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On the Number and Arrangement of Sensors for the Multiple Covering of Bounded Plane Domains

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Abstract—We propose a method for determining the number of sensors, their arrangement, and approximate lower bounds for the number of sensors for the multiple covering of an arbitrary closed bounded convex area in a plane. The problem of multiple covering is considered with restrictions on the minimal possible distances between the sensors and without such restrictions. To solve these problems, some 0–1 linear programming (LP) problems are constructed. We use a heuristic solution algorithm for 0–1 LP problems of higher dimensions. The results of numerical implementation are given and for some particular cases it is obtained that the number of sensors found can not be decreased.

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INTRODUCTION

As a rule, a wireless sensor network has many sensors for monitoring (observing, covering) of some given domain G. For constructing such networks, an important task is to determine the number and arrangement of sensors for a k-fold covering ($k \geq 1$) of G. The k-fold covering problem for sensors is also known as the problem of k-covering with circles of given radius.

Let G be a bounded closed convex area with nonempty interior in a plane P on which we introduce the Cartesian coordinates xOy. Suppose that $d(t, s)$ is the Euclidean distance between the points t and s on P and sensors have the covering radius r. Let the sensor s (the point s) be a center of a circle of radius r. In some cases, we identify a circle of radius r with a sensor with the covering radius r.

A collection of sensors $S = \{s_1, \ldots, s_m\}$ with covering radius r generates a k-fold covering of G (k-covering, for short), $1 \leq k \leq m$, if for every point $t \in G$ there exist at least k sensors s_j such that $d(s_j, t) \leq r$, $s_j \in S$.

Problems of determining the number and arrangement of sensors (circles) for k-covering of domains are studied by many authors (see, for example, [1–16]). There are many works on covering (1-covering) with circles of the entire plane, as well as its bounded parts such as squares, circles, triangles, and some other figures (see [17–29]). In those articles, as a rule, the minimization problem on the radius of the covering circles is under study provided that their number is known. Min-max-min models [26], Voronoi diagrams [17], bar models [29], etc. are used to solve these problems. Note that coverings of a triangle, a square, and a circle with circles of the radius determined up to 19–20 decimal digits are obtained in [25–27]. Though, achievement of such results for arbitrary domains is highly unlikely.

There are a few articles on k-coverings for $k \geq 2$. For k-coverings, $k \geq 2$, various methods are proposed for determining the minimum radius for given number of circles or finding the number of circles of given radius (see [11, 12, 15, 16, 18, 30, 31]). These covering problems are NP-complete (for instance, see [32–36]).

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There are many publications on monitoring and multimonitoring with use of the sensor networks (see [1–4, 6–8, 10, 11, 14–16]). In these networks, the following can be taken as a criterion: minimization of the number of sensors, minimization of energy consumption, system communicative abilities guarantee, detection upon accumulation of the received signal, and many others. In general, the problem of monitoring is of high priority. There are various methods for solving the problems of monitoring of connected domains G. One of the widely-used consists in constructing a grid on G and monitoring the nodes of the grid instead of G. In some monitoring problems the goal is to cover given bounded set of points with zones controlled by the sensors which, in their turn, are placed at the points from some finite set. For instance, in [15] the arrangement of sensors at some of the vertices of a (finite) graph is under study for monitoring all vertices of the graph.

In this article, we propose a method for determining an approximate number of circles (sensors) and their arrangement for k-covering, $k \geq 1$, of an arbitrary bounded closed convex set G with nonempty interior on the plane P. Since placing of several sensors at one point is undesirable, the constraints on minimal distances between sensors are introduced. To solve the problems of monitoring G with or without the constraints, a rectangular grid is constructed whose nodes form a finite set T_Δ on G. Then with use of T_Δ we construct 0–1 linear programming (LP) problems, and solving these problems leads to estimation of the number of circles needed and their arrangement. We obtain some approximate lower estimates on the number of circles for the given covering of G. The numerical implementation shows that in some cases the estimates are attainable and thus unimprovable.

1. MATHEMATICAL MODELS OF THE PROBLEM

Given a set G, let Δ be the grid size on G. With $\Delta x = \Delta y = \Delta$ we construct a rectangular grid on G whose lines form squares C_Δ of side Δ. The nodes of the grid in G generate the finite set

$$T_\Delta = \{t_1, \dots, t_n\}, \qquad t_i \in G.$$

If the square C_Δ lies entirely in G then each vertex of it belongs to T_Δ. If C_Δ is partially contained in G, so that the area of $(C_\Delta \cap G)$ is positive, then the points of entrance of the boundary $\operatorname{fr} G$ of G into C_Δ and the points of exit of $\operatorname{fr} G$ from C_Δ are included into the set T_Δ. If C_Δ is partially contained in G and $D = C_\Delta \cap G$ includes the center s of the square C_Δ then s is added to T_Δ. If the point s is not in D then we draw line segments l_1 and l_2 of length Δ through s which lie in C_Δ and are parallel to the sides of C_Δ. To the set T_Δ we add the intersection points of $\operatorname{fr} G$ with l_1 and l_2. Moreover, if $\operatorname{fr} G$ intersects with l_j, $j = 1, 2$, by some line segment $[d_1, d_2]$ then only d_1 and d_2 are included in T_Δ.

If the grid size equals $\Delta/2$ then $T_{\Delta/2}$ is constructed by analogy to T_Δ with including all points of T_Δ in the set of the nodes obtained for $\Delta/2$; therefore, $T_\Delta \subset T_{\Delta/2}$.

Let G, k, Δ, and r be given and the finite set T_Δ be constructed so that T_Δ has n elements. Consider the problems:

Problem P1. *Find a k-covering of G with circles of radius r such that to minimize the number of covering circles.*

Problem P2. *Find a k-covering of G with circles of radius r such that to minimize the number of covering circles under condition that, for each of the covering circles, the center coincides with a point of the set T_Δ and every point of T_Δ coincides with at most one of the possible centers of a circle.*

Problem P3. *Find a k-covering of T_Δ with circles of radius r such that to minimize the number of covering circles and place the center of each covering circle in T_Δ, while every point of T_Δ coincides with at most one of the possible centers of a circle.*

Firstly consider Problem P3. Let Δ be selected and the set T_Δ be constructed on G. Introduce some parameter α and put

$$a_{ij} = \begin{cases} 1, & \text{if } \ d(t_i, t_j) \leq r - \alpha, \\ 0, & \text{if } \ d(t_i, t_j) > r - \alpha. \end{cases} \tag{1}$$

For (1) be correct, we need $\alpha < r$. In what follows we assume $\Delta \leq r/4$. Define the next variables: Given i, $1 \leq i \leq n$, let z_i be the number of circles of radius $r - \alpha$ whose centers coincide with the point t_i. Consider the problem

$$z_1 + z_2 + \cdots + z_n \to \min \tag{2}$$

under the following restrictions:

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n &\geq k, \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n &\geq k, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots& \\ a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n &\geq k. \end{aligned} \tag{3}$$

$$z_i \in \{0, 1\}, \qquad 1 \leq i \leq n. \tag{4}$$

Conditions (3) guarantee that each point t_i is covered with at least k circles.

If we put $\alpha = 0$ in (1) then (2)–(4) becomes the problem of k-covering of T_Δ with the minimal number of circles of radius r such that the center of every circle coincides with some point of T_Δ. This assertion easily follows, for example, from [37, p. 71] if the subsets S_j of T_Δ consist of the points of T_Δ belonging to the closed circle of radius r with center $t_j \in T_\Delta$. Hence, for solving Problem P3 we should solve system (2)–(4) with $\alpha = 0$.

Consider Problem P2. Covering of T_Δ with circles of radius r does not guarantee covering of the given set G. The square C_Δ generated by the rectangular grid has the diagonal length equal to $\sqrt{2}\Delta$. If we reduce the radius of the covering circles by half of the diagonal length (i.e., by $\alpha_0 = \Delta\sqrt{2}/2$) and the circles of radius $r - \alpha_0$ generate a k-covering of T_Δ then the circles of the initial radius r form a k-covering of G. The parameter α_0 can be replaced by α, $\alpha_0 \leq \alpha \leq 2\alpha_0$.

Lemma 1. *Let Δ be selected, $\alpha = \Delta\sqrt{2}$, and T_Δ and $T_{\Delta/2}$ be constructed. Then each k-covering of T_Δ with circles of radius $r - \alpha$ generates a k-covering of $T_{\Delta/2}$ with circles of radius $r - \alpha/2$.*

Proof. Suppose that we solved problem (2)–(4) for $\alpha = \Delta\sqrt{2}$ and obtained the minimal possible number of circles with centers $c_1, c_2, \ldots, c_m$ ($c_i \in T_\Delta$) forming a k-covering of T_Δ with circles of radius $r - \alpha$. Then for every $t_i \in T_\Delta$ there exists at least k points c_l such that $d(t_i, c_l) \leq r - \alpha$.

For every point $s_i \in T_{\Delta/2}$ there is $t_j \in T_\Delta$ such that $d(s_i, t_j) \leq \alpha/2$ and for t_j there exists at least k points c_l such that $d(t_j, c_l) \leq r - \alpha$. Hence, we have

$$d(s_i, c_l) \leq d(s_i, t_j) + d(t_j, c_l) \leq \alpha/2 + r - \alpha = r - \alpha/2;$$

therefore, the number ($n_{\alpha/2}$) of the circles of radius $r - \alpha/2$ generating the k-covering of $T_{\Delta/2}$ does not exceed the number (n_α) of the circles forming the k-covering of T_Δ (with circles of radius $r - \alpha$).

In result we have

$$n_{\alpha/2} \leq n_\alpha. \tag{5}$$

The proof of Lemma 1 is complete. □

Let r and Δ be available, and put $\alpha = \Delta\sqrt{2}$.

Theorem 1. *There is m such that a solution of* (2)–(4) *for*

$$a_{ij} = \begin{cases} 1, & \textit{if} \quad d(t_i, t_j) \leq r - \alpha/2^m, \\ 0, & \textit{if} \quad d(t_i, t_j) > r - \alpha/2^m, \end{cases}$$

gives the minimum number of circles $M = n_{\alpha/2^m}$ and for all $j \geq 1$ we have $n_{\alpha/2^{m+j}} = M$. These M circles of radius $r - \alpha/2^m$ generate the k-covering of the set $T_{\Delta/2^m}$, and, moreover, for the radius r generate the k-covering of G.

Proof. Let Δ be chosen and T_Δ be built. If we put $\alpha = \Delta\sqrt{2}$ in (1) and solve (2)–(4) then we obtain the minimal number (n_α) of circles of radius $r - \alpha$ generating a k-covering of T_Δ. Note that if these n_α circles have a new radius r then they generate a k-covering of the set G.

Consecutively put $\Delta := \Delta/2^m$ ($\alpha := \alpha/2^m$), build $T_{\Delta/2^m}$, $m = 0, 1, 2, 3, \dots$, and solve (2)–(4), where substituting $\alpha/2^m$ for α in (1). As a result, we obtain the numbers $n_{\alpha/2^m}$ of the circles of radius $r - \alpha/2^m$ which form a k-covering of $T_{\Delta/2^m}$. From (5) we have

$$n_\alpha \geq n_{\alpha/2} \geq n_{\alpha/2^2} \geq n_{\alpha/2^3} \geq \dots \geq n_{\alpha/2^m} \geq \dots . \tag{6}$$

The sequence of integers (6) is nonincreasing and bounded below by a number $M^* > 0$. Hence, starting with some m, all $n_{\alpha/2^{m+j}}$ are the same for every $j \geq 1$; say $n_{\alpha/2^{m+j}} = M$. If we put the radius of these circles equal to r then they generate a k-covering of the set G. Theorem 1 is proved. □

Thus, to find an approximate solution to Problem P2, we do not need to keep making Δ smaller because, starting with some grid size ($\Delta := \Delta/2^m$), any further decreasing of Δ does not change the number of the circles sufficient for the covering of G. Then M is the least estimate for the number of circles of radius r which generate a k-covering of G and are obtained as a solution of (2)–(4).

Theorem 1 implies that Problem P2 can be solved by means of the 0–1 LP problem (2)–(4) with an appropriate choice of Δ ($\Delta := \Delta/2^m$). Obviously, by solving the indicated 0–1 LP problem for the chosen Δ we obtain an upper estimate for the number of circles for covering of G.

Consider, for example, the problem of covering a unit square with circles of radius 0.3. If we choose Δ successively equal to 0.1, 0.05, 0.025, and 0.0125 then the upper estimates for the number of circles for a covering (1-covering) are equal to 9, 8, 7, and 7 respectively. It is well known that the minimum possible number of circles equals 6.

The question of choosing Δ remains open. It is clear that Δ should be chosen sufficiently small, but this increases the dimension of the indicated problem. Equality of n_α and $n_{\alpha/2}$ can be a good test for appropriate choice of Δ. We had to choose Δ taking into account our computer's capacity to solve 0–1 LP problems. It is clear that in the general case we obtain an approximate solution of P2. Note that henceforth we give a lower estimate for the number of sensors under some condition.

We take the approximate solution of Problem P2 as the approximate solution of Problem P1.

2. INTRODUCTION OF CONSTRAINTS ON DISTANCES BETWEEN SENSORS

As was already mentioned, placing several sensors in the same point is undesirable; therefore, we introduce constraints on the minimal distance between sensors. Observe that in the statements of Problems P2 and P3 it is mentioned that at most one sensor can be placed at one point. In the mathematical model this is written in the form of condition (4): $z_i \in \{0, 1\}$, $1 \leq i \leq n$. Studying problems of k-covering of a set by the minimal number of circles, it is obvious that there cannot be more than k centers of circles at every point. If we consider the problem of covering by the minimal number of circles provided that the centers of the circles can coincide then in problem (2)–(4) condition (4) is replaced by

$$z_i \in \{0, 1, \dots, k\}, \qquad 1 \leq i \leq n. \tag{7}$$

To solve problem (2), (3), (7), we can use, for example, CPLEX library, but solving takes more time then. For example, when we used system (2), (3), (7) instead of (2)–(4) for the problem of 3-covering of a unit square with circles of radius 0.3, our calculations were approximately 1.5 times slower.

If the given minimal distance between sensors is greater than Δ then some additional conditions should be introduced. To this end, in [11], for example, a cumbersome mathematical model is built. We introduce this condition in the following way:

Suppose that the distance between sensors should be at least some λ. For of every two circles nearest to each other, all points of the line segment connecting their centers should be covered; therefore, the value of λ cannot be greater than $2r$, where r is the radius of the covering circles. Moreover, from the condition of 1-covering of an arbitrary G it is easy to find that $\lambda \leq r\sqrt{3}$. For k-coverings, $k > 1$, it is obvious that $\lambda \leq r$. Thus, for 1-covering we choose λ such that $0 < \lambda \leq r\sqrt{3}$ and for k-covering, $k > 1$, we take $0 < \lambda \leq r$. Depending on the covered domain G, various conditions on λ can appear. For example, for 3-covering of an equilateral triangle W with side c, it is easy to see that $\lambda \leq c$; otherwise the centers of the covering circles are outside W. We represent a general procedure for considering constraints on the distance between the centers of circles as follows [38]:

Suppose that for each point t_i in T there is p_i points t_j ($i \neq j$, $1 \leq j \leq n$) such that $d(t_i, t_j) < \lambda$. Define coefficients

$$b_{ij} = \begin{cases} 1, & \text{if } d(t_i, t_j) < \lambda \\ 0, & \text{if } d(t_i, t_j) \geq \lambda \end{cases}, \qquad i \neq j, \quad 1 \leq i, j \leq n,$$

$$b_{ii} = p_i, \qquad 1 \leq i \leq n.$$

Introduce the constraints:

$$\begin{aligned} &b_{11}z_1 + b_{12}z_2 + \cdots + b_{1n}z_n \leq p_1, \\ &b_{21}z_1 + b_{22}z_2 + \cdots + b_{2n}z_n \leq p_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &b_{n1}z_1 + b_{n2}z_2 + \cdots + b_{nn}z_n \leq p_n. \end{aligned} \tag{8}$$

Constraints (8) guarantee that the distance between sensors is at least λ. Therefore, by solving problem (2)–(4) with constraints (8), we can obtain an arrangement of the sensors taking the minimal distances between them into account. Assume that λ is such that problem (2)–(4), (8) has a solution.

The constructed problems are 0–1 LP problems and can be solved by an integer-valued LP problems solver. Decrease of Δ increases the dimension n of the problem under consideration and so the program execution time can become inappropriate. And as many other authors, we propose a possible version of a heuristic algorithm for the case of large n.

First of all, we observe that solving an LP problem takes less time than solving an integer LP problem. If in some 0–1 LP problem we replace the condition $z_i \in \{0, 1\}$, $1 \leq i \leq n$, by $0 \leq z_i \leq 1$, $1 \leq i \leq n$, then the obtained problem is called a *relaxed* LP problem.

3. A HEURISTIC ALGORITHM OF SOLVING PROBLEMS (2)–(4)

One of the heuristic ways of solving 0–1 LP problems is using the solutions of the relaxed LP problem (for instance, see [30, 39–42]). In some cases, for solving the original problem, only a relaxed problem is posed, while in others, some additional new 0–1 LP problem is considered along with the relaxed. By analogy with the above-cited articles, we use both the relaxed and the new 0–1 LP problems. Represent the procedure as the following

Algorithm 1

Step 1. For problem (2)–(4), we build the relaxed problem by taking constraints $0 \leq z_i$, $1 \leq i \leq n$, instead of assumption of integer values z_i, which is the only difference from the initial statement.

Step 2. Solve the relaxed LP problem. Suppose that the solution is found and $z_1^*, z_2^*, \ldots, z_n^*$ are the optimal values of the variables. If all z_i^*, $1 \leq i \leq n$, equal either 0 or 1 then the initial 0–1 LP problem is solved. If among z_i^* there are some nonintegers then go to Step 3.

Step 3. Arrange z_i^*, $1 \leq i \leq n$, in nonincreasing order. Choose some number q and take q first values from the ordered array z_i^*, $1 \leq i \leq n$. Among these q values there could be only $v < p$ different from zero. Add to these v nonzero values $q - v$ zero values z_i^*, $1 \leq i \leq n$, which are chosen randomly (by uniform distribution law). We act by analogy when it is necessary to choose one from several identical values. Suppose that the values $z_j^* : z_{m1}^*, z_{m2}^*, \ldots, z_{mq}^*$ are chosen in that way. Introduce the variables y_l, $1 \leq l \leq q$, and coefficients $b_{il} = a_{i,ml}$, where $1 \leq i \leq n$ and $1 \leq l \leq q$. Henceforth we build a new 0–1 LP problem, the so-called *core* problem:

$$\begin{gathered} y_1 + y_2 + \cdots + y_q \to \min \\ b_{11}y_1 + \cdots + b_{1q}y_q \geq k, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{n1}y_1 + \cdots + b_{n1}y_q \geq k, \\ y_i \in \{0, 1\}, \qquad 1 \leq i \leq q. \end{gathered}$$

Step 4. Solve the core problem. Suppose that the core problem is solved and the optimal values of all y_i are found: $y_1^*, y_2^*, \dots, y_q^*$. For every $y_l^* = 1$ let the respective z_{il}^* equal 1; and for all other y_l^* let $z_{il}^* = 0$. The obtained values are taken as the solution of (2)–(4).

The value of q is chosen so that the core problem can be solved by some precise method, without any heuristics or random procedures. In this article, Δ is mainly chosen equal to 0.01 under which the number of variables in the so-obtained 0–1 LP problems approximately equals 10000. In some cases we solved problems with 14000 variables choosing q equal to 300, by which we obtained the appropriate solutions.

Introducing constraints (8), problem (2)–(4), (8) was solved without a heuristic. We can solve it using the above-mentioned heuristic, but for construction of the core problem it is necessary to transform constraints (8) using new variables y_l, $1 \le l \le q$.

4. LOWER ESTIMATE FOR NUMBER OF SENSORS

Suppose that r, $\Delta > 0$, k $(1 \le k \le 4)$, and $\lambda = 3\Delta$ are chosen, $\beta = \Delta\sqrt{2}$ is defined and T_Δ is constructed.

Theorem 2. *Suppose that n_{opt} be the minimal number of circles of radius r which guarantees k-covering of the set G when the minimal distance between sensors at least λ, n_1 is the minimal number of circles of radius $r + \beta$ obtained as a solution to problem (2)–(4) for $\alpha = 0$ (and generating some k-covering of T_Δ). Then*

$$n_1 \le n_{\text{opt}}. \tag{9}$$

Proof. Assume that for the chosen set G there exists the minimal number n_{opt} of circles of radius r which guarantees the k-covering of G when the minimal distance between sensors is at least λ. Let the centers $m = n_{\text{opt}}$ $(m \ge 1)$ of the circles be placed at $c_1, \dots, c_m$, $c_i \in G$, $1 \le i \le m$. Since

$$\min\{d(c_i, c_j) \ge \lambda \mid 1 \le i, j \le n,\ i \ne j\},$$

all points $c_1, \dots, c_m$ are different. Put $C = \{c_1, \dots, c_m\}$.

Given Δ, we find T_Δ (independently of C). To find T_Δ, we build a grid on G of grid size Δ, and so, the distance from $c_i \in C$ to the nearest node of the grid is at most β. Suppose that by shift at most β from c_i we can reach some t from T_Δ. Then c_i belongs to the closed circle K of radius β with center t. Since the diameter of K is less than 3Δ, in K there are no other points from $C\backslash\{c_i\}$ reachable from t. Hence, to each point from T_Δ not coinciding with points from C, we can move at most one point from C. For the indicated translations the minimal possible distance between the centers of the circles can decrease, but will not be less than Δ.

Thus, if c_i does not coincide with any point of T_Δ then move c_i to the nearest point from T_Δ in which there are no points of C. The indicated translation is done by at most β. In addition to translations, we increase the radii of the circles by β. The circles of radius r with centers $c_1, \dots, c_m$ generate the k-covering of G. After translating the centers and increasing the radii by β, obviously, these m circles with centers T_Δ generate a k-covering of G and hence of T_Δ; moreover, the minimal distance between the centers of circles is at least Δ. Given Δ and T_Δ, solve problem (2)–(4) for the radii $r + \beta$ of the circles (for $\alpha = 0$). In result, we obtain the minimal number n_1 of the circles of radius $r + \beta$ generating a k-covering of T_Δ. Thus, we arrive at (9). The proof of Theorem 2 is complete. $\square$

The value of n_{opt} is unknown yet, although, n_1 is found as a solution of (2)–(4). The value of n_1 obtained in (9) depends on Δ. In the same way as in proof of Theorem 1 we can show that, starting at some Δ, further decrease of Δ has no effect on the value of n_1.

We obtain the following procedure of determining the approximate lower estimate for the number of circles for k-covering $(1 \le k \le 4)$ of the given domain G, provided that the minimal possible distance between the centers of circles is not less than 3Δ: Solve problem (2)–(4) for radii $r + \beta$, where coefficients a_{ij} are determined by (1) for $\alpha = 0$. The number of nonzero z_i provides the lower estimate for the number of circles for the indicated k-covering of G.

For covering a unit square with circles of radius 0.3, we choose Δ equal to 0.1, 0.05, 0.025, and 0.0125 consecutively. Then the lower estimates for the numbers of circles for 1-coverings are equal to 4, 5, 5, and 6 respectively. It is well known that the minimal possible number of circles is equal to 6.

Consider the case with constraints (8) when $\lambda \geqslant 3\Delta$. Assume that there exists the least number M^* of circles of radius r guaranteeing a k-covering of G such that the least distance between the centers of circles is not less than λ. Choose Δ and construct T_Δ. The centers of the circles that are not lying in points from T_Δ we move to the nearest points from T_Δ not containing a center of a covering circle; let this translation value be at most $\beta = \Delta\sqrt{2}$. Increase the radii of the circles by β. Then M^* circles of radius $r + \beta$ guarantee a k-covering of G, and thus a k-covering of T_Δ. After the indicated translations, conditions (8) hold for some $\lambda^* \in [\lambda - 2\beta, \lambda + 2\beta]$. If we solve problem (2)–(4), (8) with λ replaced by λ^* then we obtain number $n_1(\lambda^*)$ of the circles such that

$$n_1(\lambda^*) \le M^* \tag{10}$$

In problem (2)–(4), (8) we minimize the number of circles, and hence, constraint (8) can only increase the number of circles. To obtain the lower estimate of the number of circles, we choose $\lambda^* = \lambda - 2\beta$. Thus, (10) gives an estimate on the unknown number M^*.

5. NUMERICAL RESULTS

In many articles (for instance, see [19, 21–29]) we can find the number of circles needed for 1-covering of some figures, for example a square, triangle, and circle, with circles of the least possible radii. As for k-covering, $k \ge 2$, only few results are known [12, 13, 15, 16, 30, 31]. We now obtain estimates for the number of circles for k-covering $(1 \le k \le 4)$ of an arbitrary set G and, moreover, can provide some constraints on the minimal distances between circles.

For numerical implementation, we took a unit square, a rectangle with sides 1.22 and 0.82, and a circle of radius 0.5642 as domains for covering. The sizes are chosen so that the areas of the figures are almost the same. Henceforth symbol / is used as a separator. The minimal admissible distance λ between the centers of circles is given by constraints (8) for $\lambda > \Delta$ and by constraints (4) for $\lambda = \Delta$. The obtained numbers of circles coincide with their upper estimates; therefore, we do not mention the upper estimates.

In the table we give the numbers of circles and their approximate lower estimates for the square, rectangle, and circle for the chosen values of radii r of the covering circles and the multiplicity of covering k equal to 1, 2, and 3. In the upper row (for the chosen radius r), the lower estimate for the number of circles for $\lambda \geqslant 3\Delta$ and then the obtained number of circles for $\lambda \ge \Delta$ are presented. In the lower row (for the chosen r) in parentheses, the lower estimate for the number of circles for $\lambda \ge r/2$ and then the obtained number of circles for $\lambda \ge r/2$ are written.

The table demonstrates that for some coverings the obtained number of circles coincides with its lower estimate, hence these results are unimprovable. For other coverings, the difference between the estimate and the obtained number is small.

Let us discuss the results for 1-covering of the square. It follows from [17, 27–29] that for radii of circles equal to 0.5, 0.45, 0.4, 0.35, and 0.25, the minimal possible number of circles for covering of the square is equal to 4, 4, 4, 5, and 9 respectively, which coincide with the same numbers obtained in this article. In this article, for radius 0.3, the number of circles equals 7, although a square can be covered by 6 circles. Choosing Δ less than 0.01, the number of circles could be decreased by 1, but our computer does not cope with this. The results in the table turn out to be appropriate since the given approach allows us to find the necessary number of circles for an arbitrary bounded convex set G rather than only for some domains.

There are a few articles in which distances between sensors are taken into account (see [5, 11]). Such constraints can be introduced to consider effect of sensors on each other or for some other reason. Introduction of constraints on the centers of circles in some cases does not change the number of covering circles, while in others the number of circles and their arrangement change. For example, the 3-covering of the square with circles of radius 0.25 takes 27 circles, whereas after the introduction of constraint (8) $\lambda = r/2$ we need 28 circles. Similarly, the 2-coverage of the rectangle with circles of radius 0.25 requires 18 circles, while it takes 19 circles after the introduction of condition (8) $\lambda = r/2$. For the

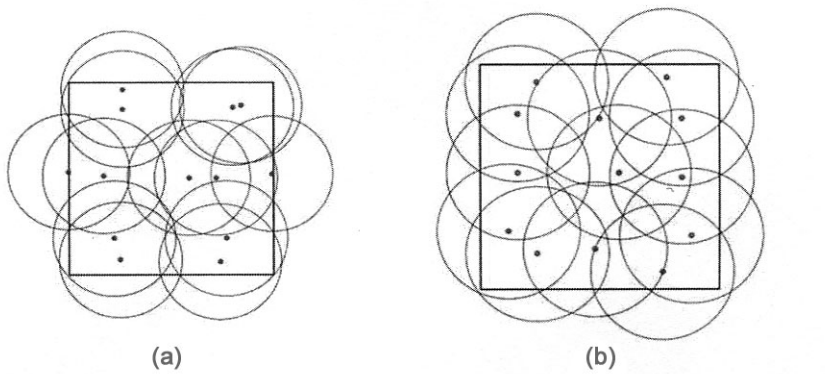

Fig. 1. 2-Coverings of the square with circles of radius 0.3 without and with constraints on the distance between the centers of circles.

cases when the number of circles does not coincide with its lower estimates, the difference between them is small.

The table confirms that, depending on the type of figure, estimates for the number of circles vary, though inessentially since the areas of the figures are almost the same. For example, the 2-covering of the square with circles of radius 0.3 is performed with 13 circles, while for the rectangle, it takes 12 circles.

We obtain the arrangement of circles on the chosen domain solving either problem (2)–(4) or problem (2)–(4), (8) under given radius of the circles and given value λ for (8). In Figs. 1–3 some examples are shown of 2-coverings with circles of radius 0.3 in the cases of the square 1×1, the

Numbers of circles (sensors) and their estimates for covering of a square, a rectangle, and a circle under given values of radii r of circles (zones of monitoring by sensors)

	Multiplicity of covering (k)								
	The square 1×1			The rectangle 1.22×0.82			The circle of radius 0.5642		
r	1	2	3	1	2	3	1	2	3
0.5	3/4 (3/4)	6/8 (6/8)	10/12 (10/12)	3/3 (3/3)	6/6 (6/6)	9/9 (9/10)	3/3 (3/3)	6/6 (6/6)	9/9 (9/10)
0.45	4/4 (4/4)	8/8 (8/8)	12/12 (12/12)	3/4 (3/4)	7/8 (7/8)	10/12 (10/12)	4/4 (4/4)	7/7 (7/7)	11/11 (10/11)
0.4	4/4 (4/4)	8/8 (8/8)	12/12 (12/12)	4/4 (4/4)	8/8 (8/8)	12/12 (12/13)	4/4 (4/5)	8/8 (8/9)	12/12 (12/13)
0.35	4/5 (4/5)	8/10 (8/10)	12/15 (13/16)	4/5 (4/5)	9/10 (9/10)	15/15 (15/16)	5/5 (5/5)	10/10 (9/11)	14/14 (14/16)
0.3	6/7 (6/7)	12/13 (12/14)	17/21 (17/21)	6/6 (6/6)	12/12 (12/12)	18/18 (18/19)	7/7 (6/7)	12/14 (12/14)	18/19 (18/21)
0.25	9/9 (9/9)	17/18 (17/18)	25/27 (25/28)	8/9 (8/9)	16/18 (16/19)	22/26 (23/28)	8/8 (8/9)	15/15 (15/18)	23/24 (23/28)

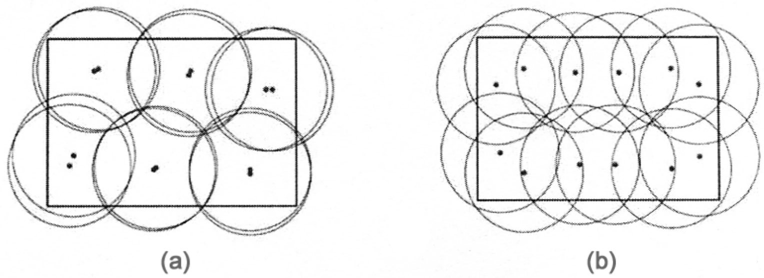

Fig. 2. 2-Coverings of the rectangle with circles of radius 0.3 without and with constraints on the distance between the centers of circles.

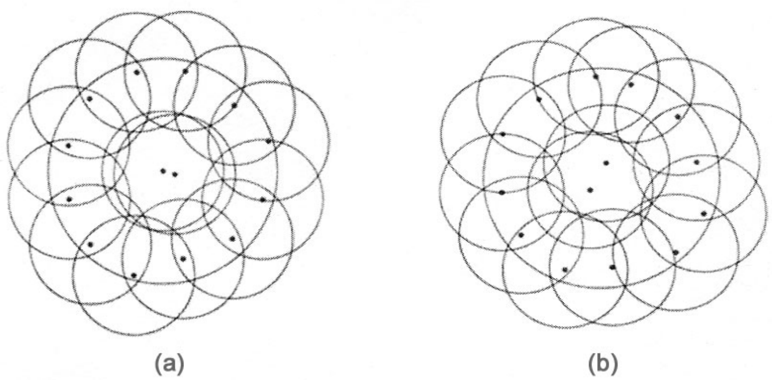

Fig. 3. 2-Coverings of the circle with circles of radius 0.3 without and with constraints on the distance between the centers of circles.

rectangle 1.22×0.82, and the circle of radius 0.5642 without restrictions (8) (see the left plots (a)) and with restrictions (8) (the right plots (b)) for $\lambda = 0.5r$, where r is the radius of the covering circles, $r = 0.3$.

The numerical implementation was carried out by CPLEX−12.6.3 library on Intel Core i7−4790K, RAM 8 GB, Windows 10x64 computer. To obtain the data in the table, the computations took time from several seconds to several minutes. For covering of the square, for example, it took approximately 10 seconds to obtain the estimate for the number of circles with $r = 0.3$ and $k = 2$. Since these results must be computed a priori rather than on-line, here we do not give the computational time for each case.

CONCLUSION

In this article, we presented a mathematical model and method for determining the number of circles (sensors), their arrangement, and the approximate lower estimates for k-covering $(1 \leq k \leq 4)$ of an arbitrary bounded convex domain G with nonempty interior on the plane. We found out that for construction of grids with grid size Δ on G (discretization of the problem) there is no need to decrease Δ infinitely because there exists Δ after which the results do not change.

We presented an effective and simple way of taking the minimal possible distance between the centers of the covering circles (sensors) into account. We gave the approximate lower estimates for the numbers of circles for k-coverings of the given domains provided that such covering exists. All results are obtained for k-covering of a sufficiently arbitrary bounded domain. The numerical results demonstrate effectiveness of the method and, for the known cases, coincide with those published earlier. Our method is based on solving of 0−1 LP problems and can be extended on the case of k-covering in the three-dimensional space.

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