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The Fundamental Solution of the Steady-State Two-Velocity Hydrodynamics Equation with Phase Equilibrium Pressure in the Dissipative Approximation

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Abstract—A fundamental solution is constructed for describing three-dimensional steady-state flows of viscous fluids of a two-velocity continuum with phase equilibrium pressure in the dissipative approximation.

Keywords: *two-velocity hydrodynamics, viscous fluid, fundamental solution, overdetermined system, friction coefficient*

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INTRODUCTION

The study of physical and technical processes in continuum mechanics begins with the construction of a mathematical model. Taking into account the presence of partial melts in the upper layers of the mantle has acquired an important role in the geophysical literature. The assumption about the formation of a partial melt in a phase transition of the first kind allowed V.N. Dorovsky to explain the localization of substantial masses of such a substance in space under dynamic conditions [1]. Here the effect of volumetric magma generation was neglected. The account of magma generation under conditions of shear deformation of mantle strata was made in [2]. In these papers, the continuous medium on the geological time scale is a viscous fluid-1 and, owing to its own viscosity or for other reasons, it reaches the necessary thermodynamic conditions for the phase transition to occur. Magma begins to accumulate along grain boundaries and intergranular nodes; this is fluid-2 with a viscosity inherent in melts known in geology. Such a melt is included in the process of joint heat and mass transfer and is filtered through the system that generated it. In other words, this theory represents the dynamics of heat and mass transfer of the mutual penetration of one less viscous fluid through a more viscous medium as a kind of filtration process, or, by analogy with the Navier–Stokes equation, it can be called a two-velocity system of Navier–Stokes equations or an equation of two-velocity hydrodynamics.

The study of flows of viscous compressible/incompressible fluids based on the solution of the complete system of equations of two-velocity hydrodynamics seems to be relevant. A very limited number of cases are known in the literature that allow analytical integration of the Navier–Stokes equations [3–5]. The aim of the present paper is to construct fundamental solutions of a time-invariant system of equations of two-velocity hydrodynamics with phase equilibrium pressure in the dissipative approximation determined by the phase viscosity coefficients and the intercomponent friction coefficient. These formulas can be useful for testing numerical methods for solving two-velocity hydrodynamics equations.

1. EQUATIONS OF TWO-VELOCITY HYDRODYNAMICS WITH ONE PRESSURE

In the papers [6, 7], a nonlinear two-velocity model of fluid motion through a deformable porous medium was constructed based on conservation laws, the invariance of equations with respect to Galileo transformations, and the thermodynamic consistency condition. The two-velocity two-fluid hydrodynamic theory with the condition of pressure equilibrium of subsystems was constructed in the paper [2]. The equations of motion of a two-velocity medium in the dissipative case with one pressure for the isothermal case have the form [2]

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \\ &\frac{\partial \tilde\rho}{\partial t} + \operatorname{div}(\tilde\rho \tilde{\mathbf{v}}) = 0, \end{aligned} \tag{1}$$

$$\bar\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v},\nabla)\mathbf{v}\right) = -\nabla p + \nu\Delta\mathbf{v} + (\nu/3+\mu)\nabla\operatorname{div}\mathbf{v} + \frac{\tilde\rho}{2}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2 - b\bar\rho\frac{\tilde\rho}{\rho}(\mathbf{v}-\tilde{\mathbf{v}}) + \bar\rho\mathbf{f}, \tag{2}$$

$$\bar\rho\left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\tilde{\mathbf{v}},\nabla)\tilde{\mathbf{v}}\right) = -\nabla p + \tilde\nu\,\Delta\tilde{\mathbf{v}} + (\tilde\nu/3+\tilde\mu)\nabla\operatorname{div}\tilde{\mathbf{v}} - \frac{\rho}{2}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2 + b\bar\rho\,(\mathbf{v}-\tilde{\mathbf{v}}) + \bar\rho\mathbf{f}, \tag{3}$$

where $\tilde{\mathbf{v}}$ and $\mathbf{v}$ are the velocity vectors of the subsystems that make up the two-velocity continuum with the corresponding partial densities $\tilde\rho$ and ρ; ν (μ) and $\tilde\nu$ $(\tilde\mu)$ are the corresponding shear (bulk) viscosities; $b = \chi\,\tilde\rho$; χ is the interphase friction coefficient; $\bar\rho = \tilde\rho + \rho$ is the total density of the two-velocity continuum; $p = p(\bar\rho, (\tilde{\mathbf{v}}-\mathbf{v})^2)$ is the equation of state for the two-velocity continuum; and $\mathbf{f}$ is the vector of mass force per unit mass.

Let us rewrite Eqs. (2) and (3) in the equivalent form

$$\begin{aligned} \bar\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(v^2) - \mathbf{v}\times\operatorname{rot}\mathbf{v}\right) &= -\nabla p + \nu\,\Delta\mathbf{v} \\ &\quad + (\nu/3+\mu)\nabla\operatorname{div}\mathbf{v} + \frac{\tilde\rho}{2}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2 - b\bar\rho\frac{\tilde\rho}{\rho}(\mathbf{v}-\tilde{\mathbf{v}}) + \bar\rho\mathbf{f}, \end{aligned} \tag{4}$$

$$\begin{aligned} \bar\rho\left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \frac{1}{2}\nabla(\tilde v^2) - \tilde{\mathbf{v}}\times\operatorname{rot}\tilde{\mathbf{v}}\right) &= -\nabla p + \tilde\nu\,\Delta\tilde{\mathbf{v}} \\ &\quad + (\tilde\nu/3+\tilde\mu)\nabla\operatorname{div}\tilde{\mathbf{v}} - \frac{\rho}{2}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2 + b\bar\rho\,(\mathbf{v}-\tilde{\mathbf{v}}) + \bar\rho\mathbf{f}, \end{aligned} \tag{5}$$

From these equations, one can derive other equations that determine the time evolution of vortices. To this end, we apply the operator rot to both sides of Eqs. (4), (5). As a result, we obtain

$$\begin{aligned} \frac{\partial \mathbf{\Omega}}{\partial t} - \operatorname{rot}(\mathbf{v}\times\mathbf{\Omega}) &= -\operatorname{rot}\left(\frac{\nabla p}{\bar\rho}\right) + \nu\Delta\mathbf{\Omega} \\ &\quad + \operatorname{rot}\left(\frac{\nu/3+\mu}{\bar\rho}\nabla\operatorname{div}\mathbf{v}\right) + \operatorname{rot}\left(\frac{\tilde\rho}{2\,\bar\rho}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2\right) - b\bar\rho\frac{\tilde\rho}{\rho}(\mathbf{\Omega}-\widetilde{\mathbf{\Omega}}) + \operatorname{rot}\mathbf{f}, \\ \frac{\partial \widetilde{\mathbf{\Omega}}}{\partial t} - \operatorname{rot}(\tilde{\mathbf{v}}\times\widetilde{\mathbf{\Omega}}) &= -\operatorname{rot}\left(\frac{\nabla p}{\bar\rho}\right) + \tilde\nu\Delta\widetilde{\mathbf{\Omega}} \\ &\quad + \operatorname{rot}\left(\frac{\tilde\nu/3+\tilde\mu}{\bar\rho}\nabla\operatorname{div}\tilde{\mathbf{v}}\right) - \operatorname{rot}\left(\frac{\rho}{2\bar\rho}\nabla(\tilde{\mathbf{v}}-\mathbf{v})^2\right) + b\bar\rho\,(\mathbf{\Omega}-\widetilde{\mathbf{\Omega}}) + \operatorname{rot}\mathbf{f}. \end{aligned}$$

2. LINEAR SYSTEM OF EQUATIONS FOR TWO-VELOCITY HYDRODYNAMICS OF A COMPRESSIBLE MEDIUM

With no mass forces $\mathbf{f} = 0$, system (1)–(3) has the solution $\mathbf{v} = 0$, $\tilde{\mathbf{v}} = 0$, $\rho = \rho^0$, $\tilde{\rho} = \tilde{\rho}^0$ for a mixture of fluids at rest with uniform pressure $p = p^0$, partial densities ρ^0 and $\tilde{\rho}^0$, and temperature T (see [8]).

Let us linearize Eqs. (2), (3) about the hydrodynamic background $\mathbf{v} = 0$, $\tilde{\mathbf{v}} = 0$, $\rho = \rho^0$, $\tilde{\rho} = \tilde{\rho}^0$, $p = p^0$; i.e.,

$$\mathbf{v} = \mathbf{v}^1, \quad \tilde{\mathbf{v}} = \tilde{\mathbf{v}}^1, \quad \rho = \rho^0 + \rho^1, \quad \tilde{\rho} = \tilde{\rho}^0 + \tilde{\rho}^1, \quad p = p^0 + p^1.$$

We substitute these expressions into (1)–(3) and, for brevity, write $\mathbf{v}$, $\tilde{\mathbf{v}}$, ρ, and $\tilde{\rho}$ rather than $\mathbf{v}^1$, $\tilde{\mathbf{v}}^1$, ρ^1, and $\tilde{\rho}^1$ in what follows. As a result, we obtain

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho^0 \operatorname{div} \mathbf{v} = 0, \\ &\frac{\partial \tilde{\rho}}{\partial t} + \tilde{\rho}^0 \operatorname{div} \tilde{\mathbf{v}} = 0, \end{aligned} \tag{6}$$

$$\bar{\rho}^0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \nu \Delta \mathbf{v} + (\nu/3 + \mu)\nabla \operatorname{div} \mathbf{v} - b\bar{\rho}^0 \frac{\tilde{\rho}^0}{\rho^0} (\mathbf{v} - \tilde{\mathbf{v}}) + \bar{\rho}^0 \mathbf{f}, \tag{7}$$

$$\bar{\rho}^0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\nabla p + \tilde{\nu} \Delta \tilde{\mathbf{v}} + (\tilde{\nu}/3 + \tilde{\mu})\nabla \operatorname{div} \tilde{\mathbf{v}} + b\bar{\rho}^0 (\mathbf{v} - \tilde{\mathbf{v}}) + \bar{\rho}^0 \mathbf{f}. \tag{8}$$

3. LINEAR STEADY-STATE SYSTEM OF EQUATIONS OF TWO-VELOCITY HYDRODYNAMICS IN THE CASE OF CONSTANT VOLUME SATURATION VALUES

In the steady-state case of $(\dot{\rho}, \dot{\tilde{\rho}}, \dot{\mathbf{v}}, \dot{\tilde{\mathbf{v}}}) = \mathbf{0}$, system (6)–(8) has the form [9, 10]

$$\begin{aligned} &\operatorname{div} \mathbf{v} = 0, \\ &\operatorname{div} \tilde{\mathbf{v}} = 0, \end{aligned} \tag{9}$$

$$\nu \Delta \mathbf{v} = \nabla p + b\bar{\rho}^0 \frac{\tilde{\rho}^0}{\rho^0} (\mathbf{v} - \tilde{\mathbf{v}}) - \bar{\rho}^0 \mathbf{f}, \tag{10}$$

$$\tilde{\nu} \Delta \tilde{\mathbf{v}} = \nabla p - b\bar{\rho}^0 (\mathbf{v} - \tilde{\mathbf{v}}) - \bar{\rho}^0 \mathbf{f}. \tag{11}$$

This system is an overdetermined system of partial differential equations. The papers [11–13] deal with the study of boundary value problems for such overdetermined systems of partial differential equations. In [12], the existence of a generalized solution of system (9)–(11) in a bounded domain was proved in the dissipative-free approximation with inhomogeneous boundary conditions.

The components of the matrices $G_{ij}(\mathbf{r}, \mathbf{r}')$ and $\widetilde{G}_{ij}(\mathbf{r}, \mathbf{r}')$, $i, j = 1, 2, 3$, and a vector $P(\mathbf{r}, \mathbf{r}')$ with components P_j, $j = 1, 2, 3$, of the fundamental solution of system (9)–(11) are found from the following system of differential equations:

$$\begin{aligned} &\partial_m G_{mj}(\mathbf{r}, \mathbf{r}') = 0, \\ &\partial_m \widetilde{G}_{mj}(\mathbf{r}, \mathbf{r}') = 0, \end{aligned} \tag{12}$$

$$\nu \Delta G_{ij}(\mathbf{r}, \mathbf{r}') - \partial_i P_j(\mathbf{r}, \mathbf{r}') - b\bar{\rho}^0 \frac{\tilde{\rho}^0}{\rho^0} (G_{ij}(\mathbf{r}, \mathbf{r}') - \widetilde{G}_{ij}(\mathbf{r}, \mathbf{r}')) = \delta_{ij} \delta(\mathbf{r} - \mathbf{r}'), \tag{13}$$

$$\tilde{\nu} \Delta \widetilde{G}_{ij}(\mathbf{r}, \mathbf{r}') - \partial_i P_j(\mathbf{r}, \mathbf{r}') + b\bar{\rho}^0 (G_{ij}(\mathbf{r}, \mathbf{r}') - \widetilde{G}_{ij}(\mathbf{r}, \mathbf{r}')) = \delta_{ij} \delta(\mathbf{r} - \mathbf{r}'), \tag{14}$$

where $\mathbf{r} = (x_1, x_2, x_3) \in R^3$, δ_{ij} is the Kronecker delta, $\delta(\mathbf{r})$ is the Dirac delta function, and $\partial_i = \frac{\partial}{\partial x_i}$. Summation from 1 to 3 over repeated indices is assumed.

Let $(\hat{\mathbf{v}}(\alpha),\ \hat{\tilde{\mathbf{v}}}(\alpha),\ \hat{p}(\alpha))$ be the Fourier transform of $(\mathbf{v}(\mathbf{r}),\ \tilde{\mathbf{v}}(\mathbf{r}),\ p(\mathbf{r}))$; namely,

$$\big(\hat{\mathbf{v}}(\alpha),\hat{\tilde{\mathbf{v}}}(\alpha),\hat{\mathbf{p}}(\alpha)\big)=\frac{\mathbf{1}}{(\mathbf{2}\pi)^{\mathbf{3/2}}}\int\limits_{\mathbb{R}^3}\big(\mathbf{v}(\mathbf{r}),\tilde{\mathbf{v}}(\mathbf{r}),\mathbf{p}(\mathbf{r})\big)\mathbf{e}^{-\mathbf{i}\alpha\mathbf{r}}\,\mathbf{dr}.$$

We multiply (12)–(14) by $\frac{1}{(2\pi)^{3/2}}e^{-i\alpha(\mathbf{r}-\mathbf{r}')}$ and integrate over $\mathbf{r}\in\mathbb{R}^3$ to obtain

$$\alpha_m\widehat{G}_{mj}=0,\quad \alpha_m\widehat{\widetilde{G}}_{mj}=0,\quad j=1,2,3, \tag{15}$$

$$-\nu\alpha^2\widehat{G}_{mj}+i\alpha_m\widehat{P}_j-b\bar{\rho}^0\,\frac{\tilde{\rho}^0}{\rho^0}(\widehat{G}_{mj}-\widehat{\widetilde{G}}_{mj})=\frac{1}{(2\pi)^{3/2}}\delta_{mj},\quad m,j=1,2,3, \tag{16}$$

$$-\tilde{\nu}\alpha^2\widehat{\widetilde{G}}_{mj}+i\alpha_m\widehat{P}_j+b\bar{\rho}^0(\widehat{G}_{mj}-\widehat{\widetilde{G}}_{mj})=\frac{1}{(2\pi)^{3/2}}\delta_{mj},\quad m,j=1,2,3. \tag{17}$$

Based on this, we uniquely determine the functions $\widehat{G}_{mj}$, $\widehat{\widetilde{G}}_{mj}$, and $\widehat{P}_j$,

$$\widehat{G}_{mj}=\frac{\tilde{\nu}\tilde{\rho}^0}{(2\pi)^{3/2}(\nu\rho^0+\tilde{\nu}\tilde{\rho}^0)}\times\left(\frac{1/\tilde{\nu}-1/\nu}{\alpha^2+A^2}\delta_{mj}+\frac{1/\tilde{\nu}-1/\nu}{A^2}\alpha_m\alpha_j\left[\frac{1}{\alpha^2+A^2}-\frac{1}{\alpha^2}\right]-\frac{\bar{\rho}^0/\tilde{\rho}^0}{\tilde{\nu}\alpha^2}\left[\delta_{mj}-\frac{\alpha_m\alpha_j}{\alpha^2}\right]\right),$$

$$\widehat{\widetilde{G}}_{mj}=-\frac{\nu\rho^0}{(2\pi)^{3/2}(\nu\rho^0+\tilde{\nu}\tilde{\rho}^0)}\times\left(\frac{1/\tilde{\nu}-1/\nu}{\alpha^2+A^2}\delta_{mj}+\frac{1/\tilde{\nu}-1/\nu}{A^2}\alpha_m\alpha_j\left[\frac{1}{\alpha^2+A^2}-\frac{1}{\alpha^2}\right]+\frac{\bar{\rho}^0/\rho^0}{\nu\alpha^2}\left[\delta_{mj}-\frac{\alpha_m\alpha_j}{\alpha^2}\right]\right),$$

$$\widehat{P}_j=-\frac{i\alpha_j}{(2\pi)^{3/2}\alpha^2},$$

where $A=\sqrt{b\bar{\rho}^0\left(\frac{\tilde{\rho}^0}{\rho^0}\frac{1}{\nu}+\frac{1}{\tilde{\nu}}\right)}$.

The inverse Fourier transform, formula (3.723) in [14], and the formula [15]

$$\left(\delta(\mathbf{r}-\mathbf{r}'),\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|},\frac{|\mathbf{r}-\mathbf{r}'|}{8\pi}\right)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}(1,\alpha^{-2},\alpha^{-4})e^{i\alpha(\mathbf{r}-\mathbf{r}')}d\alpha$$

give

$$G_{mj}(\mathbf{r},\mathbf{r}')=\frac{\tilde{\nu}\tilde{\rho}^0}{\nu\rho^0+\tilde{\nu}\tilde{\rho}^0}\times\left(\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}|\mathbf{r}-\mathbf{r}'|}\delta_{mj}e^{-A|\mathbf{r}-\mathbf{r}'|}-\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}A^2}\partial_m\partial_j\frac{e^{-A|\mathbf{r}-\mathbf{r}'|}-1}{|\mathbf{r}-\mathbf{r}'|}-\frac{\bar{\rho}^0/\tilde{\rho}^0}{\tilde{\nu}}\left[\frac{\delta_{mj}}{4\pi|\mathbf{r}-\mathbf{r}'|}-\partial_m\partial_j\frac{|\mathbf{r}-\mathbf{r}'|}{8\pi}\right]\right),$$

$$\widetilde{G}_{mj}(\mathbf{r},\mathbf{r}')=-\frac{\nu\rho^0}{\nu\rho^0+\tilde{\nu}\tilde{\rho}^0}\times\left(\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}|\mathbf{r}-\mathbf{r}'|}\delta_{mj}e^{-A|\mathbf{r}-\mathbf{r}'|}-\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}A^2}\partial_m\partial_j\frac{e^{-A|\mathbf{r}-\mathbf{r}'|}-1}{|\mathbf{r}-\mathbf{r}'|}+\frac{\bar{\rho}^0/\rho^0}{\nu}\left[\frac{\delta_{mj}}{4\pi|\mathbf{r}-\mathbf{r}'|}-\partial_m\partial_j\frac{|\mathbf{r}-\mathbf{r}'|}{8\pi}\right]\right),$$

$$P_m(\mathbf{r},\mathbf{r}')=\partial_m\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

From these expressions, we obtain Green's formulas for problem (9)–(11) in the form

$$G_{mj}(\mathbf{r},\mathbf{r}') = \frac{\tilde{\nu}\tilde{\rho}^0}{\nu\rho^0+\tilde{\nu}\tilde{\rho}^0}\Bigg(\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}|\mathbf{r}-\mathbf{r}'|}\delta_{mj}e^{-A|\mathbf{r}-\mathbf{r}'|} - \frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}A^2}\partial_m\partial_j\frac{e^{-A|\mathbf{r}-\mathbf{r}'|}-1}{|\mathbf{r}-\mathbf{r}'|} - \frac{\bar{\rho}^0/\tilde{\rho}^0}{8\pi\tilde{\nu}}\bigg[\frac{\delta_{mj}}{|\mathbf{r}-\mathbf{r}'|}+\frac{(x_m-x'_m)(x_j-x'_j)}{|\mathbf{r}-\mathbf{r}'|^3}\bigg]\Bigg), \tag{18}$$

$$\widetilde{G}_{mj}(\mathbf{r},\mathbf{r}') = -\frac{\nu\rho^0}{\nu\rho^0+\tilde{\nu}\tilde{\rho}^0}\Bigg(\frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}|\mathbf{r}-\mathbf{r}'|}\delta_{mj}e^{-A|\mathbf{r}-\mathbf{r}'|} - \frac{\nu-\tilde{\nu}}{4\pi\nu\tilde{\nu}A^2}\partial_m\partial_j\frac{e^{-A|\mathbf{r}-\mathbf{r}'|}-1}{|\mathbf{r}-\mathbf{r}'|} + \frac{\bar{\rho}^0/\rho^0}{8\pi\nu}\bigg[\frac{\delta_{mj}}{|\mathbf{r}-\mathbf{r}'|}+\frac{(x_m-x'_m)(x_j-x'_j)}{|\mathbf{r}-\mathbf{r}'|^3}\bigg]\Bigg), \tag{19}$$

$$P_m(\mathbf{r},\mathbf{r}') = -\frac{x_m-x'_m}{4\pi|\mathbf{r}-\mathbf{r}'|^2}. \tag{20}$$

It can be seen from these formulas and system (12)–(14) that the functions $G_{mj}(\mathbf{r},\mathbf{r}')$, $\widetilde{G}_{mj}(\mathbf{r},\mathbf{r}')$, and $P_m(\mathbf{r},\mathbf{r}')$ satisfy the adjoint system

$$\frac{\partial G_{mj}(\mathbf{r},\mathbf{r}')}{\partial x'_m} = 0, \qquad \frac{\partial \widetilde{G}_{mj}(\mathbf{r},\mathbf{r}')}{\partial x'_m} = 0, \tag{21}$$

$$\nu\Delta_{\mathbf{r}'}G_{ij}(\mathbf{r},\mathbf{r}') + \frac{\partial P_j(\mathbf{r},\mathbf{r}')}{\partial x'_i} - b\bar{\rho}^0\,\frac{\tilde{\rho}^0}{\rho^0}\big(G_{ij}(\mathbf{r},\mathbf{r}') - \widetilde{G}_{ij}(\mathbf{r},\mathbf{r}')\big) = \delta_{ij}\delta(\mathbf{r}-\mathbf{r}'), \tag{22}$$

$$\tilde{\nu}\Delta_{\mathbf{r}'}\widetilde{G}_{ij}(\mathbf{r},\mathbf{r}') + \frac{\partial P_j(\mathbf{r},\mathbf{r}')}{\partial x'_i} + b\bar{\rho}^0\big(G_{ij}(\mathbf{r},\mathbf{r}') - \widetilde{G}_{ij}(\mathbf{r},\mathbf{r}')\big) = \delta_{ij}\delta(\mathbf{r}-\mathbf{r}') \tag{23}$$

with respect to the argument $\mathbf{r}'$. The functions $G_{ij}(\mathbf{r},\mathbf{r}')$, $\widetilde{G}_{ij}(\mathbf{r},\mathbf{r}')$, and $P_i(\mathbf{r},\mathbf{r}')$ permit one to construct the volume potentials

$$v_i(\mathbf{r}) = -\bar{\rho}^0\int dr'\,G_{ij}(\mathbf{r},\mathbf{r}')f_j(\mathbf{r}'),$$
$$\tilde{v}_i(\mathbf{r}) = -\bar{\rho}^0\int dr'\,\widetilde{G}_{ij}(\mathbf{r},\mathbf{r}')f_j(\mathbf{r}'),$$
$$p(\mathbf{r},\omega) = \bar{\rho}^0\int dr'\,P_i(\mathbf{r},\mathbf{r}')f_i(\mathbf{r}'),$$

which, by virtue of system (12)–(14), satisfy the steady-state inhomogeneous system (9)–(11) of two-velocity hydrodynamics with one pressure. The nature of the singularities of the kernels $G_{ij}(\mathbf{r},\mathbf{r}')$, $\widetilde{G}_{ij}(\mathbf{r},\mathbf{r}')$, and $P_i(\mathbf{r},\mathbf{r}')$ is the same as for the singular solution $\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ of the Laplace equation and its first derivatives, respectively. Based on this, it can be seen how the solutions of system (9)–(11) depend on the physical densities, the interphase friction coefficient, and also the volume saturation values of the substances forming the two-phase continuum.

CONCLUSIONS

Thus, a fundamental solution has been constructed to describe three-dimensional steady flows of viscous fluids of a two-velocity continuum in the dissipative approximation with phase equilibrium pressure. The influence of the physical phase densities, the volume saturation values of the substances, the interphase friction coefficient, and the viscosity of the constituent two-phase continuum on the flow velocities and pressure is shown.

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