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On the Properties of the Symbols of One Class of Hypoelliptic Equations

H. A. Petrosyan*

Russian-Armenian University, ul. Hovsep Emin 123, Yerevan, 0051 Armenia

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Abstract—We consider regular hypoelliptic operators and study some properties of completely regular polyhedra. Basing on the obtained properties, we find an upper bound for the functional dimension of the solution spaces of hypoelliptic equations.

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We consider regular hypoelliptic operators $P(D)$ with constant coefficients and study the properties of completely regular polyhedra. It is well known (see, for example, [1–3]) that

(1) if a polynomial $P(\xi)$ is hypoelliptic then its characteristic polyhedron $\mathfrak{N}(P)$ is completely regular;

(2) if the characteristic polyhedron $\mathfrak{N}(P)$ of a regular polyhedron is completely regular then $P(D)$ is hypoelliptic.

It was shown in [4, 5] that the Gevrey classes to which the solutions to the equation $P(D)U = f$ belong are determined by the characteristic polyhedron of the hypoelliptic operator $P(D)$. In [3, 6–9], the upper and lower bounds for the functional dimension of the spaces of solutions were found for some classes of hypoelliptic equations. In the present article, basing on the obtained properties of completely regular polyhedra, we establish a sharper upper bound as compared to the available bounds for the functional dimension of the solution spaces of hypoelliptic equations.

1. NOTATIONS AND DEFINITIONS

Consider the n-dimensional Euclidean space $\mathbb{R}^n$ of points $\xi = (\xi_1, \dots, \xi_n)$. Let

$$|\xi| = \left(\xi_1^2 + \dots + \xi_n^2\right)^{1/2},$$

$$\mathbb{C}^n := \mathbb{R}^n \times i\mathbb{R}^n, \qquad \mathbb{R}^n_+ := \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,\ \xi_j \ge 0,\ j = 1, \dots, n\},$$

$$\mathbb{R}^n_0 := \{\xi \in \mathbb{R}^n,\ \xi_1 \ne 0, \dots, \xi_n \ne 0\}.$$

Further, $\mathbb{N}$ is the set of naturals; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; while $\mathbb{N}_0^n$ is the set of all n-dimensional multi-indices; i.e., of points $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_j \in \mathbb{N}_0$ for all $j = 1, \dots, n$.

Introduce the following notations for $\xi, \eta \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $\nu \in \mathbb{R}^n_+$, and $t > 0$:

$$\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}, \qquad (\xi, \eta) = \sum_{j=1}^n \xi_j \eta_j, \qquad |\xi|^\nu = |\xi_1|^{\nu_1} \dots |\xi_n|^{\nu_n},$$

$$t\nu = (t\nu_1, \dots, t\nu_n), \qquad |\nu| = \nu_1 + \dots + \nu_n, \qquad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad i^2 = -1, \qquad j = 1, \dots, n.$$

*E-mail: heghine.petrosyan@rau.am

Let $\mathcal{A} \subset \mathbb{R}^n_+$ be a finite collection. The *characteristic polyhedron* of $\mathcal{A}$ is the minimal convex polyhedron $\mathfrak{N}(\mathcal{A}) \subset \mathbb{R}^n_+$ containing $\mathcal{A} \cup \{0\}$.

A polyhedron $\mathfrak{N} \subset \mathbb{R}^n_+$ is called *completely regular* if

(1) the origin is a vertex of $\mathfrak{N}$,

(2) $\mathfrak{N}$ has vertices on each coordinate axis different from the origin;

(3) the components of the outward normals (with respect to $\mathfrak{N}$) of the $(n-1)$-dimensional noncoordinate faces are positive.

Let $\mathfrak{N} \subset \mathbb{R}^n_+$ be a completely regular polyhedron. Introduce the notations:

- $\mathfrak{N}^0$ is the set of the vertices of $\mathfrak{N}$,
- $\partial\mathfrak{N}$ is the set of the points of $\mathfrak{N}$ lying on the closures of the noncoordinate faces of $\mathfrak{N}$,
- $\wedge(\mathfrak{N})$ is the set of the outward normals $\lambda = (\lambda_1, \dots, \lambda_n)$ (with respect to $\mathfrak{N}$) of the noncoordinate faces of $\mathfrak{N}$ normalized so that

$$d_{\mathfrak{N}}(\lambda) := \max_{\nu \in \mathfrak{N}} (\nu, \lambda) = 1,$$

- $\wedge^{n-1}(\mathfrak{N})$ is the set of those $\lambda \in \wedge(\mathfrak{N})$ that are the normals of the $(n-1)$-dimensional (noncoordinate) faces of $\mathfrak{N}$,
- $\wedge^{n-1}_k(\mathfrak{N})$ is the set of the normals $\lambda \in \wedge^{n-1}(\mathfrak{N})$ whose corresponding $(n-1)$-face has a subface lying in some coordinate hyperplane.

Remark 1. It is not hard to observe that, for every completely regular polyhedron $\mathfrak{N} \subset \mathbb{R}^n_+$, the set

$$\tilde{\wedge}(\mathfrak{N}) := \{\mu,\ \mu = \lambda/|\lambda|,\ \lambda \in \wedge(\mathfrak{N})\}$$

coincides with the convex hull of

$$\tilde{\wedge}^{n-1}(\mathfrak{N}) := \{\mu,\ \mu = \lambda/|\lambda|,\ \lambda \in \wedge^{n-1}(\mathfrak{N})\}.$$

Remark 2. Let $\mathfrak{N} \subset \mathbb{R}^n_+$ be a completely regular polyhedron. Then, for all $\lambda = (\lambda_1, \dots, \lambda_n) \in \wedge(\mathfrak{N})$, we have $\lambda_j > 0$, $j = 1, \dots, n$.

Let $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$ be a differential operator with constant coefficients, where the sum ranges over the finite set $(P) := \{\alpha \in \mathbb{N}^n_0,\ \gamma_\alpha \neq 0\}$, and let

$$P(\xi) = \sum_{\alpha \in (P)} \gamma_\alpha \xi^\alpha$$

be the complete symbol of $P(D)$.

Definition 1. The *characteristic polyhedron* of $P(D)$ is the characteristic polyhedron of the collection (P).

Definition 2 [10]. The operator $P(D)$ (the polynomial $P(\xi)$) is called *regular* (or *nondegenerate*) if there exists a constant $C > 0$ such that, for every $\nu \in \mathfrak{N}(P)$, we have

$$|\xi^\nu| \le C(|P(\xi)| + 1) \qquad \text{for all} \quad \xi \in \mathbb{R}^n.$$

Definition 3. The differential operator $P(D)$ (the polynomial $P(\xi)$) is called *hypoelliptic* if all solutions to the equation $P(D)U = 0$ are infinitely differentiable functions.

Hörmander proved (see [12, Definition 11.1.2, Theorem 11.1.3]) that $P(D)$ is hypoelliptic if, for $\xi \in \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n_0$, $\alpha \neq 0$, we have

$$\frac{D^\alpha P(\xi)}{P(\xi)} \to 0 \qquad \text{as} \quad |\xi| \to \infty$$

(see also [11, Chapter 3]).

Let P be a regular hypoelliptic polynomial with characteristic polyhedron $\mathfrak{N}(P)$. Denote the *weight hypoellipticity set* of P by

$$\mathfrak{M}(P) := \Big\{\nu \in \mathbb{R}^n_+,\ \ (\nu, \lambda) \le 1/ \min_{1\le j\le n} \lambda_j,\ \ \lambda \in \wedge^{n-1}(\mathfrak{N})\Big\}.$$

Obviously, $\mathfrak{M}(P)$ is a completely regular polyhedron.

Let $\mathfrak{M}(P)$ be a completely regular polyhedron. Denote by $\Gamma^{\mathfrak{M}(P)}(\Omega)$ the multianisotropic Gevrey class; i.e., the set of functions $f \in C^\infty(\Omega)$ such that, for every compact set $K \subset \Omega$, there exists a constant $C = C(K, f) > 0$ such that

$$\sup_{x\in K} |D^\alpha f(x)| \le C^{j+1} j^j, \qquad \alpha \in j\mathfrak{M}(P), \qquad j = 0, 1, \dots .$$

It is known (see [9]) that if $P(D)$ is a regular hypoelliptic operator then, for every domain $\Omega \subset \mathbb{R}^n$, we have

$$E(P, \Omega) := \{U \in \mathcal{D}'(\Omega),\ P(D)U = 0 \text{ on } \Omega\} \subset \Gamma^{\mathfrak{M}(P)}(\Omega),$$

where $\mathcal{D}'(\Omega)$ is the space of generalized functions.

Definition 4 [13]. A topological vector space E is called *locally convex* if there exists a fundamental system of convex neighborhoods of zero.

Suppose that E is a locally convex space, U is a neighborhood of zero in E, and B is a set $B \subset E$.

Definition 5 [14]. Given $\varepsilon > 0$, a set F is called an *ε-net for B with respect to U* if $B \subset F + \varepsilon U$.

Given $\varepsilon > 0$, the sets B and U, denote by $N(B, \varepsilon U)$ the least number of elements in an ε-net for B with respect to U.

Definition 6 [15]. For a locally convex space E, the quantity

$$df\, E := \sup_U \inf_V \varlimsup_{\varepsilon\to 0+} \frac{\ln\ln N(V, \varepsilon U)}{\ln\ln \varepsilon^{-1}},$$

where U and V range over all neighborhoods of zero in E, is called the *functional dimension of E*.

2. PRELIMINARY RESULTS

Let $\mathfrak{N}$ be a regular polyhedron. Put

$$\tilde{\wedge}^{n-1}_k(\mathfrak{N}) := \big\{\mu,\ \mu = \lambda/|\lambda|,\ \lambda \in \wedge^{n-1}_k(\mathfrak{N})\big\}.$$

Proposition. *Let $\mathfrak{N} \subset \mathbb{R}^n_+$ be a completely regular polyhedron and put*

$$\mathfrak{M} := \big\{\nu \in \mathbb{R}^n_+\ (\nu, \lambda) \le 1,\ \lambda \in \wedge^{n-1}_k(\mathfrak{N})\big\}.$$

Then

$$\tilde{\wedge}(\mathfrak{M}) \setminus \tilde{\wedge}^{n-1}_k(\mathfrak{M}) = \tilde{\wedge}(\mathfrak{N}) \setminus \tilde{\wedge}^{n-1}_k(\mathfrak{N}).$$

Proof. It is not hard to observe that, by the definition of $\mathfrak{M}$,

$$\tilde{\wedge}^{n-1}_k(\mathfrak{M}) = \tilde{\wedge}^{n-1}_k(\mathfrak{N}). \tag{2.1}$$

Since, by Remark 1, $\tilde{\wedge}(\mathfrak{M})$ coincides with the convex hull of $\tilde{\wedge}^{n-1}_k(\mathfrak{M})$; therefore, $\tilde{\wedge}(\mathfrak{M}) \subset \tilde{\wedge}(\mathfrak{N})$. By (2.1), this gives

$$\tilde{\wedge}(\mathfrak{M}) \setminus \tilde{\wedge}^{n-1}_k(\mathfrak{M}) \subset \tilde{\wedge}(\mathfrak{N}) \setminus \tilde{\wedge}^{n-1}_k(\mathfrak{N}).$$

Prove the reverse inclusion. Let $\mu \in \tilde{\Lambda}(\mathfrak{N}) \setminus \tilde{\Lambda}_k^{n-1}(\mathfrak{N})$. Since, by the definition of $\mathfrak{M}$, the polyhedron $\mathfrak{N} \subset \mathfrak{M}$ and the set of vertices in $\mathfrak{M}$ lying on the coordinate hyperplanes coincides with the set of vertices in $\mathfrak{N}$ also lying on the coordinate hyperplanes; therefore, we infer that

$$d_{\mathfrak{M}}(\mu) = \max_{\nu \in \mathfrak{M}} (\nu, \mu) = d_{\mathfrak{N}}(\mu),$$

$$(\alpha, \mu) < d_{\mathfrak{N}}(\mu) \quad \text{for all} \quad \alpha \in \mathfrak{M}^0$$

lying in some coordinate hyperplane. Hence,

$$\{\nu \in \mathfrak{M}, (\nu, \mu) = d_{\mathfrak{M}}(\mu)\} \subset \mathbb{R}_0^n \cap \mathbb{R}_+^n,$$

and so $\mu \in \tilde{\Lambda}(\mathfrak{M}) \setminus \tilde{\Lambda}_k^{n-1}(\mathfrak{M})$. The proposition is proved. □

Corollary 1. *For every completely regular polyhedron $\mathfrak{N} \subset \mathbb{R}_+^n$, the set $\tilde{\Lambda}(\mathfrak{N})$ coincides with the convex hull of $\tilde{\Lambda}_k^{n-1}(\mathfrak{N})$.*

Since $\tilde{\Lambda}_k^{n-1}(\mathfrak{N}) = \tilde{\Lambda}^{n-1}(\mathfrak{M})$, the proof is immediate from the above proposition and Remark 1.

Lemma 1. *Let $\mathfrak{N} \subset \mathbb{R}_+^n$ be a completely regular polyhedron. Then*

$$\max_{\lambda \in \tilde{\Lambda}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda) = \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda).$$

Proof. Suppose the contrary:

$$\max_{\lambda \in \tilde{\Lambda}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda) \neq \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda).$$

Since $\tilde{\Lambda}_k^{n-1}(\mathfrak{N}) \subset \tilde{\Lambda}(\mathfrak{N})$, this means that there exists $\mu \in \tilde{\Lambda}(\mathfrak{N}) \setminus \tilde{\Lambda}_k^{n-1}(\mathfrak{N})$ for which

$$d_{\mathfrak{N}}(\mu) > \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda). \tag{2.2}$$

Consider the completely regular polyhedron

$$\mathfrak{M} := \left\{\nu \in \mathbb{R}_+^n,\ (\nu, \lambda) \leq d_{\mathfrak{N}}(\lambda),\ \lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{N})\right\}.$$

Since $\tilde{\Lambda}_k^{n-1}(\mathfrak{M}) = \tilde{\Lambda}_k^{n-1}(\mathfrak{N})$ (see (2.1)) and obviously

$$\mathfrak{N} \subset \mathfrak{M}, \qquad d_{\mathfrak{M}}(\lambda) = d_{\mathfrak{N}}(\lambda) \quad \text{for all} \quad \lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M});$$

therefore, we have

$$d_{\mathfrak{M}}(\mu) > \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} d_{\mathfrak{M}}(\lambda) = \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{N})} d_{\mathfrak{N}}(\lambda). \tag{2.3}$$

Consequently, $\mu \in \tilde{\Lambda}(\mathfrak{M})$.

Since, by 2.1, $\tilde{\Lambda}(\mathfrak{M})$ is the convex hull of $\tilde{\Lambda}_k^{n-1}(\mathfrak{M})$, there exist some numbers

$$\theta(\lambda) \in [0, 1], \qquad \sum_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} \theta(\lambda) = 1$$

for every $\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})$ such that

$$\mu = \sum_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} \theta(\lambda)\lambda.$$

Let $\nu^0 \in \mathfrak{M}$ be such that $(\nu^0, \mu) = d_{\mathfrak{M}}(\mu)$. Then

$$d_{\mathfrak{M}}(\mu) = (\nu^0, \mu) = \sum_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} \theta(\lambda)(\nu^0, \lambda) \leq \sum_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} \theta(\lambda) d_{\mathfrak{M}}(\lambda)$$

$$\leq \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} d_{\mathfrak{M}}(\lambda) \sum_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} \theta(\lambda) = \max_{\lambda \in \tilde{\Lambda}_k^{n-1}(\mathfrak{M})} d_{\mathfrak{M}}(\lambda).$$

This contradicts (2.3) and validates Lemma 1. □

Corollary 2. *Given a completely regular polyhedron $\mathfrak{N} \subset \mathbb{R}^n_+$, we have*

$$\min_{\lambda\in\wedge(\mathfrak{N})} |\lambda| = \min_{\lambda\in\wedge_k^{n-1}(\mathfrak{N})} |\lambda|.$$

Proof. Since $\lambda/|\lambda| \in \tilde{\wedge}(\mathfrak{N})$ for every $\lambda \in \wedge(\mathfrak{N})$, by the definition of $\wedge(\mathfrak{N})$, we infer $d_{\mathfrak{N}}(\lambda/|\lambda|) = 1/|\lambda|$. Hence, Lemma 1 yields

$$\max_{\lambda\in\wedge(\mathfrak{N})} \frac{1}{|\lambda|} = \max_{\lambda\in\wedge_k^{n-1}(\mathfrak{N})} \frac{1}{|\lambda|}$$

or, which is the same,

$$\min_{\lambda\in\wedge(\mathfrak{N})} |\lambda| = \min_{\lambda\in\wedge_k^{n-1}(\mathfrak{N})} |\lambda|.$$

Corollary 2 is proved. □

Lemma 2. *For every completely regular polyhedron $\mathfrak{N} \subset \mathbb{R}^n_+$, we have*

$$\max_{\lambda\in\wedge(\mathfrak{N})} \max_{1\le j\le n} \frac{|\lambda|}{\lambda_j} = \max_{\lambda\in\wedge_k^{n-1}(\mathfrak{N})} \max_{1\le j\le n} \frac{|\lambda|}{\lambda_j}.$$

Proof. Since $\tilde{\wedge}(\mathfrak{N}) = \{\mu = \lambda/|\lambda|,\ \lambda \in \wedge(\mathfrak{N})\}$, it suffices to demonstrate that

$$\max_{\lambda\in\tilde{\wedge}(\mathfrak{N})} \max_{1\le j\le n} \frac{1}{\lambda_j} = \max_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \max_{1\le j\le n} \frac{1}{\lambda_j}. \tag{2.4}$$

Let $\mu \in \tilde{\wedge}(\mathfrak{M})$. Then, by Corollary 2, there are

$$\theta(\lambda) \in [0,1], \qquad \lambda \in \tilde{\wedge}_k^{n-1}(\mathfrak{N}), \qquad \sum_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \theta(\lambda) = 1$$

such that

$$\mu = \sum_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \theta(\lambda)\lambda,$$

whence

$$\mu_j = \sum_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \theta(\lambda)\lambda_j, \qquad j = 1, \dots, n.$$

Consequently,

$$\mu_j \ge \min_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \lambda_j \sum_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \theta(\lambda) = \min_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \lambda_j, \qquad j = 1, \dots, n,$$

whence

$$\max_{1\le j\le n} \frac{1}{\mu_j} \le \max_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \max_{1\le j\le n} \frac{1}{\lambda_j}.$$

Since $\tilde{\wedge}_k^{n-1}(\mathfrak{N}) \subset \tilde{\wedge}(\mathfrak{N})$, the arbitrariness of $\mu \in \tilde{\wedge}(\mathfrak{M})$ implies (2.4), and thus the proof of Lemma 2 is complete. □

Let P be a polynomial of n variables. Put

$$D(P) := \{\zeta \in \mathbb{C}^n,\ P(\zeta) = 0\},$$

$$d_P(\xi) := \inf_{\zeta\in D(P)} \|\xi - \zeta\|, \qquad \xi \in \mathbb{R}^n,$$

$$E(P,\Omega) := \{U \in \mathcal{D}',\ P(D)U = 0 \text{ on } \Omega\},$$

where $\mathcal{D}'$ is the space of generalized functions.

Theorem 1 [3, Theorem 2]. *Let $P(D)$ be a hypoelliptic operator and let*

$$\mathfrak{M} \subset \mathfrak{M}_P := \{\nu \in \mathbb{R}^n_+,\ \exists\, C_\nu > 0\ |\xi|^\nu \le C_\nu(d_P(\xi)+1),\ \xi \in \mathbb{R}^n\}$$

be a completely regular polyhedron. Then

$$df\, E(P,\Omega) \le 1 + \max_j \sup_{\lambda\in\wedge(\mathfrak{M}_j)} \frac{|\lambda|}{d_{j,\lambda}},$$

where

$$d_{j,\lambda} = \sup_{\nu\in\mathfrak{M}_j} (\nu,\lambda), \qquad \mathfrak{M}_j = \{\nu^j \in \mathbb{R}^n_+,\ (\nu_1,\dots,\nu_{j-1},0,\nu_{j+1},\dots,\nu_n) \in \mathfrak{M}\},$$

and the maximum is taken over those j for which there exists $\alpha \in (P)$ such that $|\alpha| = \operatorname{ord} P$ with $\alpha_j \ne 0$.

3. THE MAIN RESULTS

Given a regular hypoelliptic polynomial $P(\xi) = P(\xi_1,\dots,\xi_n)$ with constant coefficients, denote by $\mathfrak{M}(P)$ and $E(P,\Omega)$ the quantities

$$\mathfrak{M}(P) := \{\nu \in \mathbb{R}^n_+,\ (\nu,\lambda) \le \min_{1\le j\le n} 1/\lambda_j,\ \lambda \in \tilde{\wedge}(\mathfrak{N})\},$$

$$E(P,\Omega) := \{U \in \mathcal{D}',\ P(D)U = 0 \ \text{ on } \Omega\},$$

where $\mathcal{D}'$ is the space of generalized functions.

Theorem 2. *If $P(\xi) = P(\xi_1,\dots,\xi_n)$ is a regular hypoelliptic polynomial with constant coefficients then*

$$df\, E(P,\Omega) \le \max_{\lambda\in\tilde{\wedge}_k^{n-1}(\mathfrak{N})} \max_{1\le j\le n} \frac{1}{\lambda_j}. \tag{3.1}$$

Proof. By (2.4), it suffices to show that

$$df\, E(P,\Omega) \le \max_{\lambda\in\tilde{\wedge}(\mathfrak{N})} \max_{1\le j\le n} \frac{1}{\lambda_j}. \tag{3.2}$$

Since $P(\xi)$ is a regular hypoelliptic operator and

$$d_{\mathfrak{N}}(\lambda) = \max_{\nu\in\mathfrak{N}} (\nu,\lambda) = 1,$$

the proof of (3.2) is immediate from Theorem 1, which validates (3.1). □

Let $\mathfrak{B}$ denote the set of those completely regular polyhedra $\mathfrak{N} \subset \mathbb{R}^n_+$ for which

$$\varnothing \ne \mathfrak{N}^0 \setminus \{\nu^j\}_{j=1}^n \subset \mathbb{R}^n_0.$$

Here $\nu^j = (0,\dots,0,\nu^j_j,0,\dots,0)$, $j = 1,\dots,n$, are the vertices of the polyhedron $\mathfrak{N}$ lying on the coordinate axes.

Obviously, if $\mathfrak{N} \in \mathfrak{B}$ then it has the $(n-1)$-dimensional noncoordinate face passing through the points $\{\nu^j\}_{j=1, j\neq l}^n$, $l = 1, \dots, n$. Let $\lambda^l \in \wedge_k^{n-1}(\mathfrak{N})$ denote the normal to the $(n-1)$-face passing through $\{\nu^j\}_{j=1, j\neq l}^n$, $l = 1, \dots, n$.

Theorem 3. *Let* $\mathfrak{N} \in \mathfrak{B}$. *Then*

$$\min_{\lambda\in\wedge(\mathfrak{N})} |\lambda| = \min_{1\le l\le n} |\lambda^l|.$$

Proof. In view of Corollary 2, it suffices to show that

$$\min_{\lambda\in\wedge_k^{n-1}(\mathfrak{N})} |\lambda| = \min_{1\le l\le n} |\lambda^l|. \tag{3.3}$$

Suppose the contrary; i.e., (3.3) fails. Since $\{\lambda^l\}_{l=1}^n \subset \wedge_k^{n-1}(\mathfrak{N})$; therefore, this means that there exists $\mu \in \wedge_k^{n-1}(\mathfrak{N})$ for which

$$|\mu| < \min_{1\le l\le n} |\lambda^l|. \tag{3.4}$$

The condition $\mu \in \wedge_k^{n-1}(\mathfrak{N})$ implies that an $(n-1)$-face with normal μ passes through a vertex of $\mathfrak{N}$ lying in some coordinate hyperplane. The definition of $\mathfrak{B}$ (since $\mathfrak{N} \in \mathfrak{B}$) implies that these vertices lie on the coordinate axes. Suppose that an $(n-1)$-dimensional noncoordinate face with normal μ passes only through the vertices $\nu^1, \dots, \nu^r$ of $\mathfrak{N}$ lying on the coordinate axes. The condition $\mathfrak{N} \in \mathfrak{B}$ implies that $1 \le r \le n-1$. Therefore, by assumption, $(\nu^j, \mu) = 1$ for $j = 1, \dots, r$ and $(\nu^j, \mu) < 1$ for $j = r+1, \dots, n$. The definitions of $\wedge^{n-1}(\mathfrak{N})$ and $\{\lambda^l\}_{l=1}^n$ imply that

$$\mu_j = 1/\nu_j^j = \lambda_j^l, \qquad j = 1, \dots, r, \qquad l = r+1, \dots, n.$$

Let $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$. Put $\nu' = (\nu_1, \dots, \nu_r)$ and $\nu'' = (\nu_{r+1}, \dots, \nu_n)$. Hence, by (3.4),

$$|\mu''| < \min_{r+1\le j\le n} |\lambda^{j,''}|. \tag{3.5}$$

Consider the polyhedron

$$\mathfrak{M} := \{\nu'' \in \mathbb{R}_+^{n-r},\ (\nu'', \lambda^{j,''}) \le 1,\ j = r+1, \dots, n,\ (\nu'', \mu'') \le 1\}.$$

Obviously, the vertices of $\mathfrak{M}$ lying on the coordinate hyperplanes are

$$(\nu_{r+1}^{r+1}, 0, \dots, 0), \dots, (0, \dots, 0, \nu_n^n) \in \mathbb{R}_+^{n-r}.$$

Moreover, since $(\nu^j, \mu) < 1$, $j = r+1, \dots, n$, a face of $\mathfrak{M}$ with normal μ'' does not pass through these vertices. Consequently,

$$\wedge_k^{n-r-1}(\mathfrak{M}) = \{\lambda^{j,''}\}_{j=r+1}^n \qquad \mu'' \notin \wedge_k^{n-r-1}(\mathfrak{M}).$$

Hence, by Lemma 1, we conclude that

$$|\mu''| \ge \min_{r+1\le j\le n} |\lambda^{j,''}|.$$

This contradicts estimate (3.5) and hence estimate (3.4).

The so-obtained contradiction proves Theorem 3. □

Theorem 4. *Suppose that, for* $\mathfrak{N} \in \mathfrak{B}$, *we have* $\mathfrak{N}^0 \subset \mathbb{N}_0^n$. *Then*

$$\max_{\lambda\in\wedge^{n-1}(\mathfrak{N})} |\lambda| \le 1.$$

Proof. Since, by hypothesis, $\mathfrak{N} \in \mathfrak{B}$ and $\mathfrak{N}^0 \cap \mathbb{R}_0^n \neq \varnothing$; therefore, the condition $\mathfrak{N}^0 \subset \mathbb{N}_0^n$ implies that $\alpha := (1, \dots, 1) \in \mathfrak{N}$. Hence, $|\lambda| = (\alpha, \lambda) \le d_{\mathfrak{N}}(\lambda) = 1$ for every $\lambda \in \wedge^{n-1}(\mathfrak{N})$, Theorem 4 is proved. □

Corollary 3. *If, under the conditions of Theorem* 4, *the polynomial*

$$P(\xi) := \sum_{\alpha\in\mathfrak{N}^0} \xi^\alpha$$

is regular then

$$\max_{\lambda\in\wedge^{n-1}(\mathfrak{N})} |\lambda| \leq \frac{1}{2}.$$

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REFERENCES

1. S. Gindikin and L. Volevich, *The Method of Newton's Polyhedron in the Theory of Partial Differential Equations* (Kluwer Acad. Publ., 1992).
2. S. M. Nikol'skii, "The First Boundary Value Problem for a General Linear Equation," Dokl. Akad. Nauk SSSR **144** (4), 767–769 (1962).
3. V. N. Margaryan, "Some Above Estimates of Functional Dimension of the Solution Space of Hypoelliptic Equations," Izv. Natsion. Akad. Nauk Armenii **37** (5), 27–39 (2002).
4. V. N. Margaryan and G. G. Kazaryan, "Estimates of Solutions of Nonhypoelliptic Equations with a Given Support of Hypoellipticity," Izv. Akad. Nauk Armyansk. SSR **22** (5), 315–336 (1987).
5. V. N. Margaryan and G. H. Hakobyan, "On Gevrey Class Solutions of Hypoelliptic Equations," J. Contemp. Math. Anal. **33** (1), 1–13 (1998).
6. Z. Zielezni, "On the Functional Dimension of Solution of PDE," J. Differential Equations **18** (2), 340–345 (1975).
7. M. Langenbruch, "On the Functional Dimension of Solution Spaces of Hypoelliptic Partial Differential Operators," Math. Ann. **272**, 217–229 (1985).
8. M. Langenbruch, "P-Functionale und Raudwere zu Hypoelliptischen Differential Operatoren," Math. Ann. **239**, 313–324 (1979).
9. V. N. Margaryan and G. G. Kazaryan, "On Functional Dimension of the Solution Spaces of Hypoelliptic Equations," Mat. Sb. **115** (157), 614–631 (1981).
10. V. P. Mikhailov, "On the Behavior at Infinity of a Class of Polynomials," Dokl. Akad. Nauk SSSR **164** (3), 571–573 (1965).
11. G. E. Shilov, *Mathematical Analysis. The Second Special Course* (Nauka, Moscow, 1965) [in Russian].
12. L. Hörmander, *Analysis of Linear Differential Operators in Partial Derivatives*, Vol. 2 (Mir, Moscow, 1986) [in Russian].
13. A. Pich, *Nuclear Locally Convex Spaces* (Mir, Moscow, 1967) [in Russian].
14. L. Pontrjagin and L. Schnirelmann, "Sur une propriete metrique de la dimension," Ann. Math. **33**, 156–162 (1932).
15. A. N. Kolmogorov and V. M. Tikhomirov, "ε-Entropy and ε-Capacity of Sets in Function Spaces," Uspekhi Mat. Nauk **2** (96), 3–86 (1959).