ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2015, Vol. 36, No. 2, pp. 211–214. © Pleiades Publishing, Ltd., 2015.

About One Algorithm for Solving Scheduling Problem

O. N. Shulgina* and N. K. Shcherbakova**

(Submitted by F. M. Ablayev)

Kazan Federal University, Kremlevskaya ul. 18, Kazan, Tatarstan, 420008

Received June 5, 2014

Abstract—In this paper we proved the new properties optimal schedules for unknown strongly NP-complete scheduling problem of minimizing maximum lateness on a single machine, not allowing preemption. Pseudopolynomial implementation of the general scheme for solving that problem based on these properties is developed.

DOI: 10.1134/S1995080215020171

Keywords and phrases: *scheduling, lateness, sequencing, pseudopolynomial algorithm, properties, NP-completeness, complexity.*

One of the known scheduling problems is the problem of minimizing maximum lateness on a single machine. That problem is NP-complete [7] in the strong sense. It means that there is no pseudopolynomial algorithm for its solution by assuming that classes NP and P do not coincide. The problem of minimizing maximum lateness can be solved from polynomial time if preemption in the process of any job is allowed [1, 8, 9].

Algorithms complexity $O(n \log n)$ for solving the problem in the case of equal release dates and equal due dates were proposed in [9, 10]. A polynomial solution for the case of identical processing times is obtained in [8, 11, 12]. In the article [6] algorithm pseudopolynomial complexity for solution NP— complete special case of the problem where the jobs can be renumber simultaneously in non-decreasing due dates and non-increasing release dates developed and proved. The general scheme for solving that problem is proposed and proved in the [4].

Formulate the problem. n jobs have to be processed on a single machine not earlier than the time t. Machine can process at most one job at a time and preemption in the process for any job is not allowed. Assume that the jobs are specified numbers from 1 to n. Assume $N = \{1, 2, ..., n\}$. The following data spesified for each job j, $j \in N$: a release date r_j; a processing time $p_j \geq 0$; a due date d_j. Numbers t, r_j, p_j, d_j are integer. We will understand a permutation of any subset of the set N as the schedule. We will denote $\Pi(N', t')$ the set schedules on the set of jobs $N' \subseteq N$ which are begining at time $t' \geq t$. Schedule of any subset of $N' \subseteq N$ will be called a partial on the set N'. Let $\pi = (j_1, j_2, ...j_{n'}) \in \Pi(N', t')$, where $n' = |N'|$ is a number of elements in the set N', j_k is a number of jobs, which is served by place number k in schedule π. Completion time $t_{j_k}(\pi)$ of job j_k, $k = \overline{1, n'}$, is as follows: $t_{j_1}(\pi) = \max\{t', r_{j_1}\} + p_{j_1}$; $t_{j_k}(\pi) = \max\{t_{j_{k-1}}(\pi), r_{j_k}\} + p_{j_k}$, $k = 2, 3, ..., n'$. We denote the lateness $L_j(\pi)$ for job $j \in N'$ in schedule π, that is $L_j(\pi) = t_j(\pi) - d_j$.

Let $\pi^* \in \Pi(N', t')$ be schedule at which the function $F(\pi) = \max\limits_{j \in N'} L_j(\pi)$, $\pi \in \Pi(N', t')$, reaches a minimal value on the set $\Pi(N', t')$. We named π^* optimal schedule on the set $\Pi(N', t')$. If $N' = \oslash$ then assume $F(\pi) = -\infty$, $\pi \in \Pi(N', t')$, and schedule on an empty set will be denoted as $\pi^{\oslash}$. Thus, the problem of minimizing maximum lateness on a single machine consists in finding the optimal schedule on the set $\Pi(N, t)$.

*E-mail: onshul@mail.ru
**E-mail: nata6060@mail.ru

We introduce the necessary notation in the future. Let $N' \subseteq N$, $N' \neq \oslash$, $t' \geq t$, $\pi \in \Pi(N', t')$. Assume $r_{min}(N') = \min\limits_{i \in N'} r_i$; $r_{max}(N') = \max\limits_{i \in N'} r_i$; $p_{max}(N') = \max\limits_{i \in N'} p_i$; $T(\pi) = \max\limits_{j \in N'} t_j(\pi)$; $J^*(\pi) = \{j \in N' : F(\pi) = L_j(\pi)\}$; $J_d(N') = \{j \in N' : d_j = \min\limits_{i \in N'} d_i\}$; $\Pi^*(N', t') = \{\pi^* \in \Pi(N', t') : F(\pi^*) = \min\limits_{\pi \in \Pi(N', t')} F(\pi)\}$ be a set of optimal schedules on the $\Pi(N', t')$; $\vec{\Pi}_r(N', t')$, $\vec{\Pi}_d(N', t')$ be a sets of schedules for the jobs from N' at time t' in which the jobs are ordered nondecreasing release dates and decreasing due dates respectively. We will write $i \xrightarrow{\pi} j$ if the job i is precedes the job j, $i \neq j$, in the schedule π . Notation $i \xrightarrow{\pi} \bar{N}$, where $\bar{N} \subseteq N'$, $i \notin \bar{N}$, means that $i \xrightarrow{\pi} j$ $\forall$ $j \in \bar{N}$, and notation $\bar{N} \xrightarrow{\pi} \bar{\bar{N}}$, where $\bar{N}, \bar{\bar{N}} \subseteq N'$, $\bar{N} \cap \bar{\bar{N}} = \oslash$, means that for all the pairs i, j such that $i \in \bar{N}$, $j \in \bar{\bar{N}}$ we have the following relation $i \xrightarrow{\pi} j$.

Lemma 1 [5]. *Let $N' \subseteq N$, $N' \neq \oslash$, $t' \geq t$, $j_d \in J_d(N')$, $\pi \in \Pi(N', t')$. Then there is such a job $j^* \in J^*(\pi)$ that $j^* \in \{j \in N' : j_d \xrightarrow{\pi} j\} \cup \{j_d\}$.*

Lemma 2 [4]. *Let $N' \subseteq N$, $N' \neq \oslash$, $t' \geq t$, $\pi^* \in \Pi^*(N', t')$, $j_d \in J_d(N')$, the sets $N^1, N^2 \subseteq N'$ such that $N^1 \xrightarrow{\pi^*} j_d \xrightarrow{\pi^*} N^2$, $N^1 \cup N^2 \cup \{j_d\} = N'$. Then schedule $(\vec{\pi}_r^1, j_d, \pi^{2*}) \in \Pi(N', t')$, where $\vec{\pi}_r^1 \in \vec{\Pi}_r(N^1, t')$, $\pi^{2*} \in \Pi^*(N^2, T(\vec{\pi}_r^1, j_d))$, is optimal on the set $\Pi(N', t')$.*

Further, we formulate and prove the properties of optimal schedules relating to the procedure service the jobs in a partial schedules from which consists the optimal schedule adding specific jobs to the original set.

Let $N' \subseteq N$ and in the set N exists such jobs j' and j'' that

$$r_{j'} \leq r_j, \quad d_{j'} \geq d_j \quad \forall j \in N', \tag{1}$$

$$r_{j''} \geq r_j, \quad d_{j''} \leq d_j \quad \forall j \in N'. \tag{2}$$

Indexes j' and j'' will be denoted $j^0(N')$ and $j^{n+1}(N')$, respectively. If in the set N there are no j' and j'' that satisfy (1), (2) then we define fictitious jobs j' and j'' and assume such $r_{j'}$, $r_{j''}$, $d_{j'}$, $d_{j''}$ that to satisfy (1), (2). Denote them as above through the $j^0(N')$ and $j^{n+1}(N')$, respectively.

Lemma 3. *Let $\bar{N} \subseteq N$, $t' \geq t$, $N' = \bar{N} \cup \{j^{n+1}(\bar{N})\}$, where $j^{n+1}(\bar{N})$ chosen by (2). Then there are such subsets $N^1, N^2 \subseteq N'$ that $N^1 \cup N^2 = N' \setminus \{j^{n+1}(\bar{N})\}$, and schedule $\pi^* = (\vec{\pi}_r^1, j^{n+1}(\bar{N}), \vec{\pi}_d^2) \in \Pi(N', t')$ for any $\vec{\pi}_r^1 \in \vec{\Pi}_r(N^1, t')$, $\vec{\pi}_d^2 \in \vec{\Pi}_d(N^2, T(\vec{\pi}_r^1, j^{n+1}(\bar{N})))$ is optimal.*

Proof. By (2) $j^{n+1}(\bar{N}) \in J_d(N')$. According to Lemma 2 we choose such the subset $N^1, N^2 \subseteq N' \subseteq N \cup \{j^{n+1}(\bar{N})\}$ that $N^1 \cup N^2 = \bar{N}$ and schedule $\pi = (\vec{\pi}_r^1, j^{n+1}(\bar{N}), \pi^{2*})$ is optimal, where $\vec{\pi}_r^1 \in \vec{\Pi}_r(N^1, t')$, $\pi^{2*} \in \Pi^*(N^2, T(\vec{\pi}_r^1, j^{n+1}(\bar{N})))$.

By (2) follows that $r_{j^{n+1}(\bar{N})} \geq r_j$, $j \in N^2$. It means that all jobs from the set N^2 ready to process to the start process job $j^{n+1}(\bar{N})$. Consequently, schedule $\vec{\pi}_d^2 \in \vec{\Pi}_d(N^2, T(\vec{\pi}_r^1, j^{n+1}(\bar{N})))$ in which jobs from the set N^2 are sheduled in order of nondecreasing due dates is optimal ([3], p. 110) for start time $T(\vec{\pi}_r^1, j^{n+1}(\bar{N}))$. Therefore, $F(\pi^{2*}) = F(\vec{\pi}_d^2)$, and $(\vec{\pi}_r^1, j^{n+1}(\bar{N}), \vec{\pi}_d^2)$ is optimal schedule. □

Lemma 4. *Let $\bar{N} \subseteq N$, $t' \geq t$, $N' = \bar{N} \cup \{j^0(\bar{N}), j^{n+1}(\bar{N})\}$, where $j^0(\bar{N})$, $j^{n+1}(\bar{N})$ choosen by (1), (2). Then exists an optimal schedule $\pi^* \in \Pi(N', t')$ in which rightly $j^0(\bar{N}) \xrightarrow{\pi^*} j$ or $j \xrightarrow{\pi^*} j^0(\bar{N})$ for each $j \in N' \setminus \{j^0(\bar{N})\}$.*

Proof. According to Lemma 3 we choose such the subset $N^1, N^2 \subseteq N' \subseteq N \cup \{j^0(\bar{N}), j^{n+1}(\bar{N})\}$ that $N^1 \cup N^2 = N' \setminus \{j^{n+1}(\bar{N})\}$ and schedule $\pi = (\vec{\pi}_r^1, j^{n+1}(\bar{N}), \vec{\pi}_d^2) \in \Pi(N', t')$, is optimal, where $\vec{\pi}_r^1 \in \vec{\Pi}_r(N^1, t')$, $\vec{\pi}_d^2 \in \vec{\Pi}_d(N^2, T(\vec{\pi}_r^1, j^{n+1}(\bar{N})))$.

Let $j^0(\bar{N}) \in N^1$. By (1) follows $r_{j^0(\bar{N})} \leq r_j$ $\forall$ $j \in N^1$. Then exists the schedule $\bar{\pi}^1 \in \vec{\Pi}_r(N^1, t')$ in which job $j^0(\bar{N})$ is to process the first, that is $j^0(\bar{N}) \xrightarrow{\bar{\pi}^1} j$ $\forall$ $j \in N^1$. Construct the schedule $\pi' = (\bar{\pi}^1, j^{n+1}(\bar{N}), \vec{\pi}_d^2)$ which differs from the π the order of process jobs from the set N^1 and in which $j^0(\bar{N}) \xrightarrow{\pi'} j$ $\forall$ $j \in N' \setminus \{j^0(\bar{N})\}$. We will show that π' is optimal schedule. By (2) follows $d_{j^{n+1}(\bar{N})} \leq d_j$ $\forall$

$j \in N' \setminus \{j^{n+1}(\bar{N})\}$, so $j^{n+1}(\bar{N}) \in J_d(N')$. Then from Lemma 1 follows that exists the jobs $j^* \in J^*(\pi)$ and $j'^* \in J^*(\pi')$ that $j^*, j'^* \in N^2 \cup \{j^{n+1}(\bar{N})\}$, therefore,

$$F(\pi) = \max_{j \in N^2 \cup \{j^{n+1}(\bar{N})\}} L_j(\pi) \tag{3}$$

and

$$F(\pi') = \max_{j \in N^2 \cup \{j^{n+1}(\bar{N})\}} L_j(\pi'). \tag{4}$$

Since the $\vec{\pi}^1_r, \bar{\pi}^1 \in \vec{\Pi}_r(N^1, t')$ then $T(\vec{\pi}^1_r) = T(\bar{\pi}^1)$. Also the order of process the jobs from the set $N^2 \cup \{j^{n+1}(\bar{N})\}$ same in the schedules π and π'. It follows considering (3), (4) $F(\pi) = F(\pi')$, therefore, the schedule π' is optimal.

Let $j^0(\bar{N}) \in N^2$. By (1) $d_{j^0(\bar{N})} \geq d_j \ \forall \ j \in N^2$ then exists schedule $\bar{\pi}^2 \in \vec{\Pi}_d(N^2, T(\vec{\pi}^1_r, j^{n+1}(\bar{N})))$ in which job $j^0(\bar{N})$ to process latest, that is $j \xrightarrow{\bar{\pi}^2} j^0(\bar{N}) \ \forall \ j \in N^2$. Construct the schedule $\pi'' = (\vec{\pi}^1_r, j^{n+1}(\bar{N}), \bar{\pi}^2)$, which differs from the π the order of process jobs from the set N^2 and satisfies $j \xrightarrow{\pi''} j^0(\bar{N}) \ \forall \ j \in N' \setminus \{j^0(\bar{N})\}$. We will show that π'' is optimal schedule. By (2) follows that $d_{j^{n+1}(\bar{N})} \leq d_j \ \forall \ j \in N' \setminus \{j^{n+1}(\bar{N})\}$, therefore, $j^{n+1}(\bar{N}) \in J_d(N')$. Then from lemma 1 follows that exist such jobs $j^* \in J^*(\pi)$ and $j'^* \in J^*(\pi'')$, $j^*, j'^* \in N^2 \cup \{j^{n+1}(\bar{N})\}$, therefore, satisfy (3) and

$$F(\pi'') = \max_{j \in N^2 \cup \{j^{n+1}(\bar{N})\}} L_j(\pi'').$$

Since the $\vec{\pi}^2_d, \bar{\pi}^2 \in \vec{\Pi}_d(N^2, T(\vec{\pi}^1_r, j^{n+1}(\bar{N})))$ then schedules $\vec{\pi}^2_d, \bar{\pi}^2$ in which the jobs from the set N^2 scheduled in order nondecreasing due dates is optimal ([3], p. 110) for start time $T(\vec{\pi}^1_r, j^{n+1}(\bar{N}))$. Therefore, $F(\vec{\pi}^2_d) = F(\bar{\pi}^2)$. Furthermore, $t_j(\pi) = t_j(\pi'') \ \forall \ j \in N^1 \cup \{j^{n+1}(\bar{N})\}$. Hence, considering (3), (4) $F(\pi) = F(\pi'')$, consequently, schedule π'' is optimal.

Thus, from the transformation arbitrarily chosen optimal schedule π we received optimal schedules π', π'' satisfying the lemma. □

Let $N' \subseteq N, N' \neq \oslash, t' \geq t, j_d \in J_d(N'), \pi \in \Pi(N', t')$. Denote $J_{max}(\pi, j_d) = \{j \in N' \setminus \{j_d\} : r_j \geq r_{j_d}, j_d \xrightarrow{\pi} j\}$, $J_{min}(\pi, j_d) = \{j \in N' \setminus \{j_d\} : r_j < r_{j_d}, j_d \xrightarrow{\pi} j\}$. Note that the set $J_{max}(\pi, j_d)$, $J_{min}(\pi, j_d)$ can be empty. In the [4] proved the existence an optimal schedule $\pi^* \in \Pi^*(N', t')$ for which $J_{max}(\pi^*, j_d) = \{j \in N' \setminus \{j_d\} : r_j \geq r_{j_d}\}$ and propose a general scheme for finding that schedule assuming that the set $J_{min}(\pi^*, j_d)$ can be found by some algorithm A complexity $O(x(n))$, where $x(n)$ is a function of the dimension of the problem.

Scheme [4]. *Initially assume* $t_1 = \max\{r_{min}(N), t\}$*,* $N_1 = N$*,* $\pi_1 = \pi^{\oslash}$*. Suppose that known* t_k*,* N_k*,* π_k*, and* $k \geq 1$.

If $N_k = \oslash$ *then* $\pi_k \in \Pi^*(N, t)$*, and the process ends. Otherwise, choose* $j^k_d \in J_d(N_k)$*,* $J^k_{max} = \{j \in N_k \setminus \{j^k_d\} : r_j \geq r_{j^k_d}\}$*, find a set* $J^k_{min} = J_{min}(\pi^*, j^k_d)$*,* $\pi^* \in \Pi^*(N_k, t_k)$ *using of some algorithm* A*, and assume* $N_{k+1} = J^k_{min} \cup J^k_{max}$*,* $N^k = N_k \setminus (J^k_{max} \cup J^k_{min} \cup \{j^k_d\})$*,* $\pi_{k+1} = (\pi_k, \vec{\pi}^k_r, j^k_d)$*, where* $\vec{\pi}^k_r \in \vec{\Pi}_r(N^k, t_k)$*,* $t_{k+1} = T(\pi_{k+1})$.

The complexity of the scheme $O(n^2 + nx(n))$ [4]. Despite the theoretical character of this scheme it is possible to use it to solve new special cases of the problem which will be able to develop an algorithm for finding the set $J_{min}(\pi^*, j_d)$. Also scheme can be use for development a approximation algorithms for solving the problem, different choice the set J^k_{min} on the each iteration of the scheme.

In this paper we propose a pseudopolynomial realization scheme using the lemmas 3, 4.

Describe the procedure $J(N', t', j')$ which we will use to build the set J^k_{min} at the each iteration of the scheme. The procedure consists from n iterations. Inside each iteration for each integer time point the number of which does not exceed $P = \max\{r_{max}(N), t\} + \sum_{j=1}^{n} p_j - \max\{r_{min}(N), t\}$ the following schedule is obtained by adding to the already constructed schedule job it have maximum release date among nonsequencing jobs. Adding jobs is performed so that the schedules are constructed procedure will satisfy the properties formulated in lemmas 3, 4.

Procedure $\mathbf{J}(\mathbf{N}', \mathbf{t}', \mathbf{j}')$. *If $N' = \oslash$ then assume $\bar{\bar{N}} = \oslash$, and the process ends. Otherwise, enumerate jobs from the set N' so that j' was the first and the follows inequalities are valid $d_1 \le d_2 \le \ldots \le d_n$. Assume $N_1 = \{1\}$, $P_1 = \max\{r_1, t'\} + \sum\limits_{j \in N'} p_j - p_1$, $\pi_1^i = (1)$, $\pi_1^i \in \Pi(N_1, i)$, $\bar{N}_1^i = \oslash$, $\bar{\bar{N}}_1^i = \oslash \; \forall\, i = \overline{t', P_1}$. Suppose that known N_k, P_k, π_k^i, $\bar{N}_k^i$, $\bar{\bar{N}}_k^i$ for each $i = \overline{t', P_k}$, and $1 \le k < n$.*

Assume $N_{k+1} = N_k \cup \{k+1\}$, $P_{k+1} = P_k - p_{k+1}$, $\pi_i' = (\vec{\pi}_r, 1, \vec{\pi}_d)$, $\vec{\pi}_r \in \vec{\Pi}_r(\bar{N}_k^i \cup \{k+1\}, i)$, $\vec{\pi}_d \in \vec{\Pi}_d(\bar{\bar{N}}_k^i, t_1(\pi_i'))$, $\quad \pi_i'' = (\pi_k^i, k+1)$, $\quad \pi_{k+1}^i = \arg\min\{T(\pi) : \pi \in \Pi_i\}$, $\quad \Pi_i = \{\pi \in \{\pi_i', \pi_i''\} : F(\pi) = \min\limits_{\bar{\pi} \in \{\pi_i', \pi_i''\}} F(\bar{\pi})\}$, $\pi_i', \pi_i'', \pi_{k+1}^i \in \Pi(N_{k+1}, i) \;\; \forall \;\; i = \overline{t', P_{k+1}}$. If $\pi_{k+1}^i = \pi_i'$ then $\bar{N}_{k+1}^i = \bar{N}_k^i \cup \{k+1\}$, $\bar{\bar{N}}_{k+1}^i = \bar{\bar{N}}_k^i$. If $\pi_{k+1}^i = \pi_i''$ then $\bar{N}_{k+1}^i = \bar{N}_k^i$, $\bar{\bar{N}}_{k+1}^i = \bar{\bar{N}}_k^i \cup \{k+1\}$.

If $k = n'$ then $\bar{\bar{N}} = \bar{\bar{N}}_n^{t'}$, and the process ends.

Theorem. *Suppose that on each iteration of the scheme is faithful equality $J_{\min}^k = \bar{\bar{N}}$, where the $\bar{\bar{N}}$ is constructed by procedure $J(N_k, t_k, j_d^k)$. Then the complexity realization of the scheme is $O(n^2P)$ operation, where $P = \max\{r_{\max}(N), t\} + \sum\limits_{j \in N'} p_j$.*

Proof. To evaluate the complexity for the procedure $J(N', t', j')$. The $O(n \log n)$ operations enough for renumber jobs in nondecreasing due dates ([2], chapter 5). Number of construction schedules π_{k+1}^i is no more than P on the each iteration of procedure. To calculate values $F(\pi_i')$, $F(\pi_i'')$, $T(\pi_i')$, $T(\pi_i'')$, obviously, required $O(n)$ operations. Therefore, to construct each schedule π_{k+1}^i enough $O(n)$ operations, and to build all schedules π_{k+1}^i on iteration k require $O(nP)$ operations. Consequently, the complexity of the one iteration is $O(nP)$. Since the procedure $J(N', t', j')$ consists from n iteration then it complexity is $O(n^2P)$ operations.

Considering complexity of the scheme is $O(n^2 + nx(n))$ operations, then its realization using procedure $J(N', t', j')$ require $O(n^2P)$ operations. □

Realization of the scheme using the procedure $J(N', t', j')$ showed good experimental results. From 1000 conducted experiments which had dimensions $3 \le n \le 15$ in 243 examples were built optimal schedule, in the remaining 767 cases $F(\pi_J)/F(\pi^*) < \frac{21}{20}$, where $\pi^* \in \Pi^*(N, t)$, shedule $\pi_J \in \Pi(N, t)$ is constructed by scheme using the procedure $J(N', t', j')$.

REFERENCES

1. N. B. Lebedinskaya, Zap. Nauch. Seminarov Leningrad. Otd. Mat. Inst. Akad. Nauk SSSR, 117–124 (1978).
2. D. Knut, *The art of programming for computers*. Vol. 3: *Sorting and Searching* (Mir, Moscow, 1973) [in Russian].
3. V. S. Tanaev ,V. S. Gordon, and Ya. M. Shafranskiy, *Scheduling Theory. Single-Stage System* (Nauka, Moscow, 1984) [in Russian].
4. O. N. Shulgina, Avtom. Telemekh. **3**, 108 (2004).
5. O. N. Shulgina, J. Issled. Prikl. Mat. **23**, 150 (2001).
6. O. N. Shulgina, Sbor. Issl. Prikl. Mat. Inform. **25**, 148 (2004).
7. P. Bruker, J. K. Lenstra, and A. H. G. Rinnooy Kan, Ann. Disc. Math. **1**, 343 (1977).
8. W. A. Horn, Nav. Res. Log. Quart. 21, **1**, 177 (1974).
9. B. J. Lageweg, J. K. Lenstra, and A. H. G. Rinnooy Kan, Statist. Neer. **1**, 25 (1976).
10. J. R. Jackson, Res. Report 43, Manag. Sci. Res Project, USLA, Jan. 1955.
11. G. N. Frederickson, Inform. Process. Lett. **16**, 171 (1983).
12. B. A. Simons, 19th Ann. Symp. Found. Comput. Sci., Ann. Arbor, Mich. New York (1978), p. 246.