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On an Inequality Involving the Complex Polynomial and Its Derivative

N. Prasanna Kumar*

(Submitted by S. N. Tronin)

Department of Mathematics, Birla Institute of Technology and Science, Pilani K K Birla Goa Campus, Goa, 403726 India

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Abstract—Let $P(z)$ be a complex polynomial of degree n of lacunary type having zeros in $|z| < 1$, and $1 \leq r < R$. In this paper, we shall estimate a new type of bound in more generalized form for $\max\limits_{|z|=1} |P'(z)|$ in terms of $\max\limits_{|z|=Rr} |P(z)|$ and $\max\limits_{|z|=R^2} |P(z)|$.

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1. INTRODUCTION

The subject of Geometry of Polynomials is quite interesting and vast. Hundreds of results are available in the literature on Markov's and Bernstein's theorems and their generalizations in different metrics. We consider the problem of inequalities involving polynomials and their derivatives. The unfolding of this area of research is very well explained in [13, 15]. Firstly we present the well-known theorem of Turan [16] on the complex polynomials having all its zeros in the unit disc.

If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is sharp and the equality holds in (1.1) if all zeros of $P(z)$ lie on the unit circle.

More generally, if the polynomial $P(z)$ has all its zeros in $|z| \leq K \leq 1$, then Malik [12] proved the following inequality.

If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The result is sharp and the equality holds in (1.2) if $P(z) = z^n + K$.

Govil [8] considered the problem of determining the maximum modulus of the derivative of a complex polynomial whose zeros are in the disc $|z| \leq K$, $K \geq 1$. Infact he proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |P(z)|. \tag{1.3}$$

The result is sharp and the equality holds in (1.3) if $P(z) = z^n + K^n$.

There are many improvements, generalizations, and different versions of inequalities analogous to (1.3) [1, 2, 3, 5, 8, 10, 11]. The bound in (1.3) depends only on the zeros of largest modulus and not on any other zeros, even if many of them are of smaller moduli. This case was very well handled

*E-mail: prasannakornaya@rediffmail.com

by Govil [7]. So the inequality in (1.3) can be further modified if we obtain a bound which depends on atleast some of the nonzero coefficients of $P(z)$, the location of the zeros of the given polynomial and more importantly, the modulus of each of the zeros. It is quite natural to ask how does the inequality look if all the zeros are on and within a disk of radius K. A paper due to Prasanna Kumar [14] answered this question partially in one way if $K \geq 1$. Now the next question is how does this inequality may be transformed if the maximum modulus of the derivative is sought in terms of the maximum modulus of the polynomial on a circle exterior to $|z| = K$ and $K < 1$? Could we get a more generalized inequality of this kind? This paper makes an attempt to proceed in this direction and proves a new inequality in generalized form. In fact, we state the theorem more generally as follows.

Theorem 1.1. *If* $P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k)$ *is a polynomial of degree* $n > 3$ *having all its zeros in* $|z| < R$*, then for* $1 \leq r < R$*,*

$$B \max_{|z|=1} |P'(z)| \geq \frac{1}{R^{2n}} \sum_{k=1}^{n} \frac{1}{R + |z_k|} \max_{|z|=R^2} |P(z)|$$
$$- \frac{1}{R^n r^n} \sum_{k=1}^{n} \frac{1}{R + |z_k|} \max_{|z|=Rr} |P(z)| + 2|a_{n-1}| \sum_{k=1}^{n} \frac{1}{R + |z_k|} \left(\frac{(R^n - r^n)}{n(n+1)} - \frac{R - r}{n+1} \right)$$
$$+ 2|a_{n-2}| \sum_{k=1}^{n} \frac{1}{R + |z_k|} \left[\frac{R^n - r^n}{n(n-1)} - \frac{R^{n-2} - r^{n-2}}{(n-2)(n-3)} + \frac{2(R - r)}{(n-1)(n-3)} \right]$$
$$+ \frac{B}{R^{n-1}} \left(2|a_1| \frac{(R^{(n-1)} - 1)}{(n+1)} + 2|a_2| \left[\frac{R^{(n-1)} - 1}{(n-1)} - \frac{R^{(n-3)} - 1}{(n-3)} \right] \right). \tag{1.4}$$

where $B = \frac{R^n - r^n}{n} \sum_{j=1}^{n} \frac{1}{(R - |z_j|)}$*. In the limiting case* $r \to R$*, ineqality* (1.4) *becomes equality.*

2. LEMMAS

For the proof of the theorems, we need the following lemmas. First lemma is due to Girox, Rahman and Schmeisser [6].

Lemma 2.1. *If* $P(z) = a_n \prod_{k=1}^{n} (z - z_k)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq 1$*, then*

$$\max_{|z|=1} |P'(z)| \geq \sum_{k=1}^{n} \frac{1}{1 + |z_k|} \max_{|z|=1} |P(z)|. \tag{2.1}$$

Equality in (2.1) *holds if every* z_k *is positive real.*

Dewan et al. [4] derived the following lemma, which we will use in proving our results.

Lemma 2.2. *If* $P(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree* $n > 2$*, then for any* $R \geq 1$*,*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - 2|a_0| \frac{(R^n - 1)}{n+2} - |a_1| \left[\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right]. \tag{2.2}$$

Equality holds in (2.2) *if* $P(z) = z^n + 1$*.*

Now we prove a new inequality involving the derivative of polynomials.

Lemma 2.3. *If* $P(z)=\sum_{k=0}^{n}a_kz^k=a_n\prod_{k=1}^{n}(z-z_k)$ *is a polynomial of degree* n *then for any* $R>|z_j|, 1\le j\le n$,

$$\max_{|z|=R}|P'(z)|\le\sum_{k=1}^{n}\frac{1}{R-|z_k|}\max_{|z|=R}|P(z)|. \tag{2.3}$$

There is an equality in (2.3) *if all the zeros of* $P(z)$ *lie at the origin.*

Proof. The proof is analogous to the proof of Lemma 2.1 in the paper due to Govil et al. [9]. We have $P(z)=a_n\prod_{j=1}^{n}(z-|z_j|e^{i\theta_j})$.

For $0\le\theta<2\pi$, we have

$$\left|\frac{1}{Re^{i\theta}-|z_j|e^{i\theta_j}}\right|^2=\frac{1}{R^2+|z_j|^2-2R|z_j|\cos(\theta-\theta_j)}\le\frac{1}{(R-|z_j|)^2}.$$

But then

$$\left|\frac{P'(Re^{i\theta})}{P(Re^{i\theta})}\right|\le\sum_{j=1}^{n}\left|\frac{1}{Re^{i\theta}-|z_j|e^{i\theta_j}}\right|\le\sum_{j=1}^{n}\frac{1}{(R-|z_j|)}.$$

A simple observation and rearrangement give the desired result. □

Remark 2.4. One can observe that, if the difference between R and $|z_j|$ is too small, then the inequality proved in the above lemma becomes too loose! The lemma is useful and justified whenever the value of $R-|z_j|$ is considerably greater. The result is more interesting if $R-|z_j|=n$ for every j.

Now we prove a lemma which will be used in proving our theorem.

Lemma 2.5. *If* $P(z)=\sum_{k=0}^{n}a_kz^k=a_n\prod_{k=1}^{n}(z-z_k)$ *is a polynomial of degree* $n>3$ *having all its zeros in* $|z|<1$, *then for any* $1\le r\le R$,

$$\max_{|z|=R}|P(z)|\le\frac{R^n-r^n}{n}\sum_{j=1}^{n}\frac{1}{(1-|z_j|)}\max_{|z|=1}|P(z)|+\max_{|z|=r}|P(z)|-2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)$$
$$-2|a_2|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]. \tag{2.4}$$

The above result is sharp for any polynomials having zeros inside the unit disc when $R=r$.

Proof. Let the polar representation of z be $z=te^{i\theta}$. Then for every θ with $0\le\theta<2\pi$, we have

$$|p(Re^{i\theta})-p(re^{i\theta})|\le\int_r^R|p'(te^{i\theta})|dt.$$

Now applying Lemma 2.2 and then Lemma 2.3 to the polynomial $P(z)$ which is of degree greater than 3, we get

$$|p(Re^{i\theta})-p(re^{i\theta})|\le\int_r^R\left[t^{n-1}\max_{|z|=1}|P'(z)|-2|a_1|\frac{(t^{n-1}-1)}{n+1}-2|a_2|\left[\frac{t^{n-1}-1}{n-1}-\frac{t^{n-3}-1}{n-3}\right]\right]dt$$
$$=\left(\frac{R^n-r^n}{n}\right)\max_{|z|=1}|P'(z)|-2|a_1|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{(R-r)}{n+1}\right)$$
$$-2|a_2|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]$$

$$\leq \frac{R^n - r^n}{n} \sum_{j=1}^{n} \frac{1}{(1-|z_j|)} \max_{|z|=1} |P(z)| - 2|a_1| \left(\frac{(R^n - r^n)}{n(n+1)} - \frac{(R-r)}{n+1} \right)$$
$$- 2|a_2| \left[\frac{R^n - r^n}{n(n-1)} - \frac{R^{n-2} - r^{n-2}}{(n-2)(n-3)} + \frac{2(R-r)}{(n-1)(n-3)} \right].$$

Simple application of the property of difference between moduli of two functions to the left hand side of the above inequality and minor rearrangements of the terms will yield the inequality (2.4). Hence the proof is compete. □

Remark 2.6. Lemma 2.5 connects the bound for the maximum moduli of polynomials on two different circles with respect to the unit disc containing all the zeros of the polynomials. It is always interesting that maximum modulus of a polynomial in any arbitrary circles can be related by the location of zeros of the polynomials and such inequalities.

3. PROOF OF THE THEOREM

Since the zeros of $P(z)$ are z_k $(1 \leq k \leq n)$ the zeros of the polynomial $q(z) = p(Rz)$ are $\frac{z_k}{R}$ $(1 \leq k \leq n)$. Also observe that, the zeros of $q(z)$ all lie in $|z| < 1$, since all the zeros of $P(z)$ satisfy $|z_j| < R,\ \ 1 \leq j \leq n$. Now by Lemma 2.1 we have

$$\max_{|z|=1} |q'(z)| \geq \sum_{k=1}^{n} \frac{1}{1 + \frac{|z_k|}{R}} \max_{|z|=1} |q(z)|$$

which gives

$$\max_{|z|=R} |P'(z)| \geq \sum_{k=1}^{n} \frac{1}{R + |z_k|} \max_{|z|=R} |P(z)|. \tag{3.1}$$

As the degree of $P(z)$ is greater than 3, the degree of $P'(z)$ is greater than 2. Therefore applying Lemma 2.2 to $P'(z)$ with $R > 1$,

$$\max_{|z|=R} |P'(z)| \leq R^{n-1} \max_{|z|=1} |P'(z)| - 2|a_1| \frac{(R^{(n-1)} - 1)}{(n+1)} - 2|a_2| \left[\frac{R^{(n-1)} - 1}{(n-1)} - \frac{R^{(n-3)} - 1}{(n-3)} \right]. \tag{3.2}$$

From equations (3.1) and (3.2) it follows that

$$\sum_{k=1}^{n} \frac{1}{R + |z_k|} \max_{|z|=R} |P(z)| \leq R^{n-1} \max_{|z|=1} |P'(z)| - 2|a_1| \frac{(R^{(n-1)} - 1)}{(n+1)}$$
$$- 2|a_2| \left[\frac{R^{(n-1)} - 1}{(n-1)} - \frac{R^{(n-3)} - 1}{(n-3)} \right].$$

Equivalently,

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{R^{(n-1)}} \sum_{k=1}^{n} \frac{1}{R + |z_k|} \max_{|z|=R} |P(z)|$$
$$+ \frac{1}{R^{n-1}} \left(2|a_1| \frac{(R^{(n-1)} - 1)}{(n+1)} + 2|a_2| \left[\frac{R^{(n-1)} - 1}{(n-1)} - \frac{R^{(n-3)} - 1}{(n-3)} \right] \right). \tag{3.3}$$

Let $S(z) = z^n P(\frac{1}{z})$. Note that the polynomial $S(\frac{1}{Rz})$ has all its zeros in $|z| < 1$. Also observe that

$$\max_{|z|=R} \left| S\left(\frac{1}{Rz}\right) \right| = \frac{1}{R^{2n}} \max_{|z|=R^2} |P(z)|, \quad \max_{|z|=r} \left| S\left(\frac{1}{Rz}\right) \right| = \frac{1}{r^n R^n} \max_{|z|=Rr} |P(z)|$$

and

$$\max_{|z|=1} \left| S\left(\frac{1}{Rz}\right) \right| = \frac{1}{R^n} \max_{|z|=R} |P(z)|.$$

Now applying Lemma 2.5 to the polynomial $S(\frac{1}{Rz})$, we get

$$\max_{|z|=R}\left|S\left(\frac{1}{Rz}\right)\right| \leq \frac{R^n-r^n}{n}\sum_{j=1}^{n}\frac{R}{(R-|z_j|)}\max_{|z|=1}\left|S\left(\frac{1}{Rz}\right)\right| + \max_{|z|=r}\left|S\left(\frac{1}{Rz}\right)\right|$$
$$-2|a_{n-1})|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)-2|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]$$

$\Rightarrow$

$$\frac{1}{R^{2n}}\max_{|z|=R^2}|P(z)| \leq \frac{R^n-r^n}{nR^n}\sum_{j=1}^{n}\frac{R}{(R-|z_j|)}\max_{|z|=R}|P(z)| + \frac{1}{r^nR^n}\max_{|z|=Rr}|P(z)|$$
$$-2|a_{n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)-2|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]$$

$\Rightarrow$

$$\frac{R^n-r^n}{n}\sum_{j=1}^{n}\frac{R}{(R-|z_j|)}\max_{|z|=R}|P(z)| \geq \frac{1}{R^n}\max_{|z|=R^2}|P(z)| - \frac{1}{r^n}\max_{|z|=Rr}|P(z)|$$
$$+2R^n|a_{n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right)+2R^n|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right]$$

$\Rightarrow$

$$\begin{aligned}\max_{|z|=R}|P(z)| &\geq \frac{1}{BR^{n+1}}\max_{|z|=R^2}|P(z)| \\ &- \frac{1}{BRr^n}\max_{|z|=Rr}|P(z)| + 2\frac{R^{n-1}}{B}|a_{n-1}|\left(\frac{(R^n-r^n)}{n(n+1)}-\frac{R-r}{n+1}\right) \\ &+ 2\frac{R^{n-1}}{B}|a_{n-2}|\left[\frac{R^n-r^n}{n(n-1)}-\frac{R^{n-2}-r^{n-2}}{(n-2)(n-3)}+\frac{2(R-r)}{(n-1)(n-3)}\right],\end{aligned} \tag{3.4}$$

where $B = \frac{R^n-r^n}{n}\sum_{j=1}^{n}\frac{1}{(R-|z_j|)}$. Equations (3.3) and (3.4) together gives the desired result. Thus the proof is complete.

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