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A Transformation Scheme for Infinitary First-Order Combinatorics Presenting Computational Level of Expressiveness in Predicate Logic

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Abstract—We continue to develop a new first-order combinatorial approach presenting a conceptual framework for investigations concerning expressive power of first-order logic. In this work, we consider the case of infinitary first-order combinatorics. Based on the universal construction of finitely axiomatizable theories, we introduce some common scheme yielding finitely axiomatizable theories with pre-assigned sets of model-theoretic properties. At an initial stage, a maximum common Turing's computation is performed (one can say, computable Brute Force). Starting from an input block of parameters, the computation yields a computably axiomatizable theory T. Finally, by applying an available version of the universal construction, the theory T is transformed into a finitely axiomatizable theory F that inherits model-theoretic properties of T within the infinitary semantic layer. We also give three demonstrations showing possibilities of this method.

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INTRODUCTION

Universal construction of finitely axiomatizable theories, cf. [5, Ch. 6], can control model-theoretic properties within the infinitary semantic layer MQL whose fundamental nature is established in the framework of a new combinatorial approach for first-order predicate logic, [7]. There are two types of first-order combinatorics: finitary versus infinitary. In this work, we present a general method intended for decision of a class of problems concerned with the infinitary layer of expressiveness of first-order predicate logic by way of constructions of finitely axiomatizable theories with pre-assigned sets of model-theoretic properties. Three typical demonstrations are given showing possible variants of application of this method. At an initial stage, we create an abstract project of Stone space of a theory together with a design of a so-called skeleton for the computably axiomatizable theory T we are going to construct; after that, we specify axiomatic of the theory that would provide the required block of model-theoretic properties of different complete extensions of this theory. Finally, we perform a transformation from the theory T to a finitely axiomatizable theory F in correspondence with the standardized scheme for infinitary first-order combinatorics. Adequate coordination of details of the construction with the purposes of the problem under consideration successfully realizes a solution to the problem.

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PRELIMINARIES

Theories in first-order predicate logic with equality are considered. General concepts of model theory, algorithm theory, Boolean algebras, and constructive models can be found in Hodges [2], Rogers [8], and Ershov and Goncharov [1]. Generally, *incomplete* theories of finite or enumerable signatures are considered.

A finite signature is called *rich*, if it contains at least one $n$ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0, \dots, x_{k-1}$, $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ. An entry $\mathrm{Nom}\,\Psi$ denotes number of the sentence Ψ in a fixed Gödel numbering of the set of all sentences of a given signature. By $L(T)$, we denote the Tarski—Lindenbaum algebra of formulas of theory T without free variables, while $\mathcal{L}(T)$ denotes the Tarski—Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ; thereby, the concept of a computable isomorphism is applicable to such objects. By W_n, we denote nth computably enumerable set in Post's numbering, while W_n^t denotes a finite part of W_n that can be computed for t steps. By $\varphi_n(x)$, we denote nth partial computable function in Kleene's numbering, while $\varphi_n^X(t)$ denotes nth partial function in computation with an oracle X.

The set of all finite tuples α of the form $\alpha = \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$, $\alpha_i \in \{0,1\}$, is denoted by $2^{<\omega}$, while 2^ω denotes the set of all infinite tuples $\alpha = \langle \alpha_i; i < \omega \rangle$, $\alpha_i \in \{0,1\}$ For a Boolean algebra $\mathcal{B}$, by $\mathcal{B}[a]$, we denote the restriction of $\mathcal{B}$ on the set of all subelements of the element $a \in \mathcal{B}$ counting that $\mathbf{1} = a$ and $-x$ is defined as $a \setminus x$ in $\mathcal{B}[a]$. If b is an element of a Boolean algebra and $\alpha \in \{0,1\}$, then b^α means b for $\alpha = 1$ and $-b$ for $\alpha = 0$. Similarly, if Φ is a formula and $\alpha \in \{0,1\}$, then Φ^α means Φ for $\alpha = 1$ and $\neg\Phi$ for $\alpha = 0$.

Symbol $\mathfrak{P}(X_0, \dots, X_{\mathfrak{a}})$, shortly $\mathfrak{P}$, is specialized to denote a propositional formula of signature $\sigma^* = \{X_0, X_1, \dots, X_k, \dots; k \in \mathbb{N}\}$ (i.e., consisting of propositional variables), while $\mathfrak{a}$ points out the number of variables occurred in the formula. By PRO, we denote the set of all such formulas, while $\mathfrak{P}_i(X_0, \dots, X_{\mathfrak{a}(i)})$, $i \in \mathbb{N}$, is a fixed Gödel numbering of the set PRO. By $\mathfrak{P}(\Psi_0, \dots, \Psi_{\mathfrak{a}})$, we denote the result of substitution of arbitrary formulas $\Psi_0, \dots, \Psi_{\mathfrak{a}}$ instead of the variables in the formula $\mathfrak{P}$. A formula $\mathfrak{P}(X_0, \dots, X_{\mathfrak{a}}) \in PRO$ is said to be *primitive* if it has the form $X_{\alpha_1} \wedge \dots \wedge X_{\alpha_p} \wedge \neg X_{\beta_1} \wedge \dots \wedge \neg X_{\beta_q}$ with indices satisfying $\{\alpha_1, \dots, \alpha_p\} \cup \{\beta_1, \dots, \beta_q\} \subseteq \{0, 1, \dots, \mathfrak{a}\}$, and $\{\alpha_1, \dots, \alpha_p\} \cap \{\beta_1, \dots, \beta_q\} = \varnothing$.

Lemma 0.1. *Each propositional formula is equivalent to a finite disjunction of primitive propositional formulas.*

Proof. Immediately. □

For a set $A \subseteq \mathbb{N}$ and propositional formula $\mathfrak{P}(X_0, \dots, X_{\mathfrak{a}})$, an entry $A \models \mathfrak{P}$ denotes the value of the Boolean term $\mathfrak{P}(\chi_A(0), \chi_A(1), \dots, \chi_A(\mathfrak{a}))$, where $\chi_A(x)$ is characteristic function of the set A. In this situation, propositional formula $\mathfrak{P}$ plays the role of a table condition applicable for the set $A \subseteq \mathbb{N}$.

Formulation to the *universal construction* $\mathbb{F}\mathbb{U}$ of finitely axiomatizable theories can be found in [5, Ch. 6]. Main definitions connected with semantic layers are found in [7]. We use notation MQL for the model quasiexact semantic layer presenting *infinitary first-order combinatorics*, cf. [7].

1. GÖDEL NUMBERS AND C.E. INDICES OF THEORIES

Let σ be a signature, and Σ be a subset of $SL(\sigma)$. Denote by $[\Sigma]^\sigma$ a theory of signature σ generated by Σ as a set of its axioms. There is another variant of the definition. Let $\Sigma \subseteq SL(\sigma)$ be a set of sentences. By $[\Sigma]^\star$, we denote a theory of a signature $\sigma' \subseteq \sigma$ generated by Σ as a set of its axioms, where σ' contains only those symbols from σ that occur in formulas of the set Σ.

In further definitions we use a fixed finite rich or enumerable signature σ. We also consider a fixed enumerable maximaum large infinite signature σ^∞. Signature σ^∞ contains countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n > 0$. We use a fixed Gödel numbering Φ_k, $k \in \mathbb{N}$, for the set of sentences of signature σ, and Φ_k^∞, $k \in \mathbb{N}$, for the set of sentences of the maximum large signature σ^∞.

Based on the Post numbering of the family of all computably enumerable sets W_n, $n \in \mathbb{N}$, we construct an effective numbering for the class of all computably axiomatizable theories. If a theory T

of signature σ is defined by the set of axioms $\{\Phi_i | i \in W_m\}$, the number m is called a *computably enumerable index* or simply *index* of T; we denote this theory by $T^\sigma\{m\}$, $m \in \mathbb{N}$. Another version represents indices for c.a. theories of arbitrary enumerable signatures. Given $m \in \mathbb{N}$. Consider the set of axioms $\Sigma = \{\Phi_i^\infty | i \in W_m\}$ and construct the theory $T = [\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak index* of T; we denote this theory by $T^\star\{m\}$, $m \in \mathbb{N}$.

Any finitely axiomatizable theory F is determined by a finite system A of axioms; therefore, by a single formula Φ which is a conjunction of the formulas from A. If a theory F of signature σ is defined by an axiom Φ_m, this number m is called a *Gödel number* or *strong index* of F; we denote this theory by $F^\sigma\{m\}$, $m \in \mathbb{N}$. Another version represents universal indices for finitely axiomatizable theories of arbitrary finite signatures. Given $m \in \mathbb{N}$. If a theory F is defined as follows $F = [\Phi_m^\infty]^\star$; this number m is called a *universal Gödel number* or *universal strong index* of F; we denote this theory by $F^\star\{m\}$, $m \in \mathbb{N}$.

2. STANDARD SCHEME OF TRANSFORMATIONS OF THEORIES CORRESPONDING TO INFINITARY FIRST-ORDER COMBINATORICS

In this section, we specify a method of construction of finitely axiomatizable theories with pre-assigned model-theoretic properties. In the most common case of such a construction, the target theory depends on some block C of objects of algorithmic nature that, in fact, can be presented via a single c.e. index. Let the input parameter be an index n. We are going to construct a finitely axiomatizable theory $F = F^{(n)}$ of a given finite rich signature ς, which would satisfy definite (pre-assigned) model-theoretic properties. First, we build an intermediate computably axiomatizable theory $T = T^{(n)}$ using some particular method.

Our project of a computably axiomatizable theory T should provide a realization of two aims: (*Space*) — we have to realize a definite isomorphism type of the Tarski–Lindenbaum algebra of the theory; a particular method of parametric Stone spaces is applied here that requires to choose a computable sequence of sentences presenting a generating system for the Tarski–Lindenbaum algebra; (*Extension*) — we have to provide that different complete extensions of the theory would have definite model-theoretic properties adequate to purposes of our project; this aim is realized by choice appropriate axiomatic based on the specified generating sequence of sentences.

Let us turn immediately to a realization of the project. We use the following signature for the theory T:

$$\sigma = \{X_i | i \in \mathbb{N}\} \cup \sigma', \tag{2.1}$$

where X_i, $i \in \mathbb{N}$, is a sequence of propositional variables (i.e., nulary predicates), and σ' depends on the aims of our construction. We will count that the nulary predicates X_i given in (2.1) represent the required generating sequence of sentences for the Tarski–Lindenbaum algebra of the target theory (computability of the sequence is obvious). Thereby, the following property should be guaranteed by our construction:

$$\text{sentences } X_k, k \in \mathbb{N}, \text{ represent a generating system for } \mathcal{L}(T). \tag{2.2}$$

Axioms of the theory T consist of three following groups: *Frm* (*Frame*), *Spa* (*Space*), and *Ext* (*Extension*), each having some particular form.

Reference_Block (2.3)

Frm: represents a group of axioms describing some general form of a skeleton of the theory; these axioms depend on the aims of the construction;

Spa: formulas of the form $\mathfrak{P}(X_0, \ldots, X_\mathfrak{a})$, with $\mathfrak{P} \in PRO$;

Ext: formulas of the form $\mathfrak{P}(X_0, \ldots, X_\mathfrak{a}) \to \Psi$, with $\mathfrak{P} \in PRO$ and $\Psi \in SL(\sigma')$.

End_Ref

Further, by applying the universal construction $\mathbb{F}\mathbb{U}$, we build a finitely axiomatizable theory $F = F^{(n)} = \mathbb{F}\mathbb{U}(T, \varsigma)$ of the demanded finite rich signature ς together with a computable isomorphism

$$\mu : \mathcal{L}(T) \to \mathcal{L}(F) \tag{2.4}$$

between the Tarski–Lindenbaum algebras preserving model-theoretic properties of their completions within the infinitary semantic layer MQL.

Introduce the following notation

$$\theta_i = \mu(X_i),\ i \in \mathbb{N}. \tag{2.5}$$

For an arbitrary set $A \subseteq \mathbb{N}$, we denote

$$\begin{aligned} T[A] &= T + \{X_i | i \in A\} \cup \{\neg X_j | j \in \mathbb{N} \smallsetminus A\}, \\ F[A] &= F + \{\theta_i | i \in A\} \cup \{\neg \theta_j | j \in \mathbb{N} \smallsetminus A\}. \end{aligned} \tag{2.6}$$

By construction, the theory T we are going to build is computably axiomatizable. Therefore, the set of provable in T formulas is computably enumerable. Thus, we can find a natural number m, such that

$$W_m = \{k | T \vdash \mathfrak{P}_k(X_0, \ldots, X_{\mathfrak{a}(k)})\}. \tag{2.7}$$

After that, we can introduce the following notation:

$$\Omega(m) = \{A \subseteq \mathbb{N} | (\forall k \in \Omega(m)) A \models \mathfrak{P}_k\}. \tag{2.8}$$

The set $\Omega(m)$ is said to be *parametric Stone space* defined by an index m. The block of transformations $n \mapsto T \mapsto F$ is said to be *normalized* if the following condition is satisfied:

$$T \vdash \mathfrak{P}(X_0, \ldots, X_{\mathfrak{a}}) \ \Leftrightarrow\ T.Spa \vdash \mathfrak{P}(X_0, \ldots, X_{\mathfrak{a}}),\ \text{for all } \mathfrak{P} \in PRO. \tag{2.9}$$

All objects involved in the transformation $n \mapsto T \mapsto F$ are presented via appropriate computably enumerable indices or Gödel numbers such that the whole passage $n \mapsto T \mapsto F$ is defined by an effective operator on indices and Gödel numbers.

Now, we formulate the principal statement of the work:

Theorem 2.1. [Standardized scheme for infinitary first-order combinatorics] *Given a sequence of transformations $n \mapsto T \mapsto F$ of the form presented above. The following assertions are satisfied*:

(a) $T[A]$, $A \in \Omega(m)$, *represents the family of all complete extensions of* T,

(b) $F[A]$, $A \in \Omega(m)$, *represents the family of all complete extensions of* F,

(c) *isomorphism* μ *in* (2.4) *maps* $T[A]$ *to* $F[A]$, *for all* $A \in \Omega(m)$,

(d) *for all* $A \in \Omega(m)$, *complete theories* $T[A]$ *and* $F[A]$ *have identical model-theoretic properties within the infinitary semantic layer* MQL,

(e) *effectively in a system of axioms of* T, *one can find* $s \in \mathbb{N}$ *such that function* $\varphi_s^A(t)$ *is characteristic for the set* $\mathrm{Nom}(T[A])$, *for all* $A \in \Omega(m)$,

(f) *effectively in a system of axioms of* T, *one can find* $s \in \mathbb{N}$ *such that function* $\varphi_s^A(t)$ *is characteristic for the set* $\mathrm{Nom}(F[A])$, *for all* $A \in \Omega(m)$.

Remark A: *Signature* (2.1) *of theory* $T = T^{(n)}$ *may depend on the input parameter* n*; in this case, we have to use weak c.e. indices with the universal construction. However, signature of* T *should not depend on the set* A *presenting an oracle in this computation.*

Remark B: *Having an axiomatization of theory* T *in the form* (2.3), *we can transform it in another* (*primitive*) *form for which group Ext consists of formulas of the form* $\mathfrak{P}(X_0, \ldots, X_{\mathfrak{a}}) \to \Psi$, *where* $\mathfrak{P}$ *is primitive propositional and* $\Psi \in SL(\sigma')$.

Remark C: *In the case when the complex of transformations* $n \mapsto T \mapsto F$ *satisfies normalization condition* (2.9), *we can use group of axioms* $T.Spa$ *in definition* (2.7) *instead of the whole theory* T.

Proof. of Theorem 2.1. First, we consider the situation as a whole. Based on (2.7), introduce the following notation

$$\Sigma_0 =_{dfn} \{\mathfrak{P}_k | T \vdash \mathfrak{P}_k(X_0, \ldots, X_{\mathfrak{a}(i)})\} = \{\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)}) | i \in W_m\}. \tag{2.10}$$

Notice that, relation (2.10) together with (2.2) ensures the following equality:

$$T = \Sigma_0. \tag{2.11}$$

According to the upper notation in (2.6), for an arbitrary set $A \subseteq \mathbb{N}$ we have $T[A] = T + \Sigma_1[A]$, where $\Sigma_1[A] = \{X_i | i \in A\} \cup \{\neg X_j | j \in \mathbb{N} \smallsetminus A\}$. Inductively, by the length of a formula, one can prove that for all $i \in \mathbb{N}$:

$$\Sigma_1[A] \vdash \big(\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)})\big)^{\alpha}, \quad \text{with}$$

$$\alpha = \text{the value of "}A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\text{"}, \quad \alpha \in \{0,1\}, \tag{2.12}$$

where $\mathfrak{P}^1 = \mathfrak{P}$, $\mathfrak{P}^0 = \neg\mathfrak{P}$, cf. Mendelson's Lemma 1.12 in [3, Ch. 1, Sec. 4]. By adding all formulas of the form (2.12) with $\alpha = 1$ (provable from $\Sigma_1[A]$), we obtain a new presentation for the theory

$$T[A] = T + \Sigma_2[A], \text{ where } \Sigma_2[A] = \{\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)}) | A \models \mathfrak{P}_i(\ldots)\}. \tag{2.13}$$

Moreover, we obviously have $\Sigma_1[A] \subseteq \Sigma_2[A]$. The following dependencies take place:

$$\Sigma_2[A] \cup \bar{\Sigma}_2[A] = FL(\sigma), \quad \Sigma_2[A] \cap \bar{\Sigma}_2[A] = \varnothing,$$
$$\text{where } \bar{\Sigma}_2[A] = \{\mathfrak{P}_j(X_0, \ldots, X_{\mathfrak{a}(j)}) | A \not\models \mathfrak{P}_j(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(j)})\}. \tag{2.14}$$

By construction, either $\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(j)}) \in \Sigma_2[A]$ or $\neg\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(j)}) \in \Sigma_2[A]$ for each $i \in \mathbb{N}$. Thereby, by virtue of (2.2), presentation (2.13) ensures that $T[A]$ is a complete theory whenever it is consistent.

Let us prove that the following relation holds for the sets introduced above:

$$A \in \Omega(m) \;\Leftrightarrow\; \Sigma_0 \subseteq \Sigma_2[A]. \tag{2.15}$$

First, we assume that $A \in \Omega(m)$. Consider an arbitrary $\Phi \in \Sigma_0$. By (2.10) we have $\Phi = \mathfrak{P}_{k_0}(X_0, \ldots, X_{\mathfrak{a}(k_0)})$ for some $k_0 \in W_m$. Since $(\forall k \in W_m) A \models \mathfrak{P}_k$ by definition (2.8) and the choice of m, we obtain that $A \models \mathfrak{P}_{k_0}(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(k_0)})$; thus, by (2.13), we have $\mathfrak{P}_{k_0}(X_0, \ldots, X_{\mathfrak{a}(k_0)}) \in \Sigma_2[A]$, obtaining finally that $\Phi \in \Sigma_2[A]$. Now, we assume that $\Sigma_0 \subseteq \Sigma_2[A]$. We obtain from (2.10) and (2.13) that for all i satisfying $T \vdash \mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)})$, we have $A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})$. Applying again (2.10) we conclude that $(\forall i)\big[i \in W(m) \Rightarrow A \models \mathfrak{P}_i(\mathcal{X}_0, \ldots, \mathcal{X}_{\mathfrak{a}(i)})\big]$ obtaining finally $A \in \Omega(m)$ by definition (2.8). Thus, (2.15) is indeed satisfied.

Now, we are going to prove that

$$\text{(a)} \quad A \in \Omega(m) \;\Rightarrow\; T[A] \text{ is consistent and complete},$$
$$\text{(b)} \quad A \notin \Omega(m) \;\Rightarrow\; T[A] \text{ is contradictory}. \tag{2.16}$$

We consider two following cases.

Case 1: $A \in \Omega(m)$. From (2.13), we have $T[A] = T + \Sigma_2[A]$. Consider a finite set $\Delta = \{\Psi_0, \ldots, \Psi_{t-1}\} \subseteq \Sigma_2[A]$. We are going to show that $T + \Delta$ is consistent. Let Ψ be conjunction $\Psi_0 \wedge \cdots \wedge \Psi_{t-1}$. From $\Psi_i \in \Sigma_2[A]$, $i < t$, by rule (2.12), we obtain $\Psi \in \Sigma_2[A]$. By (2.14), we have $\neg\Psi \in \bar{\Sigma}_2[A]$, thus $\neg\Psi \notin \Sigma_2[A]$. By (2.11) and (2.15), we have $T = \Sigma_0 \subseteq \Sigma_2[A]$; thus, $\neg\Psi \notin T$. From this, we conclude that $T + \Psi$ is consistent; thereby, $T + \Delta$ is consistent as well. Applying Maltsev's Compactness Theorem, we obtain that theory $T[A]$ is consistent. By virtue of (2.13) together with (2.14) we obtain that for all $i \in \mathbb{N}$, either sentence $\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)})$ or its negation $\neg\mathfrak{P}_i(X_0, \ldots, X_{\mathfrak{a}(i)})$ belongs to $T[A]$ ensuring, by (2.2), that this theory is complete.

Case 2: $A \notin \Omega(m)$. In this case, by (2.11) and (2.15), we obtain $T \not\subseteq \Sigma_2[A]$. Let Ψ be a sentence in $T \setminus \Sigma_2[A]$. By (2.14), we have $\Psi \in \bar{\Sigma}_2[A]$; thus, $\neg\Psi \in \Sigma_2[A]$. As a result, we obtain $\Psi \in T \subseteq T[A]$ and $\neg\Psi \in \Sigma_2[A] \subseteq T[A]$. This shows that the theory $T[A]$ is contradictory.

Thereby, both implications (2.16)(a) and (2.16)(b) are indeed satisfied.

Now, we turn to proofs of the particular parts formulated in Theorem 2.1.

(a) From definition (2.6), it follows that, for any complete extension T^* of T, there is a set $A \subseteq \mathbb{N}$ such that $T[A] \subseteq T^*$. Based on this fact, together with (2.16)(a) and (2.16)(b), we immediately obtain the statement in Part (a).

(b) This statement is a simple consequence of Part (a) together with availability of an isomorphism μ between the Tarski-Lindenbaum algebras of theories T and F, cf. (2.4).

(c) Immediately, from (2.5) and (2.6), based on the fact that μ is an isomorphism between the Tarski-Lindenbaum algebras of theories T and F.

(d) Immediately, from the fact that the isomorphism μ preserves all model-theoretic properties within the semantic layer MQL.

(e) We use notations found in [8, Sec. 9.2]. For finite sets $D_u, D_v \subseteq \mathbb{N}$ of natural numbers, we introduce a notation for the following primitive propositional formula: $\mathfrak{P}_{u,v} = \bigwedge_{i \in D_u} X_i \;\wedge\; \bigwedge_{j \in D_v} \neg X_j$.

Let A be a subset of $\mathbb{N}$ and Ψ be a sentence of signature of theory T. By virtue of Part (a), theory $T[A]$ is complete for all A in $\Omega(m)$. This means that either Ψ or $\neg\Psi$ are provable in theory $T[A]$. Based on presentation of axioms for $T[A]$ in (2.6), we obtain that there is a pair of finite sets D_u and D_v with $D_u \cap D_v = \varnothing$, and a Boolean value $\alpha \in \{0,1\}$, satisfying the following relation

$$T \vdash \big(\mathfrak{P}_{u,v}(X_0,\dots,X_{\mathfrak{a}(u,v)}) \to \Psi^{\alpha}\big), \tag{2.17}$$

where $\Psi^0 = \neg\Psi$ and $\Psi^1 = \Psi$.

By construction, theory T is computably axiomatizable. Therefore, the set R of all sequences $\langle u, v, \mathrm{Nom}\,\Psi, \alpha\rangle$ satisfying condition (2.17) is computably enumerable. Thereby, we obtain the following presentation

$$T[A] \vdash \Psi^{\alpha} \;\Leftrightarrow\; (\exists u,v)\big[\langle u,v,\mathrm{Nom}\,\Psi,\alpha\rangle \in R \wedge D_u \subseteq A \wedge D_v \subseteq \mathbb{N} \smallsetminus A\big]. \tag{2.18}$$

By construction, the set R is regular relative to the cases with consistent $T[A]$, i.e., the value of α depending on Ψ in the left-hand side expression in (2.18) is uniquely determined for all $A \in \Omega(m)$; as for the cases $A \notin \Omega(m)$, the value of α in (2.18) does not matter for our purposes. Find an integer s such that $W_s = R$. It can be simply checked that the index s is found effectively in T, thereby s is found effectively in the input parameter n. Consider the passage to a normalized set $W_s \mapsto W_{\rho(s)}$, where both the term 'normalized' and the function ρ are defined in [8, Sec. 9.2]. By construction, the normalization procedure does not change cases involved in relation (2.18) with $A \in \Omega(m)$. We obtain finally the following new form of the relation which is an immediate reformulation of (2.18) for all $\Psi \in SL(\sigma)$, $A \in \Omega(m)$ and $\alpha \in \{0,1\}$:

$$T[A] \vdash \Psi^{\alpha} \;\Leftrightarrow\; (\exists u,v)\big[\langle u,v,\mathrm{Nom}\,\Psi,\alpha\rangle \in W_{\rho(s)} \wedge D_u \subseteq A \wedge D_v \subseteq \mathbb{N} \smallsetminus A\big]. \tag{2.19}$$

On the other hand, in accordance with [8, Sec. 9.2], we have the following standard presentation for computability with an oracle:

$$\varphi_s^A(t) = \alpha \;\Leftrightarrow\; (\exists u,v)\big[\langle u,v,t,\alpha\rangle \in W_{\rho(s)} \wedge D_u \subseteq A \wedge D_v \subseteq \mathbb{N} \smallsetminus A\big]. \tag{2.20}$$

Combining (2.19) and (2.20) together, we obtain the following summary relation $T[A] \vdash \Psi^{\alpha} \Leftrightarrow \varphi_s^A(\mathrm{Nom}\,\Psi) = \alpha$, for all $\Psi \in SL(\sigma)$, $A \in \Omega(m)$, $\alpha \in \{0,1\}$, that is exactly what is required for (e).

(f) This statement is a routine consequence of the established Part (e) together with availability of the isomorphism μ between the Tarski–Lindenbaum algebras of theories T and F, cf. (2.4).

Theorem 2.1 is proved.

Comment to Remark B: Consider a formula $\mathfrak{P}(X_0,\dots,X_{\mathfrak{a}}) \to \Psi$, with $\mathfrak{P} \in PRO$. By Lemma 0.1, there are primitive propositional formulas $\mathfrak{P}_1,\dots,\mathfrak{P}_t$, such that $\mathfrak{P}$ is equivalent to $\mathfrak{P}_1 \vee \dots \vee \mathfrak{P}_t$. We have the following chain of equivalences $\mathfrak{P} \to \Psi \sim (\mathfrak{P}_1 \vee \dots \vee \mathfrak{P}_t) \to \Psi \sim (\mathfrak{P}_1 \to \Psi) \wedge \dots \wedge (\mathfrak{P}_t \to \Psi)$. Thus, it is possible to omit the old axiom $\mathfrak{P} \to \Psi$ in the group *Ext*, including a finite set of new axioms $\mathfrak{P}_1 \to \Psi, \dots, \mathfrak{P}_t \to \Psi$ instead. An alternative confirmation to the statement in Remark B can be found in the proof of Part (e).

Let us consider a useful technical statement.

Lemma 2.2. *Let m be defined by rule* (2.7)*, while m' be defined by a normalized rule* (2.7) *with $T.Spa$ substituted instead of T. We have $\Omega(m) \subseteq \Omega(m')$; moreover, the following assertions are equivalent with each other*: (a) $\Omega(m) = \Omega(m')$; (b) *each extension $T[A]$, $A \in \Omega(m')$, is consistent.*

Proof. The set of sentences in (2.7) provable from $T.Spa$ is a subset of that provable from T. This ensures inclusion $\Omega(m) \subseteq \Omega(m')$. Implication (a) $\Rightarrow$ (b) is obvious. Now, assume that (b) is held. In this case, collection $T[A]$, $A \in \Omega(m')$, represents a set of complete extensions of T extending the set $T[A]$, $A \in \Omega(m)$. By Theorem 2.1, we must have $\Omega(m) = \Omega(m')$. □

3. STANDARDIZATION PRINCIPLE

Now, we formulate some complementary statement for Theorem 2.1.

Statement 3.1 [Standardization principle for infinitary first-order combinatorics]. *Given an arbitrary effective transformation* $n \mapsto T' \mapsto F'$*, where* T' *is a computably axiomatizable theory constructed by an arbitrary method from an input parameter* n*, and* F' *is a finitely axiomatizable theory of signature* ς *obtained from* T' *by the universal construction. This scheme can equivalently be transformed in the form of a standardized scheme* $n \mapsto T \mapsto F$ *(cf. Section* 2*).*

Proof. Let σ' be the signature of theory T', and Φ_i, $i \in \mathbb{N}$, be a Gödel numbering for the set $SL(\sigma')$. Consider a new extended signature $\sigma = \{X_i | i \in \mathbb{N}\} \cup \sigma'$, and add to T' all sentences of the form $X_k \leftrightarrow \Phi_k$, $k \in \mathbb{N}$, obtaining a new theory T of signature σ. Include in *Spa* all formulas of the form (2.3)(*Spa*) which are provable in T, while in *Ext*, all those formulas of the form (2.3) (*Ext*) which are provable in T. As for *Frm*, we include in this group all axioms of the theory T' (actually, each project of a theory with preassigned model-theoretic properties is based on a particular design of the frame axioms). By construction, the sequence of sentences X_i, $i \in \mathbb{N}$, must represent a generating set for the Tarski–Lindenbaum algebra $\mathcal{L}(T)$.

It can easily be checked that all required properties, cf. Section 2, for the obtained scheme of transformations $n \mapsto T \mapsto F$ are satisfied. Moreover, we can easily establish equivalence between the old scheme and the new one. □

4. DEMONSTRATION: AN EXISTENCE THEOREM

In this section, we are going to construct an example of finitely axiomatizable theory applying the standardized scheme for infinitary first-order combinatorics. We will construct a theory with preassigned values of the model-theoretic property $\mathfrak{p}$ = "theory has a model with first-order definable elements", as well as, of some close to it property. Both are controlled by the universal construction.

Theorem 4.1. *There is a finitely axiomatizable theory* F *without finite models of a given finite rich signature* ς *satisfying the following properties: the set of all complete extensions of* F *consists of a countable sequence* F_k, $k \in \mathbb{N} \cup \{\omega\}$*, such that, each of the theories* $F_0, F_1, F_2, \ldots$ *is finitely axiomatizable over* F *and has a model with first-order definable elements, while* F_ω *is not finitely axiomatizable over* F*, and has neither model with first-order definable elements nor a countable minimal model.*

Proof. No input parameter is used in this construction. We first have to construct a computably axiomatizable theory T whose properties are analogous to those posed in Theorem 4.1 for theory F.

Signature of theory T is $\sigma = \{X_n^0 | n \in \mathbb{N}\} \cup \{U_{n,i}^1(x) | n, i \in \mathbb{N}\}$. Axioms of T include the following groups of sentences:

Spa:

1°. $X_n \to \neg X_m$, for all cases satisfying $n \neq m$,

Frm:

2°. $(\forall x)\big(U_{n,i}(x) \wedge U_{n,i}(y) \to x = y\big)$, for all $n, i \in \mathbb{N}$,

3°. $(\forall x)\big(U_{n,i}(x) \wedge U_{n,j}(y) \to x \neq y\big)$, for all cases satisfying $i \neq j$,

4°. $(\exists^{\geqslant k} x)(x = x)$, for all $k \in \mathbb{N}$,

Ext:

5°. $X_n \iff (\exists x) U_{n,i}(x)$, for all $n, i \in \mathbb{N}$.

Notice that, the sentences listed in Axiom 5° can be reduced to a correct form presented in (2.3) (*Ext*). Choose an integer parameter m such that $W_m = \{k | T.Spa \vdash \mathfrak{P}_k(X_0, \ldots, X_{\mathfrak{a}(k)})\}$. Thus, we use a normalized version of the demand (2.7). According to Axiom 1°, we have $\Omega(m) = \{\varnothing, \{0\}, \{1\}, \ldots, \{n\}, \ldots : n \in \mathbb{N}\}$. Denote $T[\{n\}]$ by T_n, while $T[\varnothing]$ denote by T_ω. By the standard scheme for infinitary first-order combinatorics, the sequence T_n, $n \in \mathbb{N}$ together with T_ω, represents the set of all complete extensions of theory T provided that Lemma 2.2 is applicable. Let $n \in \mathbb{N}$. By axioms 1°, 2°, 3°, and 5°, each of the predicates $U_{n,i}(x)$, $i \in \mathbb{N}$, distinguish a single element in theory T_n; moreover, $U_{n,i}(x)$ and $U_{n,j}(x)$, with $i \neq j$, distinguish different elements in the theory. The other predicates $U_{m,j}(x)$, $m \neq n$, are identically false. Thus, theory T_n has a model with first-order definable elements. Relation $T_n = T + \{X_n\}$ ensures that T_n is finitely axiomatizable over T. As for theory $T_\omega = T[\varnothing]$, its axioms require that all predicates $U_{n,i}(x)$ are identically false; in addition, Axiom 5° requires that all models of T are infinite. Thereby, T_ω represents a theory of an equality relation on an infinite set. Thus, the theory T_n has neither models with first-order definable elements nor countable minimal models.

By applying the universal construction, we obtain a finitely axiomatizable theory F of a wished finite rich signature ς satisfying all posed properties.

Theorem 4.1 is proved. □

5. DEMONSTRATION: COMPLEXITY ESTIMATE FOR A SEMANTIC CLASS OF MODELS

In this section, we apply the standardized scheme for infinitary combinatorics to the problem of complexity of first-order theory of a semantic class of models.

Hereafter, we fix a finite rich signature ς; furthermore, Φ_n, $n \in \mathbb{N}$, is a Gödel numbering of the set of all sentences of this signature. We denote by $Def(\varsigma)$ the class of models of signature ς with first-order definable elements, and by $Dec(\varsigma)$, the class of all models of signature ς having a decidable theory. Intersection of these classes $DecDef(\varsigma) = Dec(\varsigma) \cap Def(\varsigma)$ is the main object of our further study.

Theorem 5.1. $\mathrm{Th}(DecDef(\varsigma)) \approx \Pi^0_3$.

Proof. Let us denote $K = DecDef(\varsigma)$. We are going to prove that

$$\text{“}\Phi_n \text{ has a } K\text{-model” represents a } \exists\forall\exists\text{-condition.} \tag{5.1}$$

Indeed, a formula $\Phi \in SL(\varsigma)$ has a K-model iff there are $m, n \in \mathbb{N}$ such that $\mathrm{Dom}(\varphi_m) \cap \mathrm{Dom}(\varphi_n) = \varnothing$, $\mathrm{Dom}(\varphi_m) \cup \mathrm{Dom}(\varphi_n) = \mathbb{N}$; moreover, the set of sentences $T = \{\Phi_k | k \in \mathrm{Dom}(\varphi_m)\}$ is a complete theory such that $\Phi \in T$ and for all $\psi(x) \in FL_1(\varsigma)$ satisfying $T \vdash (\exists x)\psi(x)$, there is a formula $\theta(x) \in FL_1(\varsigma)$ satisfying $T \vdash (\exists x)\theta(x)$, $T \vdash (\forall xy)[\theta(x) \wedge \theta(y) \to (x = y)]$, and $T \vdash (\forall x)[\theta(x) \to \psi(x)]$. Detailed calculation gives prefix $\exists\forall\exists$ for this condition. From this, we have $\Phi \in \mathrm{Th}(K) \Leftrightarrow \neg\Phi$ does not have a K-model, obtaining prefix $\forall\exists\forall$ as an upper estimate for the theory $\mathrm{Th}(K)$.

For the lower estimate, we will use the following m-universal in Σ^0_3 set, cf. [8, Cor. 14-XVI, p. 328]:

$$E_3 = \{n | W_n \text{ is cofinite}\}. \tag{5.2}$$

Given an integer parameter n. Effectively in n, we construct a theory $T^{(n)}$ of signature (depending on the input parameter n): $\sigma^{(n)} = \{X^0_p | p \in \mathbb{N}\} \cup \{R^1_p(x) | p \in \mathbb{N}\} \cup \{c_{p,i} | p \in W_n, p > 0, i \in \mathbb{N}\} \cup \{d_{p,i} | p, i \in \mathbb{N}\}$. Axioms of $T^{(n)}$ include the following sentences of signature $\sigma^{(n)}$ whose indices are supposed to be restricted by the condition of inclusion in the signature.

Spa:

1°. $X_p \to \neg X_q$, for all cases satisfying $p \neq q$,

Frm:

2°. $(\forall x)\big(R_p(x) \to \neg R_q(x)\big)$, for all cases satisfying $p \neq q$,

3°. $(\exists^{\geqslant k} x) R_p(x)$, for all p, k,

4°. $c_{i,j} \neq c_{k,t}$, for all cases satisfying $\langle i, j\rangle \neq \langle k, t\rangle$,

5°. $d_{i,j} \neq d_{k,t}$, for all cases satisfying $\langle i, j\rangle \neq \langle k, t\rangle$,

6°. $c_{i,j} \neq d_{k,t}$, for all i, j, k, t,

7°. $R_p(c_{p,i})$, for all p, i satisfying $p > 0$,

Ext:

8°. $\neg X_0 \wedge .. \wedge \neg X_{p-1} \to R_1(d_{q,i})$, $p, q, i \in \mathbb{N}$, $q + i \leqslant p$,

9°. $\neg X_0 \wedge .. \wedge \neg X_{p-1} \wedge X_p \to R_t(d_{q,i})$, $p, q, i \in \mathbb{N}$, $q + i > p$, $t = \min\{p, q\}$.

Propositional variables X_i, $i \in \mathbb{N}$, play the role of generating sentences for the Tarski–Lindenbaum algebra of T. Owing to Axiom 1°, we obtain the following sequence of extensions of $T = T^{(n)}$:

$$T^{(n)}_k = T^{(n)} + \{X_k\},\ k \in \mathbb{N}, \quad \text{and} \quad T^{(n)}_\omega = T^{(n)} + \{\neg X_i | i \in \mathbb{N}\}. \tag{5.3}$$

One can check that, in any of the theories (5.3), each constant $c_{i,j}$ and each constant $d_{k,t}$ is completely specified relative to regions distinguished by the unary predicates $R_i(x)$, $i \in \mathbb{N}$. Since the form of T is very simple, this means that all extensions (5.3) of T are complete theories. Particularly, the series (5.3) represents all possible complete extensions of T; thus, Lemma 2.2 is applicable. By construction, region $R_0(x)$ is free of any signature constants in the extension $T_\omega^{(n)}$. Thereby, theory $T_\omega^{(n)}$ does not have a K-model for all n.

As for the other extensions in (5.3), existence of a K-model for them depends on the input parameter n.

First, we consider the case $n \in E_3$. In this case, by (5.2), there is k_0 such that each number $k > k_0$ belongs to W_n. By construction, Axiom 7° provides that all regions $R_p(x)$, $p > k_0$ include infinitely many values of signature constants. Additionally, Axiom 9° ensures that all regions $R_q(x)$, $q \leqslant p$, include infinitely many values of signature constants in theory $T_p^{(n)}$. Thereby, for $p > k_0$, theory $T_p^{(n)}$ must have a model with first-order definable elements; moreover, this theory is decidable since it is computably axiomatizable and complete.

Now, we consider the other case $n \notin E_3$. In this case, by (5.2), there are infinitely many k such that $k \notin W_n$. Thereby, demands of Axioms 7°, 8°, and 9° are such that for any theory $T_p^{(n)}$, $p \in \mathbb{N}$, there are infinitely many regions $R_k(x)$, $k \in \mathbb{N}$, which are free of signature constants. Thereby, each theory $T_p^{(n)}$, $p \in \mathbb{N}$, cannot have a model with first-order definable elements; particularly, theory T does not have such a model. As a result, we have established that the following relation is satisfied: $n \in E_3 \;\Leftrightarrow\; T^{(n)}$ has a K-model.

We have described an effective construction of a theory $T^{(n)}$ depending on a parameter n; particularly, weak c.e. index of the theory $T^{(n)}$ is found effectively in n. Applying the universal construction, we can find, effectively in n, a finitely axiomatizable theory F of signature ς defined by a sentence $\Phi_{g(n)}$ as an axiom, together with a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$, that preserves model-theoretic properties of the infinitary semantic layer MQL, where $g(x)$ is a general computable function. The property $\mathfrak{p} =$ "theory has a model with first-order definable elements" belongs to MQL; moreover, the isomorphism μ maps decidable complete extensions of T in decidable completions of $\Phi_{g(n)}$. Thus, we obtain $n \in E_3$ iff $\Phi_{g(n)}$ has a K-model. This relation can be reformulated in the following form $n \notin E_3 \;\Leftrightarrow \neg\Phi_{g(n)}$ is satisfied on all K-models, ensuring the necessary lower estimate $\mathbb{N} \smallsetminus E_3 \leqslant_m \mathrm{Nom\,Th}(K)$.

Theorem 5.1 is proved. □

6. DEMONSTRATION: ALGORITHMIC COMPLEXITY OF THE TARSKI–LINDENBAUM ALGEBRA OF A SEMANTIC CLASS

Now, we apply the standardized scheme to a more common problem concerning complexity of the Tarski–Lindenbaum algebra of a semantic class of models. We consider the same class of models that was studied in Section 5. We use notations and conventions introduced at the beginning of Section 5.

Theorem 6.1. *The following assertions hold* :

(a) *$\mathcal{L}(DecDef(\varsigma))$ is a Boolean Π^0_3-algebra*,

(b) *computable ultrafilters of $\mathcal{L}(DecDef(\varsigma))$ represent a dense set among arbitrary ultrafilters in the algebra*,

(c) *for an arbitrary Boolean Σ^0_2-algebra $(\mathcal{B},\nu)$ with a dense set of computable ultrafilters among arbitrary ultrafilters there is a sentence Φ of signature ς, such that $(\mathcal{B},\nu) \cong (\mathcal{L}(\mathrm{Th}(\mathrm{Mod}(\Phi) \cap DecDef(\varsigma))),\gamma)$, where γ is a Gödel numbering of the set of sentences of signature ς.*

(d) *for an arbitrary Boolean Π^0_3-algebra $\mathcal{B}$ there is a sentence Φ of signature ς, such that $\mathcal{B} \cong \mathcal{L}(\mathrm{Th}(\mathrm{Mod}(\Phi) \cap DecDef(\varsigma)))$.*

Proof. By virtue of (5.1), we obtain that sentences Φ and Ψ are equivalent on the class $DecDef(\varsigma)$ if and only if $(\Phi \wedge \neg\Psi) \vee (\Psi \wedge \neg\Phi)$ does not have a model in this class. This gives prefix $\forall\exists\forall$ for (a).

Let T be a complete theory extending $\mathrm{Th}(DecDef(\varsigma))$, and $\Psi \in T$. Obviously, Ψ has a model $\mathfrak{M} \in DecDef(\varsigma)$. From this we have that complete decidable theory $T' = \mathrm{Th}(\mathfrak{M})$ presenting a computable ultrafilter in $\mathrm{St}(\mathrm{Th}(DecDef(\varsigma)))$, is found in the neighborhood Ψ of a given ultrafilter T of this Stone space. Thus, we obtain the required density property posed in (b). As for Part (d), it is a consequence of Part (c) together with Theorem 1 and Theorem 2 in [4] ensuring a possibility of reductions $\Pi^0_{n+1} \rightsquigarrow \Delta^0_{n+1} \rightsquigarrow \Sigma^0_n$ for Boolean algebras, for all $n \in \mathbb{N}$.

Now, we prove Part (c). We use notation $K = DecDef(\varsigma)$ in the proof below.

Given a numerated Boolean Σ^0_2-algebra $(\mathcal{B}, \nu)$. By Lemma 6 in [6], there is a numeration ν' of $\mathcal{B}$ such that $(\mathcal{B}, \nu')$ is a Boolean Σ^0_2-algebra whose computable ultrafilters represent a dense set in the set of all ultrafilters. For the sake of simplicity, we assume that the source algebra $(\mathcal{B}, \nu)$ has this property, that is:

$$\text{computable ultrafilters of } (\mathcal{B}, \nu) \text{ form a dense set in } \mathrm{St}(\mathcal{B}). \tag{6.1}$$

We also assume, that $\mathcal{B}$ is a nontrivial algebra. By definition, signature operations $\cup$, $\cap$ and $-$ in $\mathcal{B}$ are presentable by computable functions on ν-numbers, and the equality relation is a Σ^0_2-relation in numeration ν satisfies $\nu(x) = \nu(y) \;\Leftrightarrow\; H(x, y), \;\; H \in \Sigma^0_2$. Consequently, there exists a unary relation H^* in Σ^0_2 such that for any finite tuple of zeros and ones $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$, we have $\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \ldots \cap \nu(n)^{\alpha_n} = \mathbf{0} \;\Leftrightarrow\; \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \in H^*$, $H^* \in \Sigma^0_2$.

We will use the following m-complete in class Σ^0_2 set: $E_2 = \{n | W_n \text{ is finite}\}$, cf. [8, Th. 13-VIII, p. 264]. Since any Σ^0_2-set is m-reducible to E_2, there is a general computable function $f(x)$ such that for an arbitrary tuple $\alpha \in 2^{<\omega}$, $\alpha = \langle \alpha_0, \ldots, \alpha_k \rangle$, we have

$$\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \ldots \cap \nu(k)^{\alpha_k} = \mathbf{0} \;\Leftrightarrow\; W_{f(\alpha)} \text{ is finite}, \tag{6.2}$$

or equivalently,

$$\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \ldots \cap \nu(k)^{\alpha_k} \neq \mathbf{0} \;\Leftrightarrow\; W_{f(\alpha)} \text{ is infinite}. \tag{6.3}$$

By virtue of (6.3), Stone space of algebra $\mathcal{B}$ is presented by tuples $\alpha \in 2^\omega$, $\alpha = (\alpha_i : i < \omega)$, satisfying $(\forall k)[W_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} \text{ is infinite}]$; i.e, the pointed out sequence α presents a filter $\mathcal{F}_\alpha$ of $\mathcal{B}$ generated by elements of the form in the left-hand side of (6.3). There is a natural one-to-one correspondence between the set 2^ω and the power-set $\mathcal{P}(\mathbb{N}) = \{A | A \subseteq \mathbb{N}\}$ which is defined by the rules

$$\alpha \mapsto A_\alpha = \{i | \alpha_i = 1\}, \quad \text{and} \quad A \mapsto \alpha_A = (\alpha_i : i < \omega), \quad \alpha_i = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases} \tag{6.4}$$

It is possible to check that α is computable iff A_α is computable for all $\alpha \in 2^\omega$.

Now, we start to describe a computably axiomatizable theory T depending on an integer parameter n. Stone space of theory $T^{(n)}$ will be presented by all sets A in $\mathcal{P}(\mathbb{N})$; i.e., we do not include any demands in the group *Spa*. Our goal is to establish the following relation for an arbitrary tuple $\alpha \in 2^\omega$, $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_k, \ldots \rangle$, and corresponding set $A = A_\alpha$:

$$T[A] \text{ has a } K\text{-model} \;\Leftrightarrow\; \alpha \text{ computable} \wedge (\forall k \in \mathbb{N})[W_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} \text{ infinite}]. \tag{6.5}$$

Signature of theory T is $\sigma = \{X^0_i | i \in \mathbb{N}\} \cup \{R^1_i | i \in \mathbb{N}\} \cup \{c_{i,j} | i, j \in \mathbb{N}\}$. The following sentences are axioms of T:

Spa: this group is an empty set,

Frm:

1°. $(\forall x)[R_i(x) \to \neg R_j(x)]$, for all cases $i \neq j$,

2°. $(\exists^{\geqslant t} x) R_k(x)$, for all k, t,

Ext:

3°. $X_0^{\alpha_0} \wedge \cdots \wedge X_k^{\alpha_k} \;\to\; R_k(c_{k,t})$, if $W^{t+1}_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} \setminus W^t_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} \neq \varnothing$,

4°. $X_0^{\alpha_0} \wedge \cdots \wedge X_k^{\alpha_k} \;\to\; R_0(c_{k,t})$, if $W^{t+1}_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} \setminus W^t_{f(\langle \alpha_0, \ldots, \alpha_k \rangle)} = \varnothing$.

For complete extension $T[A]$ of T corresponding to the filter $\mathcal{F}_\alpha$ of $\mathcal{B}$, $\alpha = \alpha_A$, Axiom 3° defines in the regions $R_k(x)$ as many signature constants $c_{k,t}$ how many elements are computed in the set

$W_{f(\langle \alpha_0,\ldots,\alpha_k\rangle)}$; another Axiom 4° puts a constant in the trash region $R_0(x)$ in the dummy cases. In the case when all sets $W_{f(\langle \alpha_0,\ldots,\alpha_k\rangle)}$ are infinite, each region $R_k(x)$ gets infinitely many signature constants, thus $T[A_\alpha]$ is a complete theory having a model with first-order definable elements. In the other case, when one of the sets $W_{f(\langle \alpha_0,\ldots,\alpha_k\rangle)}$ is finite, corresponding region $R_k(x)$, includes finitely many signature constants, while Axiom 2° requires that this region is infinite. In this case, $T[A_\alpha]$ is a complete theory without a model with first-order definable elements.

Thereby, the principal property (6.5) is indeed satisfied.

Having constructed theory T, let us pass to a formula $\Phi \in SL(\varsigma)$ which is an axiom of finitely axiomatizable theory $F = \mathbb{F}\mathbb{U}(T, \varsigma)$; moreover, a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ preserving infinitary layer MQL is also available, cf. (2.4). Consider an arbitrary finite tuple $\alpha = \langle \alpha_0, \ldots, \alpha_k\rangle$ in $2^{<\omega}$. Construct an elementary intersection of elements in $\mathcal{B}$ by the rule $b_\alpha = \nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \cdots \cap \nu(k)^{\alpha_k}$, as well as an elementary conjunction of corresponding sentences (2.4) by the rule $\beta_\alpha = \theta_0^{\alpha_0} \wedge \theta_1^{\alpha_1} \wedge \cdots \wedge \theta_k^{\alpha_k}$.

Claim A. *For any $\alpha \in 2^{<\omega}$, $b_\alpha \neq \mathbf{0}$ if and only if $\Phi \wedge \beta_\alpha$ has a K-model.*

Proof. Assume that $b_\alpha \neq \mathbf{0}$. By (6.1), computable ultrafilters form a dense set among arbitrary ultrafilters in $(\mathcal{B}, \nu)$. Thus, there is an infinite sequence $\alpha^* = \langle \alpha_i | i < \omega\rangle$ extending α such that the set A related to α^* is computable, and $\nu(0)^{\alpha_0} \cap \cdots \cap \nu(i)^{\alpha_i} \neq \mathbf{0}$, for all $i \in \mathbb{N}$. By (6.3), we obtain that each set $W_{f(\langle \alpha_0,\ldots,\alpha_i\rangle)}$, $i \in \mathbb{N}$, is infinite. Thereby, theory $T[A]$ is consistent, complete, and, by (6.5), has a K-model $\mathfrak{N}$. This ensures that the formula $\Phi \wedge \beta_\alpha$ (since it is provable from $F[A]$) is satisfied in the model $\mathfrak{N}$.

Now, assume that $\Phi \wedge \beta_\alpha$ has a model $\mathfrak{N}$ in the class K. Consider the set

$$A = \{\theta_i | \mathfrak{N} \models \theta_i\}, \tag{6.6}$$

which is computable because $\mathrm{Th}(\mathfrak{N})$ is decidable. Build an infinite sequence $\alpha^* = \langle \alpha_i | i < \omega\rangle$ related with A by rule (6.4). By virtue of description in Section 2, theory $F[A]$ is consistent and complete. Moreover, this theory is decidable by Janiczak Theorem since it is computably axiomatizable. By (6.6), all axioms of $F[A]$ are satisfied in the model $\mathfrak{N}$. Thereby, we have that A is computable and $F[A]$ has a K-model. Applying (6.5), we finally obtain $b_\alpha \neq \mathbf{0}$.

Claim A is proved. □

Let us map elements $\nu(i)$, $i \in \mathbb{N}$, of Boolean algebra $\mathcal{B}$ to sentences θ_i, $i \in \mathbb{N}$, of signature ς by the rule:

$$\lambda^*(\nu(k)) = \theta_k, \quad k \in \mathbb{N}. \tag{6.7}$$

Now, we will extend the partial mapping (6.7) up to a computable isomorphism of the algebras under consideration by the rule: for an arbitrary finite sequence of finite binary tuples $\alpha_0, \alpha_1, \ldots \alpha_n \in 2^{<\omega}$, we put $\lambda(b_{\alpha_0} \cup b_{\alpha_1} \cup \cdots \cup b_{\alpha_n}) = \beta_{\alpha_0} \cup \beta_{\alpha_1} \cup \cdots \cup \beta_{\alpha_n}$.

Based on Claim A, it is a simple exercise to show that the mapping λ is an isomorphism that is computable in numerations ν and γ obtaining finally the required computable isomorphism $\lambda : (\mathcal{B}, \nu) \to \Big(\mathcal{L}\big(\mathrm{Th}(\mathrm{Mod}(\Phi) \cap K)\big), \gamma\Big)$.

Thereby, Theorem 6.1 is completely proved. □

CONCLUSION

This work describes in detail a general scheme of construction of finitely axiomatizable theories with pre-assigned properties. It shows that, in terms of natural model-theoretic properties, expressive power of finitely axiomatizable theories in first-order logic coincides with that of computably axiomatizable theories thereby presenting a good chance to solve various problems concerning finitely axiomatizable theories by way of their regular reduction to a much simpler case of computably axiomatizable theories.

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