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On Decidability of the Theory with the Transitive Closure Operator

A. S. Zolotov*

(Submitted by M.M. Arslanov)

Tver State University, ul. Zhelyabova 33, Tver, 170100 Russia

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Abstract—In this paper we consider a theory of integers with the successor function, divisibility predicates, equality and a transitive closure operator. The order relation can be expressed in this theory using the transitive closure operator.
We prove that given a formula with the transitive closure operator on a single pair of variables one can effectively eliminate it.

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1. INTRODUCTION

The decidability problem is one of the most important problems of mathematical logic. There are classical results concerning this problem. One of the first undecidable theories were found by Church (see [3, 4, 2]) and Rosser (see [6]). In 1929 Presburger proved that arithmetic without multiplication is decidable. An effective quantifier elimination method was employed. Given a formula $\exists x\phi$, a new formula $\psi \equiv \exists x\phi$ without quantifiers is effectively constructed.

It's well known that the transitive closure can not be expressed in relational algebra (see [1]). So one can extend some first order theory with the transitive closure operator and consider its decidability.

We consider a first order theory of integers with an unary successor function $s(x)$, divisibility predicates $D_p(x)$, equality and a transitive closure operator.

An applying of the transitive closure operator to formulas containing only equality and divisibility predicates is considered in [7]. It's easy to prove that the transitive closure operator on two pairs of variables makes it possible to express addition and multiplication of integers. Thus the transitive closure operator on two pairs of variables makes the theory undecidable.

In this paper we consider the transitive closure operator on a single pair of variables.

2. DEFINITIONS

Definition 1. *Let $\psi(\bar{x},\bar{y})$ be a formula, tuples $\bar{x}$ and $\bar{y}$ have the same length. Then $T_{\bar{x},\bar{y}}\left(\psi\left(\bar{x},\bar{y}\right)\right)$ is also a formula called* ***transitive closure*** *of $\psi\left(\bar{x},\bar{y}\right)$ on $\bar{x},\bar{y}$. A value of an individual variable x is denoted by $I(x)$.*

A formula $T_{\bar{x},\bar{y}}\left(\psi\left(\bar{x},\bar{y}\right)\right)$ is true iff $I(\bar{x}) = I(\bar{y})$ or there is a sequence of tuples $\bar{a}_1,\dots,\bar{a}_n$ such that $\psi(\bar{a}_1,\bar{a}_2)\wedge\psi(\bar{a}_2,\bar{a}_3)\wedge\dots\wedge\psi(\bar{a}_{n-1},\bar{a}_n)$, and $I(\bar{x}) = \bar{a}_1, I(\bar{y}) = \bar{a}_n$. We call the sequence $\bar{a}_1,\dots,\bar{a}_n$ and a formula $\psi(\bar{x}_1,\bar{x}_2)\wedge\psi(\bar{x}_2,\bar{x}_3)\wedge\dots\wedge\psi(\bar{x}_{n-1},\bar{x}_n)\wedge\bar{x}=\bar{x}_1\wedge\bar{y}=\bar{x}_n$ such that *$\psi(\bar{a}_1,\bar{a}_2)\wedge\psi(\bar{a}_2,\bar{a}_3)\wedge\dots\wedge\psi(\bar{a}_{n-1},\bar{a}_n)$* ***a path*** *from $\bar{a}_1$ to $\bar{a}_n$ with* ***a link*** *ψ. We call $\bar{a}_1,\dots,\bar{a}_n$ path* ***nodes****. We say that this path* ***proves*** *$T_{\bar{x},\bar{y}}\left(\psi\left(\bar{x},\bar{y}\right)\right)$ with $I(\bar{x}) = \bar{a}_1$ and $I(\bar{y}) = \bar{a}_n$.*

An expression $\underbrace{s(s(\dots s}_{k \text{ times}}(x)\dots)$ is be denoted by $s^k(x)$.

*E-mail: yesbody2@gmail.com

3. ORDER RELATION

Our main technical result can be represented with the following

Lemma 1. *Formula*

$$T_{x,y}\left(\left[\bigvee_{j=1}^{M_1} x < s^{t_j}(y) \wedge D_p(s^{a_j}(x)) \wedge D_p(s^{b_j}(y))\right]\right.$$

$$\left.\vee\left[\bigvee_{j=1}^{M_2} x > s^{t'_j}(y) \wedge D_p(s^{a'_j}(x)) \wedge D_p(s^{b'_j}(y))\right] \vee \left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]\right),$$

where $p > \max_i |k_i|$ *is true iff there is a path and all of the path nodes are contained in* $(\min(I(x), I(y)) - h, \max(I(x), I(y)) + h)$, $h = p(p^2 + p(M_1 + M_2) + 1)$

Proof. Let the transitive closure be true. Then there is a path C without repeats. Let a_{min} be the least node of C. Assume that $I(x) < I(y)$. If $a_{min} = I(x)$, then all of the path nodes are greater than $I(x)$. They are also greater than $I(x) - h$. If $a_{min} < I(x)$, then construct non-intersecting segments of length p from $I(x)$ to a_{min}. Let r be the amount of these segments. Let's enumerate them from right to left. There is a node u in C such that $I(x) < u$ and number of u is greater than number of a_{min} in the path C. Divide the path part from $I(x)$ to u in two.The first part is from $I(x)$ to a_{min} and the second part is from a_{min} to u. Let these parts be denoted by C' and C'' respectively.

Consider the nodes of C' and extract a monotone decreasing subsequence $(b_i)_{i=1}^{q_1}$ such that $b_1 = I(x)$ and b_{i+1} is a node less than b_i with the least number. The last element of this subsequence is a_{min}. Extract a monotone increasing subsequence $(c_i)_{i=1}^{q_2}$ from the nodes of C'' such that $c_1 = a_{min}$ and c_{i+1} is a node greater than c_i with the least number. If $b_i < b_j$, then number of b_i in C' is greater than number of b_j in C'. Similarly if $c_i > c_j$, then number of c_i in C'' is greater than number of c_j in C''. If there is no element of (b_i) in some segment, then there is a "greater than" link in C' such that all of the path nodes before this link are to the right of this segment. Then the path skipped this segment possibly returning to it afterwards. Similarly if there is no element of (c_i) in some segment, then there is a "less than" link in C'' such that all of the path nodes before the link are to the left of this segment. Then the path skipped this segment possibly returning to it afterwards. Therefore the paths C' and C'' can be shortened by shifting appropriate path nodes by p. Such shift does not change the remainder of the division by p.

We can assume that there are no empty segments. Consider a segment such that it has at least one element from (b_i), it has no elements from (c_i). If there are at least pM_2 such segments, then there is a pair of segments $o_{i_1}, o_{i_2}, (i_1 < i_2)$ such that elements $w_1 \in o_{i_1}$ and $w_2 \in o_{i_2}$ from (b_i) and $|w_1 - w_2|$ is divisible by p and the path C'' skips these segments using either a single link or the same link twice. If the same link was used twice, then it can be replaced with a single link due to the sequence monotonicity. The node after this link can be shifted by p. Therefore both C' and C'' can be shortened reducing number of segments between $I(x)$ and a_{min}. The remaining case is considered in [7].

It can be shown that similar manipulations can be performed in case there is at least one element from (c_i) but there is no element from (b_i) in the segment in order to shorten the path. One can shorten the right part of the path between $I(y)$ and a_{max} if necessary. Every operation reduces total sum of distances between neighboring nodes. It means that this process eventually ends. It's easy so see that such kind of operations can be performed till there are more than $p^2 + pM_1 + pM_2$ segments concerned.

In case $I(x) \geq I(y)$ the shortening procedure is similar. □

Remark 1. *Consider*

$$x < s^k(y) \wedge D_p(s^a(x)) \wedge D_p(s^b(y))$$

and let $d = \left[\frac{b-a-k}{p}\right] + 1$. *The least* $I(y)$ *satisfying* (1) *is* $I(y) = I(x) + a - b + dp$. *Let* (1) *be true and* $b = I(x) + a - b + dp$. *Then* $I(y) \geq b$ *and* $I(y)$ *and* b *are equivalent modulo* p.

Remark 2. *Similarly one can consider* $x > s^k(y) \wedge D_p(s^a(x)) \wedge D_p(s^b(y))$*. The constant* $d = \begin{cases} \left[\frac{b-a-k}{p}\right] - 1, & \text{if } p \text{ divides } (b-a-k), \\ \left[\frac{b-a-k}{p}\right], & \text{otherwise.} \end{cases}$

Lemma 2. *For a formula*

$$T_{x,y}\left(\underbrace{\left[\bigvee_{j=1}^{M_1} x < s^{t_j}(y) \wedge D_p(s^{a_j}(x)) \wedge D_p(s^{b_j}(y))\right]}_{(2)} \vee \underbrace{\left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]}_{(3)}\right) \quad (1)$$

there is an equivalent formula without order relations inside the transitive closure operator.

Proof. Substitute all of the order relations from (1) with equality according to remark 1 and build a formula

$$\Phi(x,y) \equiv \left(\underbrace{\left[\bigvee_{j=1}^{M_1} y = s^{a_j - b_j + d_j p}(x) \wedge D_p(s^{a_j}(x))\right]}_{(4)} \vee \underbrace{\left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]}_{(5)}\right).$$

Consider a formula

$$\underbrace{T_{x,y}\left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]}_{(6)} \vee \exists z \left(\exists u \exists v \left(\overbrace{T_{x,u}\left(\Phi(x,u)\right)}^{(8)} \wedge \overbrace{T_{v,z}\left(\Phi(v,z)\right)}^{(9)}\right.\right.$$

$$\left.\left.\wedge \left(\underbrace{\left[\bigvee_{j=1}^{M_1} u = s^{a_j - b_j + d_j p}(v) \wedge D_p(s^{a_j}(u))\right]}_{(10)} \wedge \underbrace{\left(\bigvee_{i=0}^{p-1} D_p(s^i(z)) \wedge D_p(s^i(y))\right) \wedge z \le y}_{(7)}\right)\right)\right). \quad (11)$$

We prove (11) is equivalent to (1). Let (1) be true. Let C be its path. If there are no order links in C, then it also proves (6). Otherwise we build a path C' by replacing all order links from C with corresponding equality links. We also shift the nodes. Let $I(u)$ and $I(v)$ be any pair of nodes connected with a link from (10) in C'. Note that C' proves (8)–(10). According to remark 1 the formula (7) is also true.

Let the formula (11) be true. If (6) is true, then the formula (1) is also true. Otherwise there are paths C_1' and C_2' proving (8) and (9) respectively. Both paths can be empty. Construct a new path C by appending a link (10) after C_1' and appending C_2' afterwards. Then replace every non-last occurrence of any link from (10) in C with corresponding order link without shifting the nodes. It's possible due to remark 1. Replace the last occurrence with the corresponding order link and shift the last node to make it equal to $I(y)$. It's possible due to (7). Note that the result path proves the formula (1). □

Lemma 3. *For a formula*

$$T_{x,y}\left(\underbrace{\left[\bigvee_{j=1}^{M_1} x < s^{t_j}(y) \wedge D_p(s^{a_j}(x)) \wedge D_p(s^{b_j}(y))\right]}_{(\Phi_1)}\right.$$

$$\vee \underbrace{\left[\bigvee_{j=1}^{M_2} x > s^{t'_j}(y) \wedge D_p(s^{a'_j}(x)) \wedge D_p(s^{b'_j}(y))\right]}_{(\Phi_2)} \vee \underbrace{\left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]}_{(\Phi_3)}\Bigg) \quad (12)$$

there is an equivalent formula without order relation inside the transitive closure operator.

Proof. Denote $\Phi_1 \vee \Phi_2 \vee \Phi_3$ by Φ. Note that (12) is true iff there is a path C and one of the following conditions holds:

(a) There are no order links in C.

(b) There is at least one occurrence of "less than" link in C, but there are no "greater than" links in it.

(c) There is at least one occurrence of "greater than" link in C, but there are no "less than" links in it.

(d) There are occurrences of both "less than" and "greater than" links in C.

Consider each of these conditions.

If the condition (a) holds, a formula $T_{x,y}\left[\bigvee_{i=1}^{N} x = s^{k_i}(y) \wedge D_p(s^{m_i}(x))\right]$ is true. Not that it does not contain any order relations. According to lemma 2 the condition (b) is equivalent to the following formula

$$\exists z \left(\exists u \exists v \left(T_{x,u}\left[\Phi_1(x,u) \vee \Phi_3(x,u)\right] \wedge T_{v,z}\left[\Phi_1(v,z) \vee \Phi_3(v,z)\right]\right.\right.$$

$$\left.\left.\wedge \left(\left[\bigvee_{j=1}^{M_1} u = s^{a_j - b_j + d_j p}(v) \wedge D_p(s^{a_j}(u))\right] \wedge \left(\bigvee_{i=0}^{p-1} D_p(s^i(z)) \wedge D_p(s^i(y))\right) \wedge z \le y\right)\right)\right).$$

The condition (c) is equivalent to a similar formula as mentioned above, but different in order relation type.

Consider the condition (d). Replace all order relations from Φ with equality according to remarks 1 and 2, denote the the result formula by $\Phi'(x,y)$. Consider the following formula

$$\exists z \left(\Psi(x,z) \wedge \left(\bigvee_{i=0}^{p-1} D_p(s^i(z)) \wedge D_p(s^i(y)) \right)\right),$$

where

$$\Psi(x,z) \equiv \exists u_1 \exists v_1 \exists u_2 \exists v_2 \left(\left[T_{x,u_1}[\Phi'(x,u_1)]\right] \wedge \left[T_{v_1,u_2}[\Phi'(v_1,u_2)]\right]\right.$$

$$\left.\wedge \left[T_{v_2,z}[\Phi'(v_2,z)]\right] \wedge \left[\left(\Phi'_1(u_1,v_1) \wedge \Phi'_2(u_2,v_2)\right) \vee \left(\Phi'_2(u_1,v_1) \wedge \Phi'_1(u_2,v_2)\right)\right]\right).$$

The formulas Φ'_1 and Φ'_2 are constructed by replacing order relations with equality in Φ_1 and Φ_2 respectively. It's easy to see that the condition (d) is equivalent to the formula constructed. If we replace order with equality, then we get a new end of the path equivalent to the original end modulo p. The shift itself is undefined as there are both "less than" and "greater than" links. □

4. EXTERNAL CONSTRAINTS

In this section we consider transitive closure operator for formulas containing individual variables $x, y, z_1, \ldots, z_n$ where x, y are transitive closure operator parameters. We consider order relation case.

Remark 3. *It's easy to see that it's enough to consider non-intersecting intervals as external constraints.*

Definition 2. *Given a set of non-intersecting intervals we say the link* ***preserves an interval*** *iff both nodes to the left and to the right of the link are in the same interval. We say a link* ***does***

not preserve the interval *iff these nodes are in different intervals. We also call them* ***preserving*** or ***non-preserving*** *respectively.*

Lemma 4. *Consider a formula* $T_{x,y}\Big(\underbrace{\bigvee_{j=1}^{N'} \phi'_{ij}(x,y) \wedge \psi'_{ij}}_{(1)}\Big)$, *where* $\psi'_{ij} \equiv \psi'_{i_1 j} \wedge \psi'_{i_2 j}$ *and* $\psi'_{i_1 j}$ *is either* $x < z_{i_1}$, *or* $z_{i_m} < x$, *or a conjunction* $(z_{i_j} < x) \wedge (x < z_{i_{j+1}})$, $\psi'_{i_2 j}$ *is either* $y < z_{i_1}$, *or* $z_{i_m} < y$, *or a conjunction* $(z_{i_j} < y) \wedge (y < z_{i_{j+1}})$ *and* ϕ'_{ij} *does not contain any external constraints. This formula is equivalent to a formula, where the transitive closure operator is applied to some formulas without non-preserving links.*

Proof. Let's give a brief overview of the idea behind the proof. Let there be some non-preserving links in (1). Consider the path C of the original formula. There are several options of the way C goes from one interval to another.

The first one is the following: the equality link $x = s^k(y)$ is used. It means that the path node has to be close to the border of the interval. It means that this link can be used at most k times. Otherwise C has repeats and can be shortened. Therefore this link can be taken away from the transitive closure operator by constructing the appropriate finite disjunction.

The second option is the following: some order link is used. Assume it is used more than once. The path goes from some point with fixed residue modulo p from the first interval to some point with some (but also fixed) residue modulo p from the second interval. We can assume this link is used only once. A node before the first occurrence of the link can be connected with a node after the last occurrence of the link. Therefore this link can be also taken away from the transitive closure operator. □

Lemma 5. *Let* $\xi \equiv T_{x,y}\Big(\bigvee_{i=1}^{N}(\phi_i(x,y) \wedge P(x) \wedge P(y))\Big)$ *and* $\chi \equiv T_{x,y}\Big(\bigvee_{i=1}^{N}(\phi_i(x,y))\Big)$, *where* $P(x) \equiv (z_1 < x) \wedge (x < z_2)$ *and there are no occurrences of individual variables different from* x, y *in* $\phi_i(x,y)$. *Let there be a constant* h *such that* χ *is true iff there is a path and all of this path nodes are contained in* $(\min(I(x), I(y)) - h, \max(I(x), I(y)) + h)$. *Then there is a formula equivalent to* ξ *without occurrences of order relations with* z_1 *or* z_2 *inside the transitive closure operator.*

Proof. It's easy to see that every path begins and ends with nodes from $(I(z_1), I(z_2))$. Assume $|I(z_1) - I(z_2)| \leq 3h$. Then $\bigvee_{l=1}^{3h} s^l(z_1) = z_2$ is true and every occurrence of $P(x)$ can be replaces with a finite disjunction:

$$\bigvee_{l=1}^{3h}\left(s^l(z_1) = z_2 \wedge T_{x,y}\left(\bigvee_{i=1}^{N}\left[\phi_i(x,y) \wedge \bigvee_{j=1}^{l-1} x = s^j(z_1) \wedge \bigvee_{j=1}^{l-1} y = s^j(z_1)\right]\right)\right).$$

Now assume that $|I(z_1) - I(z_2)| > 3h$ and consider all of the possible places $I(x)$ and $I(y)$ can take within the interval.

Let $I(z_1) + h \leq I(x) \leq I(z_2) - h$, $I(z_1) + h \leq I(y) \leq I(z_2) - h$ be true. Then the whole path is within the interval concerned and

$$(x \geq s^h(z_1)) \wedge (s^h(x) \leq z_2) \wedge (y \geq s^h(z_1)) \wedge (s^h(y) \leq z_2) \wedge T_{x,y}\left(\bigvee_{i=1}^{N}(\phi_i(x,y))\right)$$

denoted by Φ_1 is true.

Let q be some constant. There is a finite amount of paths with links from $\{\phi_1, \dots, \phi_N\}$ having q or less nodes, denote this amount by M_q and enumerate them somehow. Let C_i be a path with number i. Consider a formula from C_i:

$$\phi_{i_1}(x_1, x_2) \wedge P(x_1) \wedge P(x_2) \wedge \dots \wedge \phi_{i_j}(x_j, x_{j+1}) \wedge P(x_j) \wedge P(x_{j+1}) \wedge (x = x_1) \wedge (y = x_{j+1}).$$

and construct the following formula

$$\exists x_1, \dots, \exists x_{j+1}\big(\phi_{i_1}(x_1, x_2) \wedge P(x_1) \wedge P(x_2) \wedge \dots$$

$$\cdots \wedge \phi_{i_j}(x_j, x_{j+1}) \wedge P(x_j) \wedge P(x_{j+1}) \wedge (x = x_1) \wedge (y = x_{j+1})\big).$$

Denote this formula by $\eta_i^q(x,y)$, let $\Delta_q(x,y) \equiv \bigvee_{i=1}^{M_q} \eta_i^q(x,y)$. Let $I(x) \in (I(z_1), I(z_1)+h)$ and $I(y) \in (I(z_1), I(z_1)+h)$. Then we have several options. The path can have $2(h-1)$ or less nodes. Otherwise there is at least one node from $[I(z_1)+h, I(z_2)-h]$. Denote the first such node by v and the last such node by u. The path from $I(x)$ to v has $2(h-1)+1$ or less nodes. Similarly the path from u to $I(y)$ has $2(h-1)+1$ or less nodes. Let

$$\Gamma \equiv \Bigg[\Delta_{2(h-1)}(x,y) \vee \exists x' \exists y' \Bigg((x' \geq s^h(z_1)) \wedge (s^h(x') \leq z_2) \wedge (s^h(y') \leq z_2)$$
$$\wedge\ (y' \geq s^h(z_1)) \wedge \Delta_{2(h-1)+1}(x,x') \wedge \Delta_{2(h-1)+1}(y',y) \wedge T_{x',y'} \Bigg(\bigvee_{i=1}^{N} \phi_i(x',y') \Bigg)\Bigg)\Bigg]$$

Construct formula $\Phi_2 \equiv \Big[\bigvee_{i=1}^{h-1} x = s^i(z_1) \Big] \wedge \Big[\bigvee_{i=1}^{h-1} y = s^i(z_1) \Big] \wedge \Gamma.$

The remaining cases $I(x) \in (I(z_2)-h, I(z_2)) \wedge I(y) \in (I(z_2)-h, I(z_2))$, $I(x) \in (I(z_1), I(z_1)+h) \wedge I(y) \in (I(z_2)-h, I(z_2))$, $I(y) \in (I(z_1), I(z_1)+h) \wedge I(x) \in (I(z_2)-h, I(z_2))$ are considered similarly and formulas Φ_3, Φ_4 and Φ_4' are constructed respectively. Let

$$\Omega_1 \equiv \Bigg[\Delta_{2(h-1)}(x,y) \vee \exists x' \Bigg((x' \geq s^h(z_1)) \wedge (s^h(x') \leq z_2)$$
$$\wedge \Delta_{2(h-1)+1}(x,x') \wedge T_{x',y} \Bigg(\bigvee_{i=1}^{N} \phi_i(x',y) \Bigg)\Bigg)\Bigg],$$

$$\Omega_2 \equiv \Bigg[\Delta_{2(h-1)}(x,y) \vee \vee \exists x' \Bigg((x' \geq s^h(z_1)) \wedge (s^h(x') \leq z_2)$$
$$\wedge\ \Delta_{2(h-1)+1}(x',y) \wedge T_{x,x'} \Bigg(\bigvee_{i=1}^{N} \phi_i(x,x') \Bigg)\Bigg)\Bigg].$$

Let $I(x) \in (I(z_1), I(z_1)+h)$ and $I(z_1)+h \leq I(y) \leq I(z_2)-h$. Construct a formula $\Phi_5 \equiv \Big[\bigvee_{i=1}^{h-1} x = s^i(z_1) \Big] \wedge \Big[(s^h(z_1) \leq y) \wedge (s^h(y) \geq z_2) \Big] \wedge \Omega_1$.

The remaining cases $I(x) \in (I(z_2)-h, I(z_2)) \wedge I(z_1)+h \leq I(y) \leq I(z_2)-h$, $I(y) \in (I(z_2)-h, I(z_2)) \wedge I(z_1)+h \leq I(x) \leq I(z_2)-h$, $I(y) \in (I(z_1), I(z_1)+h) \wedge I(z_1)+h \leq I(x) \leq I(z_2)-h$ are considered similarly Φ_5', Φ_6 and Φ_6' are constructed. In both Φ_6 and Φ_6' the formula Ω_2 is used instead of Ω_1. Thus we have considered all of the cases. The resulting formula is the following:

$$\Big((s^{3h}(z_1) < z_2) \wedge P(x) \wedge P(y) \wedge \Big((x = y) \vee (\Phi_1 \vee \Phi_2 \vee \Phi_3 \vee \Phi_4 \vee \Phi_4' \vee \Phi_5 \vee \Phi_5' \vee \Phi_6 \vee \Phi_6') \Big) \Big)$$
$$\vee \bigvee_{l=1}^{3h} \Bigg(s^i(z_1) = z_2 \wedge T_{x,y} \Bigg(\bigvee_{i=1}^{N} \Bigg[\phi_i(x,y) \wedge \bigvee_{j=1}^{l-1} x = s^j(z_1) \wedge \bigvee_{j=1}^{l-1} y = s^j(z_1) \Bigg] \Bigg) \Bigg).$$

□

Now we can prove our main result.

Theorem 1. *The theory of integers with a successor function, divisibility predicates, the order relation and the transitive closure operator on one pair of variables is decidable.*

Proof. A way to effectively eliminate transitive closure operator is given in [7] and lemmas 2–5. Therefore the theory is decidable because Presburger arithmetic is decidable ([2]). □

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REFERENCES

1. A. Aho, J. Ulman, *Proc. of 6th Symp. on Principles of Programming Languages,* pp. 110–119 (1979).
2. G. S. Boolos and R. C. Jefferey, *Computability and Logic* (Cambridge University Press, 1994).
3. A. Church, J. of Symbolic Logic **1**, 40–41 (1936).
4. A. Church, Amer. J. of Math. **58**, 345–363 (1936).
5. M. Presburger, *Comptes Rendus du I congres de Mathematiciens des Pays Slaves*, pp. 92–101 (1929).
6. J. B. Rosser, J. of Symbolic Logic **1**, 87–91 (1936).
7. A. Zolotov, Vestnik TvGU. Seriya Prikl. Mat. **28**, 101–117 (2013).