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On the Qualitative Analysis of the Equations of Motion of a Rigid Body in a Magnetic Field

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Abstract—In the problem on the motion of a rigid body with a fixed point under the influence of a magnetic field generated by the Barnett–London effect as well as potential forces, the particular cases of existence of additional quadratic integrals are presented and the qualitative analysis of the equations of motion of the body is carried out in one of these cases.

Keywords: *rigid body, qualitative analysis, computer algebra*

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INTRODUCTION

The problem on the rotation of a rigid body about a fixed point under the action of forces of various nature (and also in the absence of forces) has a long history, but the interest in this problem still persists. An absolutely rigid body is used as a model for describing the motion of many complex technical devices, including spacecraft, industrial robots, rockets, etc. In the present paper, we consider the problem on the rotation of a rigid body with a fixed point in a uniform magnetic field taking into account the London–Barnett effect and potential forces. It is known that a "neutral" ferromagnet becomes magnetized along the rotation axis during rotation (the Barnett effect [1]). A similar phenomenon also takes place during the rotation of a superconducting solid body (the London effect [2]). The magnetic moment $\mathbf{B}$ is related to the angular velocity $\boldsymbol{\omega}$ by the formula $\mathbf{B} = B\boldsymbol{\omega}$, where B is a symmetric linear operator.

The motion of the body is described by Euler–Poisson equations of the form

$$A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} + B\boldsymbol{\omega} \times \boldsymbol{\gamma} + \boldsymbol{\gamma} \times (C\boldsymbol{\gamma} - \mathbf{s}), \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{1}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is a unit vector characterizing the direction of gravity, $\mathbf{s} = (s_1, s_2, s_3)$ is the vector of the center of mass of the body, and A, B, and C are symmetric 3×3 matrices: A is the body tensor of inertia relative to the fixed point, B is a matrix characterizing the body magnetic moment, and C is a matrix characterizing the effect of potential forces on the body.

For $C_i = \nu A_i$, $i = 1, 2, 3$, where ν is the gravitational constant, the differential equations (1) describe the motion of a body in a magnetic and a central Newtonian field.

Equations (1) admit two common first integrals

$$V_1 = A\boldsymbol{\omega} \cdot \boldsymbol{\gamma} = \varkappa, \quad V_2 = \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = 1 \tag{2}$$

and are nonintegrable in the general case.

There are quite a few papers studying the influence of the Barnett–London effect on the motion of a body in various aspects. Similar problems arise in many applications, for example, in cosmodynamics [3] and when designing devices using noncontact suspension [4]. The analysis of Eqs. (1)

from the viewpoint of their integrability and the search for particular solutions is carried out, for example, in [5–8]. A linear invariant relation of the Hess type [9] for Eqs. (1) was found in [5]. In [6, 7], the cases of their integrability are indicated where A and B are diagonal matrices and there are no potential forces. It was shown [7] that for $B = \lambda E$ ($\lambda = \text{const}$) Eqs. (1) are reduced to the Kirchhoff equations describing the motion of a rigid body in an ideal fluid.

In the present paper, it is assumed that

$$A = \operatorname{diag}(A_1, A_2, A_3), \quad C = \operatorname{diag}(C_1, C_2, C_3)$$

in Eqs. (1). A qualitative analysis of these equations is carried out in special cases where there exist additional first integrals. Linear invariant relations for Eqs. (1) were obtained in [10] using the method of indeterminate coefficients in conjunction with the method of Gröbner bases [11]. In the same way, the following quadratic integrals of the equations in question are found under certain restrictions on the problem parameters:

1. For $A_1 = A_2$, $B_{13} = B_{23} = 0$, $B_{33} = B_{11} + B_{22}$, $C_1 = C_3 = C_2$, and $s_1 = s_2 = s_3 = 0$, one has the integral

$$\begin{aligned} K_1 = \omega_1^2 + \omega_2^2 + \frac{A_3}{A_2^2}(2A_2 - A_3)\omega_3^2 - \frac{1}{A_2^2}\big(2A_2(B_{11} - B_{22})\omega_1\gamma_1 + 2A_2B_{12}\big(\omega_1\gamma_2 + \omega_2\gamma_1\big) \\ + 2A_3B_{11}\omega_3\gamma_3 - \big(B_{22}^2 - B_{11}^2\big)\gamma_1^2 + 2B_{12}(B_{11} + B_{22})\gamma_1\gamma_2 + (B_{11}^2 + B_{12}^2)\gamma_3^2\big). \end{aligned} \tag{3}$$

2. For $A_1 = A_3$, $B_{12} = B_{23} = 0$, $B_{22} = B_{11} + B_{33}$, $C_1 = C_2 = C_3$, and $s_1 = s_2 = s_3 = 0$, one has the integral

$$\begin{aligned} K_2 = \omega_1^2 - \frac{A_2}{A_3^2}(A_2 - 2A_3)\omega_2^2 + \omega_3^2 + \frac{1}{A_3^2}\Big(2A_3\big(B_{33}\omega_1\gamma_1 + B_{11}\omega_3\gamma_3 - B_{13}(\omega_1\gamma_3 + \omega_3\gamma_1)\big) \\ + \big(B_{33}^2 - B_{11}^2\big)\gamma_1^2 - \big(B_{11}^2 + B_{13}^2\big)\gamma_2^2 - 2B_{13}(B_{11} + B_{33})\gamma_1\gamma_3\Big). \end{aligned}$$

3. For $A_2 = A_3$, $B_{12} = B_{13} = 0$, $B_{33} = B_{11} - B_{22}$, $C_1 = C_2 = C_3$, and $s_1 = s_2 = s_3 = 0$, one has the integral

$$\begin{aligned} K_3 = \omega_1^2 - \frac{A_3^2\big(\omega_2^2 + \omega_3^2\big)}{A_1(A_1 - 2A_3)} + \frac{1}{A_1(A_1 - 2A_3)}\big(2A_1B_{22}\omega_1\gamma_1 - 2A_3(B_{11} - 2B_{22})\omega_2\gamma_2 \\ + 2A_3B_{23}(\omega_2\gamma_3 + \omega_3\gamma_2) + \big(B_{22}^2 + B_{23}^2\big)\gamma_1^2 - B_{11}(B_{11} - 2B_{22})\gamma_2^2 + 2B_{11}B_{23}\gamma_2\gamma_3\big). \end{aligned}$$

As can be seen, the integrals exist under the condition of dynamic symmetry of the body, and the coordinates of the center of mass of the body coincide with the coordinates of the fixed point.

Further, using generalizations of the Routh–Lyapunov method [12], a qualitative analysis of Eqs. (1) is carried out in the case where these equations admit one of the above quadratic integrals.

1. STATEMENT OF THE PROBLEM

Consider the differential equations (1) for the case in which they admit the integral K_1 in (3). The equations are written as follows:

$$\begin{aligned} A_2\dot\omega_1 &= (B_{12}\omega_1 + B_{22}\omega_2)\gamma_3 - \big((B_{11} + B_{22})\gamma_2 + (A_3 - A_2)\omega_2\big)\omega_3, \\ A_2\dot\omega_2 &= \big((B_{11} + B_{22})\gamma_1 + (A_3 - A_2)\omega_1\big)\omega_3 - (B_{11}\omega_1 + B_{12}\omega_2)\gamma_3, \\ A_3\dot\omega_3 &= (B_{11}\omega_1 + B_{12}\omega_2)\,\gamma_2 - (B_{12}\omega_1 + B_{22}\omega_2)\gamma_1, \\ \dot\gamma_1 &= \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot\gamma_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot\gamma_3 = \omega_2\gamma_1 - \omega_1\gamma_2. \end{aligned} \tag{4}$$

The integrals (2) become

$$\widetilde{V}_1 = A_2(\omega_1\gamma_1 + \omega_2\gamma_2) + A_3\omega_3\gamma_3 = \varkappa, \quad V_2 = \sum_{i=1}^{2} \gamma_i^2 = 1. \tag{5}$$

Let us pose the problem of qualitative analysis of this system. Based on the necessary conditions for the extremum of the first integrals of the problem (or some combination of them), special sets of differential equations will be found and their stability in the sense of Lyapunov will be studied. With the chosen method of analysis, the special sets are defined as stationary sets [12], i.e., sets of any finite dimension on which necessary conditions for the extremum of elements of the algebra of first integrals of the problem are satisfied. Stationary sets of dimension zero are traditionally called stationary solutions, and stationary sets of nonzero dimension are called stationary invariant manifolds.

2. ISOLATION OF STATIONARY SOLUTIONS AND INVARIANT MANIFOLDS

In accordance with the above-indicated method, we take a linear combination

$$2\Omega = \lambda_0 K_1 - 2\lambda_1 \widetilde{V}_1 - \lambda_2 V_2 \tag{6}$$

of the first integrals and write necessary conditions for the extremum of Ω with respect to the variables ω_i and γ_i,

$$\begin{aligned}
\frac{\partial\Omega}{\partial\omega_1} &= \lambda_0(\omega_1 - A_2^{-1}((B_{11} - B_{22})\gamma_1 + B_{12}\gamma_2)) - \lambda_1 A_2\gamma_1 = 0,\\
\frac{\partial\Omega}{\partial\omega_2} &= \lambda_0\big(\omega_2 - A_2^{-1}B_{12}\gamma_1\big) - \lambda_1 A_2\gamma_2 = 0,\\
\frac{\partial\Omega}{\partial\omega_3} &= \lambda_0 A_2^{-2}A_3\big((2A_2 - A_3)\omega_3 - B_{11}\gamma_3\big) - \lambda_1 A_3\gamma_3 = 0,\\
\frac{\partial\Omega}{\partial\gamma_1} &= -\lambda_0 A_2^{-2}\big(A_2(B_{11} - B_{22})\omega_1 + A_2B_{12}\omega_2 + (B_{11}^2 - B_{22}^2)\gamma_1\\
&\quad + B_{12}(B_{11} + B_{22})\gamma_2\big) - \lambda_1 A_2\omega_1 - \lambda_2\gamma_1 = 0,\\
\frac{\partial\Omega}{\partial\gamma_2} &= -\lambda_0 A_2^{-2}B_{12}\big(A_2\omega_1 + (B_{11} + B_{22})\gamma_1\big) - \lambda_1 A_2\omega_2 - \lambda_2\gamma_2 = 0,\\
\frac{\partial\Omega}{\partial\gamma_3} &= -\lambda_0 A_2^{-2}\big(A_3B_{11}\omega_3 + (B_{11}^2 + B_{12}^2)\gamma_3\big) - \lambda_1 A_3\omega_3 - \lambda_2\gamma_3 = 0.
\end{aligned} \tag{7}$$

Here the λ_i are the parameters of the family of integrals Ω.

In the case of dependent equations, the solutions of system (7) permit one to determine the invariant manifolds of the differential equations (4) corresponding to the family of first integrals Ω. To find the solutions, we use the system of computer algebra *Wolfram Mathematica*.

As can be seen, Eqs. (7) can be separated in the variables. Let us construct a lexicographic Gröbner basis with respect to $\lambda_1 > \lambda_2 > \gamma_1 > \omega_1$ for the left-hand sides depending on ω_1, ω_2, and γ_1, γ_2. As a result, we obtain a system of equations splitting into two subsystems. Below we give the lexicographic bases of these subsystems:

$$\begin{aligned}
\big(B_{12}^2 - B_{11}B_{22}\big)\gamma_2 - A_2(B_{12}\omega_1 + B_{22}\omega_2) &= 0,\\
-B_{12}\gamma_1 + B_{11}\gamma_2 + A_2\omega_2 &= 0,\\
\lambda_0\big(B_{11}^2 + B_{12}^2\big) + \lambda_2 A_2^2 &= 0,\\
-\lambda_0 B_{11} - \lambda_1 A_2^2 &= 0;
\end{aligned} \tag{8}$$

$$\begin{aligned}-B_{12}\omega_1^2 + ((B_{11}-B_{22})\omega_1 + B_{12}\omega_2)\omega_2 &= 0,\\ -\omega_1\gamma_2 + \omega_2\gamma_1 &= 0,\\ -\big(A_2^2\omega_2^2 + B_{12}(B_{11}+B_{22})\omega_1\gamma_2^2\big)\lambda_0 - A_2^2\omega_2\gamma_2^2\lambda_2 &= 0,\\ \big(B_{12}\omega_1\gamma_2 - A_2\omega_2^2\big)\lambda_0 + A_2^2\omega_2\gamma_2\lambda_1 &= 0.\end{aligned} \tag{9}$$

From the last two equations in (8), we find

$$\lambda_1 = -\frac{B_{11}}{A_2^2}\lambda_0, \quad \lambda_2 = -\frac{B_{11}^2 + B_{12}^2}{A_2^2}\lambda_0 \tag{10}$$

and substitute them into the remaining equations (7) (depending on ω_3 and γ_3). These equations are reduced to one equation $\omega_3 = 0$. It can be verified by a straightforward calculation according to the definition of invariant manifold that the equations

$$\begin{aligned}\big(B_{12}^2 - B_{11}B_{22}\big)\gamma_2 - A_2(B_{12}\omega_1 + B_{22}\omega_2) &= 0,\\ A_2\omega_2 - B_{12}\gamma_1 + B_{11}\gamma_2 &= 0,\\ \omega_3 &= 0\end{aligned} \tag{11}$$

define an invariant manifold of codimension 3 for the equations of motion (4).

The differential equations on this invariant manifold are written in the form

$$\begin{aligned}&\dot\omega_1 = \frac{B_{12}^2 - B_{11}B_{22}}{A_2^2}\gamma_2\gamma_3,\\ &\dot\gamma_2 = \omega_1\gamma_3,\\ &\dot\gamma_3 = -\omega_1\gamma_2 + (A_2\omega_1 - B_{12}\gamma_2)\left(\frac{B_{12}}{B_{22}^2}\omega_1 + \frac{B_{11}B_{22} - B_{12}^2}{A_2B_{22}^2}\gamma_2\right)\end{aligned} \tag{12}$$

and describe pendulum-like oscillations of the body.

Let us substitute λ_1 and λ_2 given by (10) into (6). By a straightforward calculation it can also be verified that the integral

$$2\Omega_1 = K_1 + \frac{2B_{11}}{A_2^2}\widetilde{V}_1 + \frac{B_{11}^2 + B_{12}^2}{A_2^2}V_2 \tag{13}$$

takes a stationary value on the invariant manifold (11).

In a similar way, based on Eqs. (9), we obtain the equations

$$\begin{aligned}-B_{12}\omega_1^2 + \big((B_{11}-B_{22})\omega_1 + B_{12}\omega_2\big)\omega_2 &= 0,\\ -\omega_1\gamma_2 + \omega_2\gamma_1 &= 0,\\ \omega_3 &= 0,\\ \gamma_3 &= 0\end{aligned} \tag{14}$$

defining an invariant manifold of codimension 4.

The differential equations $\dot\omega_2 = 0$ and $\dot\gamma_2 = 0$ on this invariant manifold have the following family of solutions:

$$\omega_2 = \omega_2^0 = \mathrm{const},\quad \gamma_2 = \gamma_2^0 = \mathrm{const}.$$

Thus, from the geometrical viewpoint, the invariant manifold (14) in the space $\mathbb{R}^6$ is associated with a surface whose each point is a fixed point in the phase space.

Using the maps of some atlas on the invariant manifold (14), one can readily show that the integral

$$\Omega_2 = -\frac{1}{4A_2^2V_2}\Big(2\widetilde{V}_1^2 - 2z_1\widetilde{V}_1V_2 - V_2\big((B_{11}+B_{22})z_1V_2 + 2A_2^2K_1\big)\Big), \tag{15}$$

where $z_1 = B_{11} - B_{22} - D$ and $D = \sqrt{(B_{11}-B_{22})^2 + 4B_{12}^2}$, takes a stationary value on the invariant manifold (14) in the map

$$\begin{aligned} \omega_1 &= \frac{z_1\omega_2}{2B_{12}}, \qquad \omega_3 = 0, \\ \gamma_1 &= \frac{z_1\gamma_2}{2B_{12}}, \qquad \gamma_3 = 0, \end{aligned} \tag{16}$$

and the integral

$$\Omega_3 = -\frac{1}{4A_2^2V_2}\Big(2\widetilde{V}_1^2 - 2z_2\widetilde{V}_1V_2 - V_2\big((B_{11}+B_{22})z_2V_2 + 2A_2^2K_1\big)\Big), \tag{17}$$

in the map

$$\begin{aligned} \omega_1 &= \frac{z_2\omega_2}{2B_{12}}, \qquad \omega_3 = 0, \\ \gamma_1 &= \frac{z_2\gamma_2}{2B_{12}}, \qquad \gamma_3 = 0. \end{aligned} \tag{18}$$

Here $z_2 = B_{11} - B_{22} + D$.

Two more invariant manifolds different from the ones above and a condition on the parameters λ_1 and λ_2 under which the integral Ω (6) takes a stationary value on these invariant manifolds can be obtained by constructing a lexicographic basis for the polynomials of the entire system (7) for $\omega_3 > \omega_1 > \omega_2 > \gamma_1 > \lambda_2 > \lambda_1$. The equations of the invariant manifold are written in the form

$$\begin{aligned} 2B_{12}\gamma_1 - (B_{11} - B_{22} \pm D)\gamma_2 &= 0, \\ 2(A_2 - A_3)\omega_2 - (B_{11} + B_{22} \mp D)\gamma_2 &= 0, \\ 2(A_2 - A_3)B_{12}\omega_1 + \big(2B_{12}^2 - B_{22}(B_{11} - B_{22} \pm D)\big)\gamma_2 &= 0, \\ 2(A_2 - A_3)\omega_3 - (B_{11} + B_{22} \mp D)\gamma_3 &= 0. \end{aligned} \tag{19}$$

The differential equations on these invariant manifolds are similar to the equations on the invariant manifolds (14),

$$\begin{aligned} \dot\gamma_2 &= 0, \\ \dot\gamma_3 &= 0. \end{aligned}$$

The integral Ω in (6) takes a stationary value on the invariant manifold (19) for

$$\begin{aligned} \lambda_1 &= \frac{\lambda_0}{2A_2^2(A_2 - A_3)}\big(A_3(B_{11} - B_{22}) + 2A_2B_{22} \mp (2A_2 - A_3)D\big), \\ \lambda_2 &= -\frac{\lambda_0}{2A_2^2(A_2 - A_3)^2}\big(2A_2^2(B_{11}^2 + B_{12}^2) - (2A_2 - A_3)A_3(B_{11} + B_{22})(B_{11} - B_{22} \pm D)\big). \end{aligned}$$

Let us study the relationship between the manifolds. We find the intersection of the invariant manifolds (11) and (14). To this end, for the polynomials of the system obtained by combining Eqs. (11) and (14) we construct the Gröbner lexicographic basis with respect to $\gamma_1 > \gamma_3 > \omega_1 > \omega_2 > \omega_3$,

$$\begin{aligned} \omega_3 &= 0, \\ (B_{12}^2 - B_{11}B_{22})\gamma_2^2 - A_2(B_{11} + B_{22})\omega_2\gamma_2 - A_2^2\omega_2^2 &= 0, \\ (B_{11}B_{22} - B_{12}^2)\gamma_2 + A_2B_{12}\omega_1 + A_2B_{22}\omega_2 &= 0, \\ \gamma_3 &= 0, \\ B_{12}\gamma_1 - B_{11}\gamma_2 - A_2\omega_2 &= 0. \end{aligned} \tag{20}$$

Equations (20), together with the integral $V_2 = 1$, determine the following solutions of the differential equations (4):

$$\begin{aligned} &\gamma_1 = \mp\frac{\sqrt{2}B_{12}}{\sqrt{DD_1}}, && \gamma_2 = \pm\frac{D_1}{\sqrt{2D}}, && \gamma_3 = 0, \\ &\omega_1 = \pm\frac{D_1\big(2B_{12}^2 - B_{22}(B_{11} - B_{22} - D)\big)}{2\sqrt{2D}A_2B_{12}}, && \omega_2 = \mp\frac{D_1(B_{11} + B_{22} + D)}{2\sqrt{2D}A_2}, && \omega_3 = 0; \end{aligned} \tag{21}$$

$$\begin{aligned} &\gamma_1 = \pm\frac{\sqrt{2}B_{12}}{\sqrt{DD_2}}, && \gamma_2 = \pm\frac{D_2}{\sqrt{2D}}, && \gamma_3 = 0, \\ &\omega_1 = \pm\frac{D_2\big(2B_{12}^2 - B_{22}(B_{11} - B_{22} + D)\big)}{2\sqrt{2D}A_2B_{12}}, && \omega_2 = \mp\frac{(B_{11} + B_{22} - D)D_2}{2\sqrt{2D}A_2}, && \omega_3 = 0. \end{aligned} \tag{22}$$

Here and in the following, $D_1 = \sqrt{B_{11} - B_{22} + D}$ and $D_2 = \sqrt{B_{22} - B_{11} + D}$.

From the mechanical viewpoint, the solutions (21), (22) correspond to permanent rotations of a body around an axis located in the Oxy-plane (in the system of axes associated with the body) with the angular velocity $\omega^2 = \big(B_{11}^2 + 2B_{12}^2 \pm B_{11}D + B_{22}(B_{22} \pm D)\big)/(2A_2^2)$.

It can be verified by a straightforward calculation that the integral

$$2\Omega_4 = K_1 + \frac{1}{A_2^2}\big(2B_{11}\widetilde{V}_1 + (B_{11}^2 + B_{12}^2)V_2\big)$$

takes a stationary value on the solutions (21) and (22).

In a similar way, it can be shown that the intersection of the invariant manifold (14) with each of the two invariant manifolds (19) is nonempty. They have common points that also correspond to permanent rotations of the body.

3. ON THE STABILITY OF STATIONARY SOLUTIONS AND INVARIANT MANIFOLDS

Let us study the stability of the invariant manifold (11) using the integral Ω_1 (13) to obtain sufficient conditions.

We introduce the deviations

$$\begin{aligned} y_1 &= \omega_1 + \frac{1}{A_2B_{12}}\big(A_2B_{22}\omega_2 + (B_{11}B_{22} - B_{12}^2)\gamma_2\big), \\ y_2 &= \gamma_1 - \frac{1}{B_{12}}(A_2\omega_2 + B_{11}\gamma_2), \\ y_3 &= \omega_3 \end{aligned}$$

and write the variation of the integral Ω_1 in a neighborhood of the solution to be studied,

$$2\Delta\Omega_1 = \frac{1}{A_2^2}\big(B_{12}^2y_2^2 + (A_2y_1 + B_{22}y_2)^2 + (2A_2 - A_3)A_3y_3^2\big).$$

Consider the restriction of $\Delta\Omega_1$ to the set defined by the first variation of the integral $\widetilde{V}_1$,

$$\delta\widetilde{V}_1 = \frac{2}{B_{12}}(A_2\omega_2 + B_{11}\gamma_2)y_2 = 0.$$

On this set, $\Delta\Omega_1$ acquires the form

$$2\Delta\widetilde{\Omega}_1 = y_1^2 + \frac{(2A_2 - A_3)A_3}{A_2^2}y_3^2.$$

The condition

$$2A_2 > A_3 \tag{23}$$

of positive definiteness of the quadratic form $\Delta\widetilde{\Omega}_1$ is sufficient for the stability of the invariant manifold studied.

Now let us analyze the stability of the invariant manifold (14) using the integral Ω_2 (15) to obtain sufficient conditions. The analysis is carried out in the map (16) on this invariant manifold.

We introduce the deviations

$$y_1 = \gamma_1 - \frac{z_1\gamma_2}{2B_{12}}, \quad y_2 = \omega_1 - \frac{z_1\omega_2}{2B_{12}}, \quad y_3 = \gamma_3, \quad y_4 = \omega_3.$$

The second variation of Ω_2 on the set defined by the first variations

$$\delta\widetilde{V}_1 = \frac{A_2 z_1}{2B_{12}}(\omega_2 y_1 + \gamma_2 y_2) = 0,$$
$$\delta V_2 = \frac{\gamma_2 z_1}{B_{12}} y_1 = 0$$

of the conditional integrals is written in the form

$$2\delta^2\Omega_2 = \frac{1}{4A_2^2}\Bigg[2(2A_2 - A_3)A_3 y_4^2 - 2A_3\bigg((B_{11} + B_{22} + D) + 2A_2\frac{\omega_2}{\gamma_2}\bigg)y_3y_4 + \bigg(2A_2^2\frac{\omega_2^2}{\gamma_2^2} - \big(B_{11}(B_{11} + D) + B_{22}(B_{22} + D) + 2B_{12}^2\big)\bigg)y_3^2\Bigg].$$

The condition

$$(A_2 - A_3)\frac{\omega_2^2}{\gamma_2^2} - A_3(B_{11} + B_{22} + D)\frac{\omega_2}{\gamma_2} - B_{11}(B_{11} + D) - B_{22}(B_{22} + D) - 2B_{12}^2 > 0,$$
$$2A_2 - A_3 > 0$$

of sign definiteness of the quadratic form $\delta^2\Omega_2$ will be sufficient for the stability of the invariant manifold in question.

Since the ratio $\Phi = \widetilde{V}_1/V_2$ of integrals on the invariant manifold (14) acquires the form $\Phi|_0 = A_2\omega_2/\gamma_2 = c = \text{const}$, we see that the last inequalities are satisfied, in particular, for

$$\begin{aligned}&(B_{11} > 0, B_{12} > 0, B_{22} > 0)\\ &\wedge\Bigg\{\Bigg[\bigg(\frac{A_3}{2} < A_2 < A_3\bigg) \wedge \bigg(\frac{A_2\bar{z}_1}{A_2 - A_3} < 2c < -\bar{z}_2\bigg)\Bigg] \vee ((A_2 = A_3) \wedge (2c < -\bar{z}_2))\\ &\vee\Bigg[(A_2 > A_3) \wedge \bigg((2c < -\bar{z}_2) \vee \bigg(2c > \frac{A_2\bar{z}_2}{A_2 - A_3}\bigg)\bigg)\Bigg]\Bigg\},\end{aligned} \tag{24}$$

where $\bar{z}_1 = B_{11} + B_{22} - D$ and $\bar{z}_2 = B_{11} + B_{22} + D$.

In the map (18), the sufficient conditions for the stability of the invariant manifold (14) become more restrictive. Under the condition of positivity of B_{11}, B_{12}, and B_{22}, they contain additional constraints on B_{11}, say, $0 < B_{11} < B_{12}^2/B_{22}$ or $B_{11} > B_{12}^2/B_{22}$.

Let us study the stability of the solutions (21). We introduce the deviations from the unperturbed solution,

$$y_1 = \gamma_1 \pm \frac{\sqrt{2}B_{12}}{\sqrt{D(B_{11} - B_{22} + D)}},$$

$$
\begin{aligned}
y_2 &= \gamma_2 \mp \frac{\sqrt{B_{11} - B_{22} + D}}{\sqrt{2D}},\\
y_3 &= \gamma_3,\\
y_4 &= \omega_1 \mp \frac{\sqrt{B_{11} - B_{22} + D}\big(2B_{12}^2 + B_{22}(B_{22} - B_{11} + D)\big)}{2\sqrt{2}A_2 B_{12}\sqrt{D}},\\
y_5 &= \omega_2 \pm \frac{\sqrt{B_{11} - B_{22} + D}(B_{11} + B_{22} + D)}{2\sqrt{2}A_2\sqrt{D}},\\
y_6 &= \omega_3.
\end{aligned}
$$

The variation of the integral Ω_4 in deviations on the set

$$
\delta V_2 = \mp \frac{\sqrt{2}\big(2B_{12}y_1 - (B_{11} - B_{22} + D)y_2\big)}{\sqrt{D(B_{11} - B_{22} + D)}} = 0
$$

is written as

$$
2\Delta\Omega_4 = \left(\frac{B_{11} + B_{22} - D}{2A_2}y_1 + y_4\right)^2 + \left(\frac{B_{12}}{A_2}\left(\frac{2B_{11}}{B_{11} - B_{22} + D} - 1\right)y_1 + y_5\right)^2 + \frac{(2A_2 - A_3)A_3}{A_2^2}y_6^2.
$$

Let us introduce the variables

$$
\begin{aligned}
\zeta_1 &= \frac{B_{11} + B_{22} - D}{2A_2}y_1 + y_4,\\
\zeta_2 &= \frac{B_{12}}{A_2}\left(\frac{2B_{11}}{B_{11} - B_{22} + D} - 1\right)y_1 + y_5.
\end{aligned}
$$

In terms of the new variables, $\Delta\Omega_4$ acquires the form

$$
2\Delta\widetilde{\Omega}_4 = \zeta_1^2 + \zeta_2^2 + \frac{(2A_2 - A_3)A_3}{A_2^2}y_6^2.
$$

Since the quadratic form $\Delta\widetilde{\Omega}_4$ is sign definite with respect to the variables occurring in it for $A_2 > A_3/2$, we conclude that the solutions studied here are stable with respect to the variables

$$
\begin{gathered}
\frac{1}{2A_2}\left(2A_2\omega_1 + (B_{11} + B_{22} - D)\gamma_1 \mp \frac{2\sqrt{2D}B_{12}}{\sqrt{B_{11} - B_{22} + D}}\right),\\
\omega_2 - \left(\frac{B_{12}\gamma_1}{A_2} \pm \frac{B_{11}(B_{11} - B_{22} - D)\left(\sqrt{2D}\sqrt{B_{11} - B_{22} + D} \pm 2B_{12}\gamma_1\right)}{4A_2B_{12}^2}\right),\\
\omega_3.
\end{gathered} \tag{25}
$$

A similar result can be obtained for the solutions (22).

The conditions for the stability of the invariant manifold (19) coincide with the condition for the stability of the invariant manifold (11).

4. ON SOLUTIONS ON A MANIFOLD

Consider the problem of finding stationary solutions and invariant manifolds of the differential equations (12). We use the same approach as in Sec. 2.

The first integrals of Eqs. (12) can be obtained by eliminating the variables ω_2, ω_3, and γ_1 from the original integrals K_1, $\widetilde{V}_1$, and V_2 by using Eqs. (11). They have the form

$$\begin{aligned}\overline{K}_1 &= B_{22}^2\big[B_{12}^2(\gamma_2^2+\gamma_3^2) - B_{11}^2(\gamma_2^2-\gamma_3^2)\big]\\ &\quad + (B_{12}\gamma_2 - A_2\omega_1)\Big[(B_{11}^2+B_{12}^2)(B_{12}\gamma_2 - A_2\omega_1) + 2B_{11}B_{22}(B_{12}\gamma_2 + A_2\omega_1)\Big] = \bar{c}_1 = \text{const},\\ \overline{V}_1 &= \big(B_{12}^2 - B_{11}B_{22}\big)\gamma_2^2 - A_2^2\omega_1^2 = \bar{c}_2 = \text{const},\\ \overline{V}_2 &= \frac{(B_{12}\gamma_2 - A_2\omega_1)^2}{B_{22}^2} + \gamma_2^2 + \gamma_3^2 = 1.\end{aligned}$$

Let us choose independent integrals from these integrals (for example, $\overline{K}_1$ and $\overline{V}_2$), form their linear combination $2\overline{\Omega} = 2\mu_0\overline{K}_1 - \mu_1\overline{V}_2$, and write the necessary conditions for the extremum of $\overline{\Omega}$ in the variables ω_1, γ_2, and γ_3,

$$\begin{aligned}\frac{\partial\overline{\Omega}}{\omega_1} &= \frac{1}{B_{22}^2}\left(\frac{B_{12}z}{A_2}\gamma_2 - (z - 2B_{11}B_{22}\mu_0)\omega_1\right) = 0,\\ \frac{\partial\overline{\Omega}}{\gamma_2} &= \frac{1}{A_2B_{22}}\left(\frac{B_{12}z}{B_{22}}\omega_1 + \frac{1}{A_2}\left(2B_{11}\big(B_{11}B_{22} - B_{12}^2\big)\mu_0 - \frac{(B_{12}^2 + B_{22}^2)\,z}{B_{22}}\right)\gamma_2\right) = 0,\\ \frac{\partial\overline{\Omega}}{\gamma_2} &= -\frac{z}{A_2^2}\gamma_3 = 0.\end{aligned} \tag{26}$$

Here μ_0 and μ_1 are parameters of the family of integrals $\overline{\Omega}$, and $z = \big(B_{11}^2 + B_{12}^2\big)\mu_0 + A_2^2\mu_1$.

Obviously, for $\mu_1 = -\big(\big(B_{11}^2 + B_{12}^2\big)\mu_0\big)/A_2^2$ Eqs. (26) have the solution

$$\omega_1 = 0, \quad \gamma_2 = 0. \tag{27}$$

A straightforward calculation using the definition of invariant manifold shows that relations (27) determine invariant manifolds of codimension 2 of the differential equations (12).

Another invariant manifold of codimension 2 was obtained by constructing a lexicographic basis for the polynomials of system (26) with respect to $\mu_1 > \omega_1 > \gamma_3$. The equations of the invariant manifold are written in the form

$$\begin{aligned}\gamma_3 &= 0,\\ A_2^2B_{12}\omega_1^2 - A_2\big(2B_{12}^2 - B_{22}(B_{11} - B_{22})\big)\omega_1\gamma_2 + B_{12}(B_{12}^2 - B_{11}B_{22})\gamma_2^2 &= 0.\end{aligned} \tag{28}$$

The differential equations $\dot\gamma_3 = 0$ ($\dot\gamma_2 = 0$) on the invariant manifold (27) (the invariant manifold (28)) have the families of solutions $\gamma_3 = \gamma_3^0 = \text{const}$ ($\gamma_2 = \gamma_2^0 = \text{const}$). Thus, from the geometrical viewpoint, the invariant manifolds found in the space $\mathbb{R}^3$ are associated with curves whose each point is a fixed point in the phase space of system (12).

Let us supplement equations (26) by the relation $\overline{V}_2 = 1$ and construct a lexicographic basis with respect to $\omega_1 > \gamma_2 > \gamma_3 > \mu_1$ for the polynomials of the resulting system. The result will be a system of equations that permits one to obtain the following solutions of the differential equations (12):

$$\omega_1 = \gamma_2 = 0, \quad \gamma_3 = \pm 1, \tag{29}$$

$$\omega_1 = \pm\frac{D_1\big(2B_{12}^2 - B_{22}(B_{11} - B_{22} - D)\big)}{2\sqrt{2D}A_2B_{12}}, \quad \gamma_2 = \pm\frac{D_1}{\sqrt{2D}}, \quad \gamma_3 = 0, \tag{30}$$

$$\omega_1 = \pm\frac{D_2\big(2B_{12}^2 - B_{22}(B_{11} - B_{22} + D)\big)}{2\sqrt{2D}A_2B_{12}}, \quad \gamma_2 = \pm\frac{D_2}{\sqrt{2D}}, \quad \gamma_3 = 0. \tag{31}$$

These solutions correspond to fixed points in the phase space of the system.

Obviously, the solutions (29) belong to the invariant manifold (27). It can readily be shown that the solutions (30) and (31) belong to the invariant manifold (28). To this end, we substitute relations (30) into Eqs. (28), which turn into identities. It follows that the solutions (30) belong to the invariant manifold (28). A similar result can also be obtained for the solutions (31).

It can be verified by a straightforward calculation that the expressions (30), together with Eqs. (11), determine solutions of the differential equations (4) coinciding with (21) in the original phase space. The solutions (31) in the original space are associated with the solutions (22).

Let us study the stability of the solutions (30) using the integral

$$2\Phi = 2\overline{K}_1 + \frac{B_{12}^2 - B_{11}(B_{22}+D)}{A_2^2}\overline{V}_2,$$

which takes a stationary value on these solutions, to obtain sufficient conditions.

In the deviations

$$y_1 = \omega_1 \mp \frac{D_1\big(2B_{12}^2 - B_{22}(B_{11}-B_{22}-D)\big)}{2\sqrt{2D}A_2B_{12}}, \quad y_2 = \gamma_2 \mp \frac{D_1}{\sqrt{2D}}, \quad y_3 = \gamma_3$$

on the linear manifold

$$\delta\overline{V}_2 = \pm\frac{\sqrt{2}}{B_{22}\sqrt{D}D_1}\Big(2A_2B_{12}\,y_1 + \big(B_{22}(B_{11}-B_{22}+D) - 2B_{12}^2\big)y_2\Big) = 0,$$

the variation of the integral Φ is written as follows:

$$\Delta\Phi = -\frac{2B_{11}\big(4B_{12}^2 + (B_{11}-B_{22})^2\big)(B_{11}-B_{22}+D)}{\big(2B_{12}^2 - B_{22}(B_{11}-B_{22}+D)\big)^2}y_1^2 - \frac{B_{11}(B_{11}+B_{22}+D)}{2A_2^2}y_3^2.$$

The quadratic form $\Delta\Phi$ will be positive definite under the following constraints on the parameters B_{11}, B_{12}, and B_{22}:

$$\left((B_{12}\neq 0)\wedge\left((B_{22}<0)\wedge\left(\frac{B_{12}^2}{B_{22}}<B_{11}<0\right)\right)\right)\vee\big((B_{22}>0)\wedge(B_{11}<0)\big). \tag{32}$$

Conditions (32) are sufficient for the stability of the solutions (30).

Let us obtain necessary conditions for the stability of the solutions (30) using the Lyapunov theorem on the stability by the first approximation [13].

In the case under consideration, the equations of the first approximation are written in the form

$$\begin{aligned}
\dot{y}_1 &= \frac{(B_{11}B_{22} - B_{12}^2)D_1}{\sqrt{2D}A_2^2}y_3,\\
\dot{y}_2 &= \pm\frac{\big(B_{22}(B_{11}-B_{22}-D) - 2B_{12}^2\big)D_1}{2\sqrt{2D}A_2B_{12}}y_3,\\
\dot{y}_3 &= \pm\left(\frac{B_{22}\big(B_{11}^2 + B_{22}(B_{22}+D)\big) + (2B_{12}^2 - B_{11}B_{22})(2B_{22}+D)}{2\sqrt{2D}A_2B_{12}B_{22}}y_2 - \frac{\sqrt{D}}{\sqrt{2}B_{22}}y_1\right)D_1.
\end{aligned} \tag{33}$$

The characteristic equation

$$\lambda\big(2A_2^2\lambda^2 + (B_{11}-B_{22})^2 + (B_{11}+B_{22})D + 4B_{12}^2\big) = 0$$

of system (33) has only zero and pure imaginary roots under the conditions

$$(B_{12}\neq 0)\wedge\left((B_{22}<0)\wedge\ \left(\frac{B_{12}^2}{B_{22}}\leq B_{11}<0\right)\vee(B_{11}>0)\right)\vee\big((B_{22}>0)\wedge(B_{11}\neq 0)\big).$$

Comparing the last inequalities with (32), we conclude that the sufficient conditions are close to the necessary ones.

Thus, the solutions (30) that are stable on the manifold correspond to the solutions (21) that are stable in part of the variables in the original phase space. The same result has also been obtained in the case of the solutions (31).

Based on the results presented, we can state the following assertion.

Assertion. *The generalizations of the Routh–Lyapunov method used to analyze the problem under consideration have made it possible to isolate special sets of differential equations* (1) *in the special case of the existence of an additional quadratic integral* K_1 (3) *of these equations—the stationary invariant manifolds* (11), (14), (19) *and the stationary solutions* (21), (22) *as the points of intersection of these invariant manifolds—and investigate the solutions found for stability. With the help of linear and nonlinear combinations of the first integrals of the problem delivering stationary values to the solutions found, sufficient stability conditions* (23) *and* (24) *have been obtained for the invariant manifolds* (11), (19) *and the invariant manifolds* (14), *respectively, and stability in terms of the variables* (25) *is proved for the stationary solutions. The approach used has also allowed carrying out a similar study of the differential equations on the invariant manifold* (11).

CONCLUSIONS

New additional quadratic integrals are indicated in the problem of the motion of a rigid body with a fixed point under the action of a magnetic field generated by the Barnett–London effect and potential forces. A qualitative analysis of a system of differential equations admitting one of these integrals is carried out. Stationary invariant manifolds of codimensions 3 and 4 are isolated, and sufficient conditions for their stability are obtained. It is shown that the intersections of the manifolds are fixed points in the phase space of the system corresponding to the permanent rotations of the body. The stability of these motions with respect to part of the variables is proved. A qualitative analysis of the differential equations has also been carried out on one of the invariant manifolds found. Stationary solutions and invariant manifolds of codimension 2 are isolated. Sufficient stability conditions are obtained for stationary solutions on a manifold and are compared with the necessary ones. We note that in the considered case of the existence of an additional quadratic integral, the parameters characterizing the influence of the potential forces do not explicitly appear in either the equations of motion of the body or the first integrals. The conditions for the stability of solutions are obtained in the form of constraints on the parameters characterizing the magnetic forces. Thus, it can be assumed that potential forces do not have a considerable effect on the motion of the body in this case.

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