebra of vector fields on π has the form

$$\begin{aligned} \hat{X} = \xi \frac{\partial}{\partial x} - (2\eta + \xi')a \frac{\partial}{\partial a} - (\xi'b + 3\zeta a + \eta b) \frac{\partial}{\partial b} + (\eta' - \xi'c - 2\zeta b) \frac{\partial}{\partial c} \\ + (\zeta' - \zeta c + \eta d - \xi'd) \frac{\partial}{\partial d}. \end{aligned}$$

Definition 1. By an *(absolute) differential invariant* of order k of the action of G on π we understand a function $I \in J^k(\pi)$ which is constant along the orbits of the prolonged action of G .

The infinitesimal version of this definition is the equality $\hat{X}^{(k)}(I) = 0$ for all $X \in \mathfrak{g}$, where $\hat{X}^{(k)}$ denotes the k -th prolongation of \hat{X} to $J^k(\pi)$. The set of all differential invariants is an algebra.

We denote the fiber coordinates in $J^k(\pi)$ by a', a'', \dots . Let

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + a' \frac{\partial}{\partial a} + b' \frac{\partial}{\partial b} + c' \frac{\partial}{\partial c} + a'' \frac{\partial}{\partial a'} + b'' \frac{\partial}{\partial b'} + c'' \frac{\partial}{\partial c'} + \dots$$

denote the total derivative operator with respect to x . We say that an *invariant derivation* is an operator

$$\nabla = A \frac{D}{Dx}, \quad A \in C^\infty(J^\infty(\pi)), \tag{5}$$

which is invariant under the prolonged action of G . This is equivalent to the fact that $[\nabla, \hat{X}] = 0$ for every $X \in \mathfrak{g}$. The coefficient A satisfies a certain PDE system, see [5]. For every differential invariant I , the function ∇I also is an invariant. Obviously, for every differential invariant J the operator $J \cdot \nabla$ also is an invariant derivation. It follows easily that any two invariant derivations (5) are proportional.

The first nontrivial differential invariant is J_1 , it appears in order 2. One can easily verify that $\nabla = \frac{s_1}{s_3^{2/3}} \frac{D}{Dx}$ is an invariant derivation. Note that invariants (3) satisfy the equality $J_2 = \nabla(J_1^{1/3}) + 15J_1$.

The submanifold $\{s_3 = 0\} \subset J^1(\pi)$ is a singular orbit for the action of G . We call the point $z_k \in J^k(\pi)$ *regular*, if $s_1 s_3 \neq 0$ at this point. In what follows we consider only the orbits of regular points.

Let J be a differential invariant such that $DJ/Dx \neq 0$ on some open interval Δ . Then for every function F on Δ one has

$$\frac{DF}{Dx} = \lambda \frac{DJ}{Dx}.$$

The coefficient λ is called the *Tresse derivative* of F and is denoted by $\lambda = DF/DJ$. The operator D/DJ is an invariant derivation (see [5]). For any invariant derivation ∇ one has $\nabla = K \cdot D/DJ$ for some invariant K . Substituting J into this equality we see that $K = \nabla J$. Thus,

$$\nabla = \nabla J \cdot \frac{D}{DJ} \quad (6)$$

for every invariant J and invariant derivation ∇ .

Theorem 2. *The algebra of differential invariants of the action of G on π is generated by the invariant J_1 and the invariant derivation ∇ . This algebra separates regular orbits.*

Proof. One can easily see that the k -th prolongation of \widehat{X} depends on the $(k+1)$ -jets of functions ξ , η , ζ . Let Ξ_i^k, H_i^k, Z_i^k denote the components of the decomposition

$$\widehat{X}^{(k)} = \sum_{i=0}^{k+1} \left(\xi^{(i)}(x) \Xi_i^k + \eta^{(i)}(x) H_i^k + \zeta^{(i)}(x) Z_i^k \right).$$

The vector fields $\Xi_i^k, H_i^k, Z_i^k, i = 0, \dots, k+1$, generate the completely integrable distribution on $J^k(\pi)$ and its integral submanifolds are exactly orbits of the action of G .

Let \mathcal{O}_k be an orbit in $J^k(\pi)$. Its projection $\mathcal{O}_{k-1} = \pi_{k,k-1}(\mathcal{O}_k) \subset J^{k-1}(\pi)$ is an orbit in $J^{k-1}(\pi)$. Let $z_{k-1} \in J^{k-1}(\pi)$ be a point such that $\widehat{X}^{(k)}$ is $\pi_{k,k-1}$ -vertical over it. Since the components of $\widehat{X}^{(k)}$ depend on $\xi^{(k+1)}, \eta^{(k+1)}, \zeta^{(k+1)}$, it follows that the bundles $\pi_{k,k-1} : \mathcal{O}_k \rightarrow \mathcal{O}_{k-1}$ are 3-dimensional for $k \geq 2$. Orbits in the space of 2-jets can be found by direct integration of 12-dimensional completely integrable distribution generating by the vector fields $\Xi_i^k, H_i^k, Z_i^k, i = 0, 1, 2, 3$.

Since the bundles $\pi_{k,k-1} : J^k(\pi) \rightarrow J^{k-1}(\pi)$ are 4-dimensional, it follows that for $k \geq 2$ there is one differential invariant of pure order k and so the dimension of algebra of differential invariants of order $\leq k$ equals $k-1$.

The invariant J_1 generates the space of differential invariants of pure order 2 and separates regular orbits in $J^2(\pi)$. Moreover, J_1 is linear in second order derivatives a'' and b'' (and does not depend on c'' , d''), and the coefficient $s_1/s_3^{2/3}$ in ∇ is the function on $J^1(\pi)$. It follows that for $k \geq 1$ the invariant $\nabla^k J_1$ is linear in $a^{(k+2)}, b^{(k+2)}$ and thus generates the space of differential invariants of pure order $k+2$ and separates regular orbits. \square

Consider the space \mathbb{R}^2 with coordinates (j_1, j_{11}) . For every Abel ODE \mathcal{E} we define the map $\sigma_{\mathcal{E}} : \Delta \rightarrow \mathbb{R}^2$ by $j_1 = J_1^{\mathcal{E}}, j_{11} = (\nabla J_1)^{\mathcal{E}}$, where $\Delta \subset \mathbb{R}$ is an open interval and the superscript \mathcal{E} means that the invariants are evaluated at the coefficients of \mathcal{E} . Clearly, the image $\Sigma_{\mathcal{E}} = \text{im}(\sigma_{\mathcal{E}}) \subset \mathbb{R}^2$ depends only on equivalence class of \mathcal{E} .

Definition 2. We say that the equation \mathcal{E} is *regular* at a point $x \in \mathbb{R}$, if:

- i) 2-jets of coefficients of \mathcal{E} belong to regular orbits;
- ii) $\sigma_{\mathcal{E}}(\Delta)$ is a smooth curve in \mathbb{R}^2 for some open interval Δ , containing x ;
- iii) one of the functions j_1, j_{11} can be chosen as a local coordinate on $\Sigma_{\mathcal{E}}$.

Theorem 3. *Two regular equations \mathcal{E} and $\overline{\mathcal{E}}$ are locally G -equivalent if and only if*

$$\Sigma_{\mathcal{E}} = \Sigma_{\overline{\mathcal{E}}}. \quad (7)$$

Proof. The necessity is obvious.

Assume that (7) holds. Let us show that \mathcal{E} and $\overline{\mathcal{E}}$ are equivalent.

Without loss of generality, we may suppose that j_1 is a local coordinate on $\Sigma_{\mathcal{E}}$. Let $J_{11}^{\mathcal{E}} = j_{11}^{\mathcal{E}}(j_1)$ on $\Sigma_{\mathcal{E}}$ and $J_{11}^{\overline{\mathcal{E}}} = j_{11}^{\overline{\mathcal{E}}}(j_1)$ on $\Sigma_{\overline{\mathcal{E}}}$. The condition (7) means that

$$J_{11}^{\mathcal{E}} = J_{11}^{\overline{\mathcal{E}}}. \quad (8)$$

One can see by (6) that the invariant derivation ∇ is proportional to the Tresse derivative D/DJ_1 with coefficient ∇J_1 . It follows from (8) and from Theorem 2 that the restrictions to \mathcal{E} and $\overline{\mathcal{E}}$ of differential

invariants of all orders coincide. Since the algebra of differential invariants separates regular orbits, it follows that \mathcal{E} and $\bar{\mathcal{E}}$ are G -equivalent. \square

Remark 1. Note that, by the inverse function theorem, the conditions ii)-iii) in the definition of a regular equation are equivalent to the condition that either $J_1(x)$ or $\nabla J_1(x)$ have non-zero derivative at a point x , that is either $\nabla J_1(x) \neq 0$ or $\nabla^2 J_1(x) \neq 0$.

3. EQUIVALENCE OF GENERALIZED Abel ODEs

By a generalized Abel differential equation we mean an ODE of the form

$$y' = a_k(x)y^k + a_{k-1}(x)y^{k-1} + \dots + a_1(x)y + a_0(x), \quad a_k(x) \neq 0.$$

We suppose all functions $a_0(x), \dots, a_k(x)$ to be of class C^∞ . We consider the problem of equivalence of such ODEs under the action of the pseudogroup of linear transformations for the cases $k = 4$ and $k = 5$. The computations of differential invariants were performed using the `Maple` packages `DifferentialGeometry` and `JetCalculus` by I.M. Anderson.

3.1. Regular Case for $k = 4$

We start with the case $k = 4$, that is, the equation

$$y' = a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + e(x), \quad a(x) \neq 0. \tag{9}$$

These equations can be identified with the sections of the 5-dimensional bundle

$$\pi : \mathbb{R}^6 \rightarrow \mathbb{R}, \quad (x, a, b, c, d, e) \rightarrow x.$$

The representation of \mathfrak{g} into the Lie algebra of vector fields on π consists of vector fields

$$\begin{aligned} \widehat{X} = & \xi \frac{\partial}{\partial x} - (3\eta + \xi')a \frac{\partial}{\partial a} - (\xi'b + 4\zeta a + 2\eta b) \frac{\partial}{\partial b} - (\xi' + 3\zeta b + \eta) \frac{\partial}{\partial c} \\ & + (\eta' - 2\zeta c - \xi'd) \frac{\partial}{\partial d} + (\zeta' - \zeta d + \eta e - \xi'e) \frac{\partial}{\partial e}. \end{aligned}$$

Definition 3. The function $F \in C^\infty(J^k \pi)$ is called the *relative differential invariant of k -th order*, if for all $g \in G$ there holds $g^*F = \mu(g) \cdot F$, where $\mu : G \rightarrow C^\infty(J^\infty \pi)$ is a smooth function, called the *weight function*. The function $F \in C^\infty(J^k \pi)$ is called the *absolute differential invariant of k -th order*, if $g^*F = F$ for all $g \in G$.

Again, we denote the fiber coordinates in $J^k(\pi)$ by a', a'' , etc. There are four basic relative invariants – two of order 0 and two of order 1:

$$\begin{aligned} I_0 = a, \quad I_1 = 8ac - 3b^2, \quad I_2 = 3(ab' - a'b) + ac^2 - 3abd + 12a^2e, \\ I_3 = 8aa'(4ac - 3b^2) + 24a^2bb' - 32a^3c' - 3b^5 + 64a^3cd - 24a^2b^2d - 32a^2bc^2 + 20ab^3c. \end{aligned}$$

Definition 4. We say that the point $z_k \in J^k(\pi)$ is *regular*, if $I_0I_1 = a(8ac - 3b^2)$ does not vanish at this point.

In this subsection we consider orbits of regular points only. First absolute invariants appear in order 1. They are

$$J_1 = I_2I_0/I_1^2, \quad J_2 = I_3/|I_1|^{5/2}.$$

One can verify that the invariant derivation is

$$\nabla = \frac{I_0^2}{|I_1|^{3/2}} \frac{D}{Dx}.$$

Theorem 4. *The algebra of differential invariants of the action of G is generated by J_1 and J_2 and the invariant derivation ∇ . This algebra separates regular orbits.*

Proof. The proof is similar to that of Theorem 2. We just mention the differences.

One expects four differential invariants in the order ≤ 2 . They are $J_1, J_2, \nabla J_1$ and ∇J_2 . These invariants are linear in first order and second derivatives respectively. Thus, they generate the space of differential invariants of order ≤ 2 and separate regular orbits.

The bundles $\pi_{k,k-1} : J^k(\pi) \rightarrow J^{k-1}(\pi)$ are 5-dimensional, hence for $k \geq 3$ there are two differential invariants of pure order k . The dimension of algebra of differential invariants of order $\leq k$ equals $2k$. It follows that for $k \geq 2$ the invariants $\nabla^k J_i, i = 1, 2$, are linear in $a^{(k+1)}, b^{(k+1)}, c^{(k+1)}$. The latter two invariants generate the space of differential invariants of pure order $k + 1$ and separate regular orbits. \square

Consider the space \mathbb{R}^4 with coordinates $(j_1, j_2, j_{11}, j_{12})$. For every generalized Abel ODE \mathcal{E} of the form (9) define the map $\sigma_{\mathcal{E}} : \mathbb{R} \supset \Delta \rightarrow \mathbb{R}^4$ by

$$j_1 = J_1^{\mathcal{E}}, \quad j_2 = J_2^{\mathcal{E}}, \quad j_{11} = (\nabla J_1)^{\mathcal{E}}, \quad j_{12} = (\nabla J_2)^{\mathcal{E}},$$

where the superscript \mathcal{E} means that the invariants are evaluated at the coefficients of \mathcal{E} .

Definition 5. We say that the equation \mathcal{E} is *regular* at a point $x \in \mathbb{R}$, if:

- i) 2-jets of coefficients of \mathcal{E} belong to regular orbits;
- ii) $\Sigma_{\mathcal{E}} = \sigma_{\mathcal{E}}(\Delta)$ is a smooth curve in \mathbb{R}^4 for some open interval Δ , containing x ;
- iii) one of the functions j_1, j_2, j_{11}, j_{12} can be chosen as a local coordinate on $\Sigma_{\mathcal{E}}$.

The proof of the following Theorem 5 is similar to that of Theorem 3.

Theorem 5. *Two regular equations \mathcal{E} and $\bar{\mathcal{E}}$ are locally G -equivalent if and only if $\Sigma_{\mathcal{E}} = \Sigma_{\bar{\mathcal{E}}}$.*

One can check that using the transformations (2) the equation (9) may be reduced to the canonical form $Y' = Y^4 + R_1 Y^2 + R_2$, $R_1 \neq 0$. The transformations $\tilde{X} = K^{-3}(X + h)$, $\tilde{Y} = KY$ map one canonical form to another one.

3.2. Singular Case for $k = 4$

Let us now consider the case when I_1 vanishes identically, that is, the class of ODEs (9) for which $8ac = 3b^2$. This class consists of ODEs of the form

$$y' = (p(x)y + q(x))^4 + r(x)y + s(x), \quad p(x) \neq 0. \quad (10)$$

These equations can be identified with the sections of the 4-dimensional bundle $\pi : \mathbb{R}^5 \rightarrow \mathbb{R}$, $(x, p, q, r, s) \rightarrow x$ and the representation of \mathfrak{g} into the Lie algebra of vector fields on π consists of vector fields

$$\begin{aligned} \hat{X} = & \xi \frac{\partial}{\partial x} - \frac{1}{4}(3\eta + \xi')p \frac{\partial}{\partial p} + \frac{1}{4}((\eta - \xi')q - 4\zeta p) \frac{\partial}{\partial q} \\ & + (\eta' - \xi'r) \frac{\partial}{\partial r} + (\zeta' + (\eta - \xi')s - \zeta r) \frac{\partial}{\partial s}. \end{aligned}$$

There are three basic relative invariants:

$$\begin{aligned} L_0 = p, \quad L_1 = q'p - p'q + p^2s - pqr, \\ L_2 = p(pq'' - qp'') + 6p'(p'q - pq') + p'p(9qr - 4ps) - 5rp^2q' - p^2qr' + p^3s' + 4p^2r(qr - ps). \end{aligned}$$

In fact, they are restrictions of relative invariants from the regular case.

We restrict ourselves to the orbits of the points for which L_1 does not vanish. Then there is one absolute invariant of order 2

$$J = \frac{L_2}{L_0^2 \cdot |L_1|^{7/2}}.$$

The invariant derivation is

$$\nabla = \frac{|L_0|^{1/2}}{|L_1|^{3/4}} \frac{D}{Dx}.$$

The following two theorems are proved in the same manner as above.

Theorem 6. *The algebra of differential invariants of the action of G is generated by J and the derivation ∇ . This algebra separates regular orbits.*

Theorem 7. *Two ODEs \mathcal{E} and $\bar{\mathcal{E}}$ of the form (10) are G -equivalent in the neighborhood of a point x if and only if*

$$J^{\mathcal{E}} = J^{\bar{\mathcal{E}}}, \quad (\nabla J)^{\mathcal{E}} = (\nabla J)^{\bar{\mathcal{E}}}$$

and one of the functions $J^{\mathcal{E}}$ or $(\nabla J)^{\mathcal{E}}$ has non-zero derivative at x .

Note that the canonical form of the ODE (10) is $Y' = Y^4 + \frac{L_1}{L_0} \exp(-r)$. Thus, the ODEs (10) having $L_1 = 0$ are all equivalent to the ODE $Y' = Y^4$.

3.3. Regular Case for $k = 5$

Now we deal with the equations of the form

$$y' = a(x)y^5 + b(x)y^4 + c(x)y^3 + d(x)y^2 + e(x)y + f(x), \quad a(x) \neq 0. \quad (11)$$

These equations can be identified with the sections of the 6-dimensional bundle

$$\pi : \mathbb{R}^7 \rightarrow \mathbb{R}, \quad (x, a, b, c, d, e, f) \rightarrow x.$$

There are following basic relative invariants of the action of G :

$$\begin{aligned} K_0 &= a, & K_1 &= 5ac - 2b^2, & K_2 &= 4b^3 - 15abc + 25a^2d, \\ K_3 &= 50ab' - 50ba' + 8b^2d + 5acd - 50abe - 3bc^2 + 250a^2f, \\ K_4 &= 2500a^2da' + 1500a^2bc' - 2500a^3d' - 1500a^2cb' + 825a^2c^2d + 6000a^2b^2f - 495abc^3 \\ &\quad + 1440ab^2cd - 3000a^2bd^2 - 288b^4d - 1500a^2bce + 7500a^3de - 15000a^3cf + 108b^3c^2. \end{aligned}$$

Definition 6. We say that the point $z_k \in J^k(\pi)$ is *regular*, if $K_0K_1 = a(5ac - 2b^2)$ does not vanish at this point. In this subsection we consider orbits of regular points only.

The action of G has the following three basic absolute invariants, one of order 0 and two of order 1:

$$J_0 = \frac{K_2^2}{K_1^3}, \quad J_1 = \frac{K_3K_0^2}{|K_1|^{5/2}}, \quad J_2 = \frac{K_4K_0^2}{|K_1|^{7/2}}$$

and the invariant derivation is

$$\nabla = \frac{K_0^3}{K_1^2} \frac{D}{Dx}.$$

Theorem 8. *The algebra of differential invariants of the action of G is generated by J_0, J_1, J_2 and the derivation ∇ . This algebra separates regular orbits.*

Theorem 9. *Two ODEs \mathcal{E} and $\bar{\mathcal{E}}$ of the form (11) are G -equivalent in the neighborhood of a point x if and only if*

$$\begin{aligned} J_0^{\mathcal{E}} &= J_0^{\bar{\mathcal{E}}}, & J_1^{\mathcal{E}} &= J_1^{\bar{\mathcal{E}}}, & J_2^{\mathcal{E}} &= J_2^{\bar{\mathcal{E}}}, \\ (\nabla J_0)^{\mathcal{E}} &= (\nabla J_0)^{\bar{\mathcal{E}}}, & (\nabla J_1)^{\mathcal{E}} &= (\nabla J_1)^{\bar{\mathcal{E}}}, & (\nabla J_2)^{\mathcal{E}} &= (\nabla J_2)^{\bar{\mathcal{E}}} \end{aligned}$$

and at least one of the above six functions has non-zero derivative at x .

The canonical form of the ODE (11) with respect to the action of G is

$$Y' = Y^5 + R_1Y^3 + R_2Y^2 + R_3, \quad R_1 \neq 0.$$

The transformations $\tilde{X} = K^{-4}(X + h), \tilde{Y} = KY$ permute the canonical forms.

3.4. Singular Cases for $k = 5$

The first singular case arises when K_1 vanishes, that is, when $5ac = 2b^2$. It corresponds to the ODEs of the form

$$y' = (p(x)y + q(x))^5 + r(x)y^2 + s(x)y + t(x), \quad p, r \neq 0. \quad (12)$$

The action of G on the class of ODEs (12) has the following basic relative invariants

$$\begin{aligned} M_0 &= p, & M_1 &= r, & M_2 &= -p'q + q'p + q^2r - pqs + tp^2, \\ M_3 &= 5p'r + 3prs - 6qr^2 - pr'. \end{aligned}$$

They provide two absolute invariants of order 1:

$$J_0 = \frac{L_2 \cdot p^{4/3}}{r^{5/3}}, \quad J_1 = \frac{L_3 \cdot p^{2/3}}{r^{7/3}}.$$

The invariant derivation is

$$\nabla = \frac{p^{5/3}}{r^{4/3}} \frac{D}{Dx}.$$

Theorem 10. *Two ODEs \mathcal{E} and $\bar{\mathcal{E}}$ of the form (12) are G -equivalent in the neighborhood of a point x if and only if*

$$J_0^{\mathcal{E}} = J_0^{\bar{\mathcal{E}}}, \quad J_1^{\mathcal{E}} = J_1^{\bar{\mathcal{E}}}, \quad (\nabla J_0)^{\mathcal{E}} = (\nabla J_0)^{\bar{\mathcal{E}}}, \quad (\nabla J_1)^{\mathcal{E}} = (\nabla J_1)^{\bar{\mathcal{E}}}$$

and at least one of the above functions has non-zero derivative at x .

The canonical form for such ODEs is $Y' = Y^5 + R_2Y^2 + R_3$. The second singular case arises when both K_1 and K_2 vanish and corresponds to the ODEs of the form

$$y' = (p(x)y + q(x))^5 + s(x)y + t(x), \quad p \neq 0. \quad (13)$$

The relative invariants are

$$\begin{aligned} M_0 &= p, & M_2 &= -p'q + q'p - pqs + tp^2, \\ M_4 &= p(pq'' - qp'') + 7p'(p'q - pq') + p^3t' - p^2qs' - 6p^2sq' + p'p(11qs - 5pt) + 5p^2s(qs - pt). \end{aligned}$$

For the subclass of ODEs for which $M_2 \neq 0$ there is one basic absolute invariant

$$J = \frac{M_4}{p^{2/5} \cdot M_2^{9/5}}$$

and the invariant derivation

$$\nabla = \frac{p^{3/5}}{M_2^{4/5}} \frac{D}{Dx}.$$

The canonical form for such ODEs is $Y' = Y^5 + R_3$ and the case $M_2 = 0$ corresponds to $R_3 = 0$.

Theorem 11. *Two ODEs \mathcal{E} and $\bar{\mathcal{E}}$ of the form (13) are G -equivalent in the neighborhood of a point x if and only if $J^{\mathcal{E}} = J^{\bar{\mathcal{E}}}$, $(\nabla J)^{\mathcal{E}} = (\nabla J)^{\bar{\mathcal{E}}}$ and at least one of the above functions has non-zero derivative at x .*

REFERENCES

1. N. Abel, *J. Reine. Angew. Math.* **4**, 309–348 (1829).
2. P. Appell, *J. de Mathématique* **5**, 361–423 (1889).
3. E. S. Cheb-Terrab and A. D. Roche, *Computer Physics Communications*. 01/2000; DOI: 10.1016/S0010-4655(00)00042-4, arXiv:math-ph/0001037v1 (2000).
4. R. Liouville, *Comptes Rendus Acad. Sci.* **105**, 460–463 (1887).
5. V. Lychagin, *Zb. Pr. Inst. Mat. NAN Ukr.* **6** (2), 288–302 (2009).
6. O. Wone, arXiv:1401.2375[math.DG] (2014).