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On the Complexity of the Vertex 3-Coloring Problem for the Hereditary Graph Classes With Forbidden Subgraphs of Small Size

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Abstract—The 3-coloring problem for a given graph consists in verifying whether it is possible to divide the vertex set of the graph into three subsets of pairwise nonadjacent vertices. A complete complexity classification is known for this problem for the hereditary classes defined by triples of forbidden induced subgraphs, each on at most 5 vertices. In this article, the quadruples of forbidden induced subgraphs is under consideration, each on at most 5 vertices. For all but three corresponding hereditary classes, the computational status of the 3-coloring problem is determined. Considering two of the remaining three classes, we prove their polynomial equivalence and polynomial reducibility to the third class.

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INTRODUCTION

A *regular vertex coloring* of a graph G is a mapping $c\colon V(G) \to \mathbb{N}$ such that $c(v_1) \neq c(v_2)$ for all adjacent vertices $v_1, v_2 \in V(G)$. A regular vertex coloring c of G is called a k-*coloring* if $c\colon V(G) \to \overline{1,k}$. If G has a k-coloring then G is called k-*colorable*. The *chromatic number* of G is the least k such that G is k-colorable. This number is denoted by $\chi(G)$.

The *vertex coloring problem* for G and k given consists in determining whether $\chi(G) \leq k$ or not. The *vertex* k-*coloring problem* (briefly, *Problem* k-VC) for a given graph G consists in determining whether $\chi(G) \leq k$ or not. Both problems are classical NP-complete problems on graphs.

A graph H is a *subgraph* of G if H can be obtained from G by removing vertices and edges. A graph H is called an *induced subgraph* of G if H can be obtained from G by removing only vertices. A *graph class* is a set of graphs closed under isomorphism. A graph class is called *hereditary* if it is closed under vertex removal. A *strongly hereditary* graph class is a herediatary graph class closed also under edge removal. As is known, each hereditary graph class $\mathcal{X}$ can be defined by the set of its forbidden induced subgraphs $\mathcal{Y}$, which is written as $\mathcal{X} = Free(\mathcal{Y})$. A strongly hereditary graph class $\mathcal{X}$ can be defined by the set of its forbidden subgraphs $\mathcal{Y}$, which is written as $\mathcal{X} = Free_s(\mathcal{Y})$. If a hereditary class can be defined by a finite set of its forbidden subgraphs then it is called *finitely defined*.

A hereditary graph class with polynomially solvable Problem 3-VC will be called 3-VC-*simple*. A hereditary graph class with NP-hard Problem 3-VC will be called 3-VC-*hard*.

The vertex coloring problem is polynomially solvable for $Free(\{H\})$ if H is an induced subgraph of the graph P_4 or of the graph $P_3 + K_1$; otherwise, it is NP-complete in this class (see [1]). But, with two forbidden induced subgraphs, complete classification is no longer possible. For example, for all but three hereditary classes with forbidden subgraphs at most 4 vertices each, the computational status

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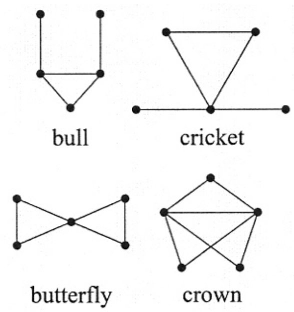

Fig. 1. The bull, cricket, butterfly, and crown graphs.

of the vertex coloring problem is known (see [2]). For the remaining three cases, this status is unknown, but for them it is possible to construct polynomial approximate algorithms [3]. Some recent results on the complexity of the vertex coloring problem in hereditary classes with forbidden subgraphs of small size are presented in [4–10].

For Problem k-VC, the complexity status remains open even for some classes with a single forbidden induced subgraph. The computational complexity of Problem 3-VC is known for all classes of the form $Free(\{H\})$ with $|V(H)| \leq 6$ (see [11]). An analogous result was obtained for Problem 4-VC and all classes of the form $Free(\{H\})$, where $|V(H)| \leq 5$ [12]. For each fixed k, Problem k-VC is solvable in polynomial time in the class $Free(\{P_5\})$ (see [13]). Problem 3-VC is polynomially solvable in the class $Free(\{P_7\})$ (see [14]). For each fixed $k \geq 5$, Problem k-VC is NP-complete in the class $Free(\{P_6\})$ (see [15]). Problem 4-VC is NP-complete in the class $Free(\{P_7\})$ (see [15]). The computational status of Problem k-VC is open for the class $Free(\{P_8\})$ and $k = 3$ and also for the class $Free(\{P_6\})$ and $k = 4$.

There are many "white spots" on the "map" of the computational complexity of the vertex coloring problem and the vertex k-coloring problem in the family of hereditary classes. There are two ways of reducing the number of these "white spots." The first is increasing the number of forbidden induced subgraphs, and the second is increasing the size of such subgraphs. A constraint on the size or number of forbidden induced subgraphs forms a subfamily of the family of hereditary classes of graphs. A possible reduction of the family of "white spots" consists in obtaining a complete complexity dichotomy for larger values of this bound.

In this article, we consider Problem 3-VC. In [16], a complete complexity dichotomy for this problem was obtained in the family of hereditary classes with a pair of forbidden induced subgraphs each of which has at most 5 vertices. In [17], a similar result was obtained for all triples of forbidden subgraphs each of which has at most 5 vertices. In this article, we consider hereditary classes with a quadruple of forbidden induced subgraphs each of which has at most 5 vertices and also, for all but three such classes, we establish the computational status of Problem 3-VC. For two of the three remaining cases, the polynomial equivalence and the polynomial reducibility to the third case are proved.

1. NOTATIONS

$N(x)$ stands for the neighborhood of a vertex x, $\deg(x)$ is the degree of x, and $\Delta(G)$ is the maximal vertex degree of a graph G.

Let P_n, C_n, K_n, and O_n denote the simple path, the simple cycle, the complete and the empty graphs on n vertices respectively. The symbol $K_{p,q}$ designates the complete bipartite graph with p vertices in one part and q vertices in the other.

Let F_k ($k \geq 3$) denote the graph that is obtained by adding to a simple path $(x_1, \ldots, x_k)$ a vertex x and edges $xx_1, xx_2, \ldots, xx_k$. The *diamond* graph is isomorphic to F_3. The *wheel* W_k ($k \geq 3$) is a graph obtained by adding a vertex x and edges xx_1, xx_2, …, xx_k to a cycle $(x_1, \ldots, x_k)$. The *odd wheel* is a member of $\{W_3, W_5, W_7, \ldots\}$.

Figs. 1 and 2 display the bull, cricket, butterfly, and crown graphs, and also spindle, kite, dart, banner, house, and sun graphs.

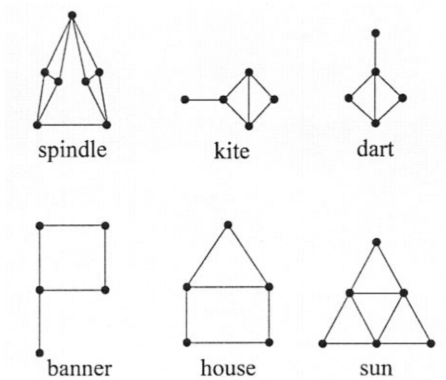

Fig. 2. The spindle, kite, dart, banner, house, and sun graphs.

Let G be a graph and let $V' \subseteq V(G)$. Then $G[V']$ is the subgraph in G induced by the subset of vertices V' and $G \setminus V'$ is the result of the removal from G of all elements of V' (together with all incident edges). Let $G_1 + G_2$ be the disjoint union of G_1 and G_2 with disjoint vertex sets. Designate the disjoint union of k copies of G as kG, and denote the complementary graph to $W_4 + K_1$ by $\overline{W_4 + K_1}$.

2. THE NP-COMPLETENESS OF THE 3-COLORING PROBLEM IN SOME GRAPH CLASSES WITH FORBIDDEN SUBGRAPHS HAVING FEW VERTICES

The following six graph classes are considered in [17, Section 2]:

- $\mathcal{X}_1^*$ is the set of all forests,
- $\mathcal{X}_2^*$ is the set of edge graphs of subcubic forests,
- $\mathcal{X}_3^*$ is the set of graphs in which every 5 vertices induce a subgraph of $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the cricket, the kite, the diamond $+ K_1\}$,
- $\mathcal{X}_4^*$ is the set of graphs in which every 5 vertices induce a subgraph of $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the kite, the diamond $+ K_1$, the butterfly, the crown$\}$,
- $\mathcal{X}_5^*$ is the set of graphs in which every 5 vertices induce a subgraph of $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the kite, the diamond $+ K_1$, the house, $C_4 + K_1, F_4, W_4$, the dart, the crown$\}$,
- $\mathcal{X}_6^*$ is the set of graphs in which every 5 vertices induce a subgraph of $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the cricket, the house, the banner, $C_4 + K_1, C_5\}$.

It was shown in [17, Lemma 4] that each of the graph classes $\mathcal{X}_3^*$–$\mathcal{X}_6^*$ is 3-VC-hard. In what follows we will prove the NP-completeness of Problem 3-VC for three more graph classes with forbidden subgraphs each of which has at most 5 vertices. To this end, consider the graphs G_1, G_2, and G_3 depicted in Figs. 3 and 4.

Lemma 1. *G_1 is 3-colorable, and in each 3-coloring of G_1, the vertices u_3 and u_4 have the same color.*

Proof. Color a_1, a_2, a_3, a_4, and v_1 with the first color; b_1, b_3, b_4, v_2, and u_2, with the second color; and b_2, c_1, c_2, v_3, u_1, u_3, and u_4, with the third color. We so obtain the 3-coloring of G_1; therefore, G_1 is 3-colorable.

Prove that, in every 3-coloring of G_1, the vertices u_3 and u_4 have identical colors. Consider some 3-coloring c of G_1. Suppose that $c(c_1) = 1$ and $c(v_3) = 2$. Then $c(v_2) = 3$ and $c(v_1) = 1$. Since $c(u_1) \neq c(u_2)$, we have $c(c_2) \neq 1$, and so $c(c_2) = 2$.

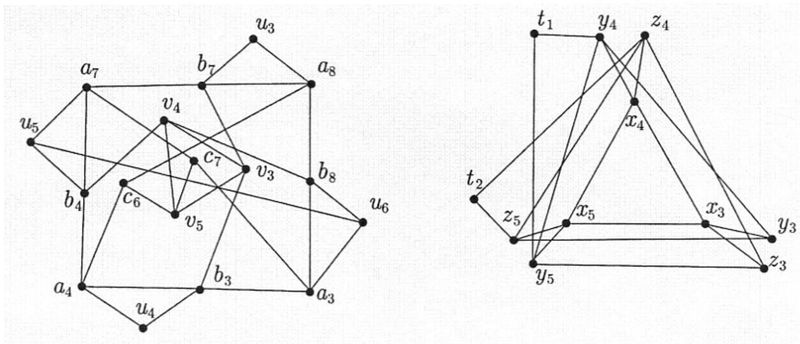

Fig. 3. The graphs G_1 and G_2.

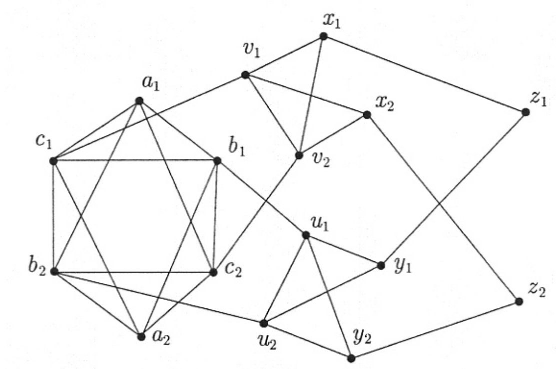

Fig. 4. The graph G_3.

Let $c(b_1) = 1$. Then, necessarily,

$$c(a_1) = 3, \qquad c(b_4) = 2, \qquad c(a_4) = 3, \qquad c(b_3) = 1, \qquad c(a_3) = 3, \qquad c(b_2) = 2.,$$

Therefore, $c(u_1) = c(u_2) = 1$; a contradiction.

Let $c(b_1) = 3$. Then

$$c(a_2) = 2, \qquad c(b_2) = 3, \qquad c(a_3) = 1, \qquad c(b_3) = 3,$$
$$c(a_4) = 2, \qquad c(b_4) = 3, \qquad c(a_1) = 1.$$

Therefore, $c(u_1) = c(u_2) = 2$; a contradiction.

If $c(c_1) = c(v_3)$ then also $c(c_1) = c(v_3) = c(u_3) = c(u_4)$. Lemma 1 is proved. □

Lemma 2. *The graph G_2 is 3-colorable, and, in each 3-coloring of G_2, the vertices t_1 and t_2 have the same color.*

Proof. Color x_3, y_1, and z_1 with the first color; x_1, y_2, and z_2, with the second; and x_2, y_3, z_3, t_1, and t_2, with the third color. Such a coloring is a 3-coloring of G_2; therefore, G_2 is 3-colorable.

Prove that, in every 3-coloring of G_2, the vertices t_1 and t_2 have identical colors. Consider some 3-coloring c of G_2. Suppose that there exists i such that $c(y_i) \neq c(z_i)$. We may assume that $i = 1$ and $c(y_1) = 1$ while $c(z_1) = 2$. Owing to the presence of the edges x_1x_2 and x_1x_3, none of the sets $\{c(y_2), c(z_2)\}$ and $\{c(y_3), c(z_3)\}$ coincides with $\{1, 2\}$. If at least one of the sets is a singleton then the color in it must be the third, and the other set must coincide with $\{1, 2\}$, which is impossible. If $\{c(y_2), c(z_2)\} = \{1, 3\}$ then $c(z_2) = 1$, $c(y_2) = 3$ and $c(x_1) = 3$, $c(x_2) = 2$, and $c(x_3) = 1$; therefore, each of the three colors is forbidden for z_3. If $\{c(y_2), c(z_2)\} = \{2, 3\}$ then $c(y_2) = 2$, $c(z_2) = 3$, and

$c(x_1) = 3$, $c(x_2) = 1$, and $c(x_3) = 2$; thus, each of the three colors is forbidden for y_3. Hence, we may assume that

$$c(y_1) = c(z_1) = 1, \qquad c(y_2) = c(z_2) = 2, \qquad c(y_3) = c(z_3) = 3;$$

therefore, $c(t_1) = c(t_2) = 3$. Lemma 2 is proved. □

Lemma 3. *The graph G_3 is 3-colorable, and, in each 3-coloring of G_3, the vertices a_1, a_2, z_1, and z_2 have the same color.*

Proof. Color the vertices a_1, a_2, v_1, u_1, z_1, and z_2 with the first color; the vertices b_1, b_2, v_2, y_1, and y_2, with the second color; and the vertices c_1, c_2, u_2, x_1, and x_2, with the third color. This yields a 3-coloring of G_3; therefore, G_3 is 3-colorable.

Prove that, in each 3-coloring of G_3, the vertices a_1, a_2, z_1, and z_2 have the same color. Indeed, it is not hard to verify that the graph $G_3[\{a_1, a_2, b_1, b_2, c_1, c_2\}]$ has a unique 3-coloring (up to a permutation of colors) in which the vertices a_1 and a_2 have the first color, b_1 and b_2 have the second color, and c_1 and c_2 have the third color. Then, in every 3-coloring of G_3, the vertices y_1 and y_2 have the second color, and x_1 and x_2 have the third color. Hence, the vertices z_1 and z_2 have the first color.

This completes the proof of Lemma 3. □

Let G be an arbitrary graph and let x be a vertex of G whose neighborhood is consists of the vertices v_1, v_2, v_3, and v_4. The operation of *G_i-bypass* consists in removing x from G, adding the graph G_i and the edges $v_1u_3,\ v_2u_3,\ v_3u_4,\ v_4u_4$ (if $i = 1$) or the edges $v_1t_1,\ v_2t_1,\ v_3t_2,\ v_4t_2$ (if $i = 2$) or the edges $v_1a_1,\ v_2a_2,\ v_3z_1,\ v_4z_2$ (if $i = 3$).

By Lemmas 1–3, the so-obtained graph is 3-colorable if and only if G is 3-colorable.

Define the following three graph classes:

- $\mathcal{X}_7^*$ is the set of graphs in which every 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the cricket, $C_5\}$,
- $\mathcal{X}_8^*$ is the set of graphs in which every 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the cricket, the banner, the house,$C_4 + K_1\}$,
- $\mathcal{X}_9^*$ is the set of graphs in which every 5 vertices induce a subgraph in $\mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \{$the kite, the diamond $+ K_1$, the dart, $C_4 + K_1$, the banner, $W_4, C_5\}$.

Each of the classes $\mathcal{X}_7^*$–$\mathcal{X}_9^*$ is hereditary.

Lemma 4. *Each of the graph classes $\mathcal{X}_7^*$–$\mathcal{X}_9^*$ is* 3-VC-*hard.*

Proof. Problem 3-VC is NP-complete in the class $\mathcal{Y}$ of connected graphs in which the degree of each vertex is equal to 4 (see [18]). Let $G \in \mathcal{Y}$. Choose $i \in \overline{1,3}$ and simultaneously apply G_i-bypass to each of the vertices of G. Denote the so-obtained graph by G_i'. The graph G_i' is 3-colorable if and only if such is G_i by Lemmas 1–3.

It is not hard to see that $G_i' \in \mathcal{X}_{i+6}^*$. Indeed, let H_i be a 5-vertex induced subgraph in G_i'. If it is disconnected then $H_i \in \mathcal{X}_1^* \cup \mathcal{X}_2^*$ or $i = 2$, $H_2 = C_4 + K_1$, or $i = 3$, $H_3 =$ diamond $+ K_1$. In the last two cases, we have $H_i \in \mathcal{X}_{i+6}^*$. If H_i is an induced subgraph in G_i then $H_i \in \mathcal{X}_{i+6}^*$. Suppose that H_i is connected but not an induced subgraph in G_i. Then one or two vertices in H_i belong to the same copy of the induced subgraph G_i of G_i', and four or three belong to another copy; therefore, $H_i \in \mathcal{X}_1^*$.

Thus, Problem 3-VC in the class $\mathcal{Y}$ is polynomially reducible to the same problem in each of the classes $\mathcal{X}_7^*$–$\mathcal{X}_9^*$. Hence, each of the graph classes $\mathcal{X}_7^*$–$\mathcal{X}_9^*$ is 3-VC-hard.

Lemma 4 is proved. □

3. SOME RESULTS CONNECTED WITH THE POLYNOMIAL REDUCIBILITY AND POLYNOMIAL SOLVABILITY OF THE 3-COLORING PROBLEM

In [17] the notion of an irreducible graph was introduced. A graph G is *irreducible* if the following are fulfilled simultaneously:

(1) G is connected and contains no vertices x and y such that $N(y) \subseteq N(x)$,

(2) G has no joints,

(3) G has no odd wheel as an induced subgraph,

(4) G does not include spindle as a subgraph,

(5) $\Delta(G) \geq 4$, and G has no vertices of degree at most 2.

It was shown in [17, Lemma 5] that for an arbitrary hereditary class $\mathcal{X}$ Problem 3-VC is polynomially reducible to the same problem for the family of reduced graphs in $\mathcal{X}$.

Lemma 5. *If* $G \in Free(\{K_{1,4}, W_3, W_4, W_5,$ the butterfly, the cricket$\})$ *then* $\Delta(G) \leq 4$. *Moreover, if* $\deg(x) = 4$ *then* $G[N(x)] \in \{K_{1,3}, P_3 + K_1, P_4\}$.

Proof. Let x^* be a vertex of maximal degree in G. Suppose that $G[N(x^*)]$ has a connected component G^* with at least four vertices. Since

$$G \in Free(\{W_3,\ W_4,\ W_5,\ \text{the butterfly}\}),$$

G^* is a tree of diameter at most 3. Clearly, $\Delta(G^*) \leq 3$. If $\Delta(G^*) = 2$ then $G^* = P_4$. If $\Delta(G^*) = 3$ and $G^* \neq K_{1,3}$ then G^* has an induced subgraph $K_2 + 2K_1$; therefore, $G \notin Free(\{\text{the cricket}\})$. Thus, $G^* = K_{1,3}$ if $\Delta(G^*) = 3$.

Since $G \in Free(\{K_{1,4}, \text{cricket}\})$, the graph $G[N(x^*)]$ has at most three connected components. Moreover, if there are exactly three of them, $G[N(x^*)]$ is empty. If there are exactly two such components then one of them is K_1 since $G \in Free(\{\text{butterfly}\})$. Suppose that $G[N(x^*)] = H + K_1$, where H is connected and $|V(H)| \geq 3$. Since $G \in Free(\{K_{1,4}, W_3, \text{the cricket}\})$, the last paragraph implies that the graph H must have exactly three vertices, which implies that $H = P_3$. Hence, $\Delta(G) \leq 4$.

If $\deg(x) = 4$ then

$$G[N(x)] \in \{K_{1,3}, P_3 + K_1, P_4\}.$$

This follows from the arguments of the previous paragraphs. Lemma 5 is proved. □

Lemma 6. *Problem* 3-VC *in the class*

$$Free(\{K_{1,4}, \text{the butterfly, the cricket, the dart}\})$$

is polynomially reducible to the same problem for the class

$$Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket, the dart}\}).$$

Proof. Obviously, the neighborhood of every vertex of a graph in $Free(\{K_{1,4}, W_4\})$ induces a subgraph in $Free(\{K_3, O_4\})$. By the Ramsey Theorem, this subgraph contains at most 8 vertices.

Let G be an irreducible graph of class $Free(\{K_{1,4}, \text{the butterfly, the cricket, the dart}\})$ containing an induced subgraph W_4. Denote the vertices of this subgraph by v, v_1, v_2, v_3, and v_4, where $C = (v_1, v_2, v_3, v_4)$ is the induced 4-cycle. We assume that the set of the vertices of G situated at distance 3 from v is nonempty. Otherwise, G contains at most $1 + 8 + 8 \cdot 7 + 8 \cdot 7^2$ vertices since this graph is connected.

Since $G \in Free(\{\text{the butterfly, the cricket}\})$, all but possibly one elements of $N(v)$ belong to

$$\widehat{V} = \bigcup_{j=1}^{4} N(v_j).$$

Let us prove that there exists an induced path of length 3 starting at the vertex v and passing through the vertices of C. Suppose the contrary. Then the induced path (v, a, b, c) contains no elements of $V(C)$. By assumption, each neighbor of every element of $\widehat{V} \setminus N(v)$ belongs to $\widehat{V} \cup \{a\}$. If a vertex $a' \notin \{v_2, v_4\}$ is a common neighbor of v and v_1 then a' must be adjacent to v_3 and simultaneously not adjacent to any of the vertices v_2 and v_4 since $G \in Free(\{W_3, \text{the dart}\})$. For the same reasons, each neighbor of a' belongs to $\widehat{V} \cup \{a\}$. Hence, the vertex a is a joint of G; therefore, G is not irreducible.

Consider an induced path (v, a_1, b_1, c_1) in which $a_1 \in V(C)$ and $c_1 \notin \widehat{V}$. Without loss of generality, we may assume that $a_1 = v_1$. Since G is irreducible and belongs to $Free(\{\text{the dart}\})$; therefore, the vertex b_1 is adjacent exactly to two vertices of the cycle C which are neighboring. This is easy to see by exhausting all cases of intersection of $N(b_1)$ and $V(C)$: one vertex, two nonadjacent vertices, and two adjacent vertices respectively. We may assume that $b_1v_2 \in E(G)$. If b_1 has a neighbor $c' \notin \{v_1, v_2, c_1\}$; then

$$c' \in N(v_1) \otimes N(v_2), \quad \text{since} \quad G \in Free(\{W_3, \text{ the butterfly, the cricket}\}).$$

By symmetry, it suffices to consider the case when $c' \in N(v_1) \setminus N(v_2)$. Since $G \in Free(\{W_3, W_5\})$, we have $c'v_4 \notin E(G)$. Then the vertices v_1, v_2, v_4, b_1, and c' induce dart.

Suppose that $b_2 \in \widehat{V} \setminus (V(W_4) \cup \{b_1\})$. The vertex b_2 cannot have exactly one neighbor on the cycle C because $G \in Free(\{\text{the dart}\})$. If

$$N(b_2) \cap V(C) \in \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_1, v_2\}\}$$

then $b_1b_2 \in E(G)$ since $G \in Free(\{\text{the butterfly, the cricket}\})$; but then $G \notin Free(\{W_3, W_5\})$. Let $N(b_2) \cap V(C) = \{v_1, v_3\}$. If $b_2b_1 \in E(G)$ then v_1, v_2, v_4, b_1, and b_2 induce dart. If $b_2b_1 \notin E(G)$ then either v_1, v_2, v_4, v, and b_2 (if $b_2v \notin E(G)$) or v_1, v_4, v, b_1, and b_2 induce dart (if $b_2v \in E(G)$). The case of $N(b_2) \cap V(C) = \{v_2, v_4\}$ is considered by analogy. In all cases, when $|V(C) \cap N(b_2)| \geq 3$, we have $b_2v \notin E(G)$ and $b_1b_2 \notin E(G)$ since $G \in Free(\{W_3, W_5\})$. Then G contains an induced subgraph dart. Thus, every element of the set $\widehat{V} \setminus (V(W_4) \cup \{b_1\})$ is adjacent in C only to v_3 and v_4.

Thus, $N(b_2) \cap V(C) = \{v_3, v_4\}$. Moreover, by the previous arguments and the fact that $G \in Free(\{W_3, \text{the cricket}\})$, we infer that $N(v_3) \cap N(v_4) = \{b_2, v\}$.

Since G is spindle$_s$-free, we have $b_1b_2 \notin E(G)$. If there exists a vertex $c_2 \in N(b_2) \setminus \{v_3, v_4\}$ then $c_2v \notin E(G)$; otherwise, $G \in Free(\{\text{the dart}\})$. Thus, $c_2 \notin \widehat{V} \cup N(v)$ and $\deg(b_2) = 3$ since

$$G \in Free(\{\text{the butterfly, the cricket}\}).$$

Hence,

$$\deg(b_1) = 3, \qquad \deg(b_2) \leq 3, \qquad \deg(v) \leq 5.$$

Remove from the graph G all vertices of $V(C)$, add to it vertices w_1 and w_2 and also the edges w_1w_2, w_1b_1, w_2b_1, w_1v, w_2v, w_1b_2, and w_2b_2. Denote the so-obtained graph by G^*. It is not hard to see that G is 3-colorable if and only if so is G^*. Moreover,

$$G^* \in Free(\{K_{1,4}, \text{the butterfly, the cricket, the dart}\}).$$

If $\widehat{V} = V(W_4) \cup \{b_1\}$ then, by analogy, we can carry a reduction in which the edges w_1b_2 and w_2b_2 are not reduced. Applying the transformation an appropriate number of times, in result we obtain a graph from $Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket, the dart}\})$.

The proof of Lemma 6 is complete. □

Lemma 7. *Problem* 3-VC *in the class* $Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket}\})$ *is polynomially reducible to the same problem in the class* $Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket, the crown}\})$. *This is also true for the classes*

$$Free(\{K_{1,4}, W_4, \text{ the butterfly, the cricket, the dart}\}),$$
$$Free(\{K_{1,4}, W_4, \text{ the butterfly, the cricket, the crown, the dart}\}).$$

Proof. Suppose that an irreducible graph G of class $Free(\{K_{1,4}, W_4,$ the butterfly, the cricket$\})$ contains an induced subgraph crown whose vertices of degree 2 we denote by x_1, x_2, and x_3. Lemma 5 implies that each of the vertices x_1, x_2, and x_3 has degree at most 3 in G. In G, contract the subgraph under consideration to a vertex x and denote the result of this contraction by G^*. Obviously, the graph G^* is 3-colorable if and only if so is G. It is also obvious that the degree of the vertex x in G^* is at most 3. If this degree is at most 2 or x is a vertex of degree 3 in the induced subgraph W_4 of G^* then G^* is 3-colorable if and only if so is the graph $G^* \setminus \{x\} = G \setminus V$(the crown). Therefore, henceforth, we assume that this case is not realized and $G^* \in Free(\{W_4\})$. Then G contains vertices y_1, y_2, and y_3 such that

$$y_i \in N(x_i) \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{3} N(x_j)$$

for all $i \in \overline{1,3}$. Clearly, in G^*, the vertices y_1, y_2, and y_3 form a neighborhood of x. Since, in passing from G to G^*, the degrees of the vertices y_1, y_2,and y_3 remain unchanged, we have $G^* \in Free(\{K_{1,4}\})$. If $G^* \notin Free(\{\text{the butterfly}\})$ then, also in G^*, the vertex x has degree 2 in an induced copy of the graph butterfly; and we may assume that x, y_1, and y_2 constitute a traingle in this subgraph butterfly and y_3 does not belong to it. But then all vertices of the butterfly subgraph but x and also the vertex x_1 induce a cricket subgraph in G. If $G^* \notin Free(\{\text{the cricket}\})$ then the vertex x in G^* is a vertex of degree 1 in the induced copy of the graph cricket (and then, obviously, $G \notin Free(\{\text{the cricket}\})$) or is a vertex of degree 2 in the induced copy of the graph cricket (and then, obviously, $G \notin Free(\{K_{1,4}\})$). Therefore,

$$G^* \in Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket}\}).$$

If in addition $G \in Free(\{\text{the dart}\})$ then an induced subgraph dart can exist in G^* only if it is induced by x, y_1, y_2, y_3, and some vertex z. We may assume that (y_1, y_2, y_3) is an induced path in G^* and

$$z \in N(y_2) \setminus (N(y_1) \cup N(y_3)).$$

Then in G the vertices y_1, y_2, y_3, x_2, and z induce the subgraph $K_{1,4}$.

Applying the above-described reduction appropriately many times, we obtain some graph

$$H_G \in Free(\{K_{1,4},\ W_4,\ \text{the butterfly, the cricket, the crown}\}),$$

where $H_G \in Free(\{\text{the dart}\})$ if $G \in Free(\{\text{the dart}\})$. Clearly, G is 3-colorable if and only if so is H_G.

The proof of Lemma 7 is complete. □

Lemma 8. *The class* $Free(\{K_{1,4},$ the butterfly, the cricket, the dart$\})$ *is* 3-VC-*simple.*

Proof. By Lemmas 6 and 7, Problem 3-VC in the class

$$Free(\{K_{1,4}, \text{the butterfly, the cricket, the dart}\})$$

is polynomially reducible to the same problem in the class

$$Free(\{K_{1,4}, W_4,\ \text{the butterfly, the cricket, the crown, the dart}\}).$$

Recall that a 2-*tree* is a graph obtainable from the graph K_3, which is regarded as the simplest 2-tree, by the same rule: Add a new vertex to the previously obtained graph and join the new vertex by edges with two adjacent vertices of the old graph. It is not hard to see that each 2-tree has a unique 3-coloring which can be found in linear time.

Let G be a 2-tree of class

$$\mathcal{X} = Free(\{K_{1,4}, W_4, \text{the butterfly, the cricket, the crown, the dart}\}).$$

Then $\Delta(G) \leq 4$ by Lemma 5. Using this and inducting on the number of vertices, it is not hard to prove each of the following three assertions:

If $G \notin \{K_3$, the diamond, F_4, the sun$\}$ then all its vertices but x_1, x_2, y_1, and y_2 have degree 4. Moreover,

$$\deg(x_1) = \deg(x_2) = 2, \qquad \deg(y_1) = \deg(y_2) = 3,$$

where $(x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(G)$, while

$$G[\{x_1, x_2, y_1, y_2\}] = 2K_2 \quad \text{or} \quad G[\{x_1, x_2, y_1, y_2\}] = P_4.$$

In a 3-coloring of G, all three colors occur among the colors of the vertices x_1, x_2, y_1, and y_3; moreover, the colors of x_1 and x_2, or of y_1 and y_2, or of x_1 and y_2 coincide. This holds also for the diamond graphs and F_4. In a 3-coloring of K_3 and the sun graphs, their vertices of degree 2 acquire pairwise distinct colors.

Let G be an irreducible graph of class $\mathcal{X}$. Refer to an inclusion maximal subgraph in G that is a 2-tree and belongs to $\mathcal{X}$ as a 2_G-*tree*. Lemma 5 implies that every two 2_G-trees do not intersect by vertices, 2_G-trees cover all vertices of degree 4 in G, and each vertex of degree 2 or 3 in a 2_G-tree has degree 3 in G.

Remove from G all vertices of degree 3 whose neighborhoods induce an empty graph and all edges ab such that

$$G[N(a)] = K_2 + K_1.$$

It is not hard to see that the result is the disjoint union of all possible 2_G-trees. Therefore, the set of all 2_G-trees can be found in polynomial time.

Consider a 2_G-tree and its 3-coloring. If G has an edge joining two vertices of the 2_G-tree of one color then G is not 3-colorable. Show that if for each 2_G-tree there is no such edge then G is 3-colorable. To this end, apply some process of graph reduction.

Let G' be the current graph; i.e., $G' = G$ at the beginning of the process. Consider G' and a 3-coloring of some of its 2_G-trees. Remove from G' the 2_G-tree under consideration, then add a triangle and, for each $i \in \overline{1,3}$, join the vertex of the triangle with index i with exactly those vertices of the obtained graph to which the vertices of color i of the remote 2_G-tree were adjacent. The triangle must contain a vertex of degree 2. After eliminating all 2_G-trees, remove all vertices of degree 2 from the so-obtained graph, and denote the resulting graph by G^*. The graph G^* contains no induced copy of K_4 and has maximal vertex degree at most 3. Clearly, G is 3-colorable if and only if so is G^*. By the Brooks Theorem (see [19]), G^* is 3-colorable. Hence, G is 3-colorable.

The proof of Lemma 8 is complete. □

4. THE MAIN RESULT AND ITS PROOF

Introduce the notations

$$\mathcal{X}'_1 = Free(\{K_{1,4}, \text{the butterfly, the cricket}, C_4\}),$$
$$\mathcal{X}'_2 = Free(\{K_{1,4}, \text{the butterfly, the cricket}, C_4 + K_1\}),$$
$$\mathcal{X}'_3 = Free(\{K_{1,4}, \text{the butterfly, the cricket}, W_4\}).$$

Theorem. *Let $\mathcal{X}$ be a graph class with at most four forbidden induced subgraphs each of which has at most* 5 *vertices; and let $\mathcal{X}$ be different from each of the graph classes $\mathcal{X}'_1$–$\mathcal{X}'_3$. Then $\mathcal{X}$ is* 3-VC-*hard if $\mathcal{X}$ includes at least one of the classes $\mathcal{X}^*_1$–$\mathcal{X}^*_9$; otherwise, $\mathcal{X}$ is* 3-VC-*complete. Problem* 3-VC *in the class $\mathcal{X}'_1$ is polynomially equivalent to the same problem in $\mathcal{X}'_2$; and Problem* 3-VC *in the class $\mathcal{X}'_2$ is polynomially reducible to the same problem in $\mathcal{X}'_3$.*

Proof. It was proved in [20] that a finitely defined graph class that includes at least one of the graph classes $\mathcal{X}^*_1$ or $\mathcal{X}^*_2$ is 3-VC-hard. Therefore, if $\mathcal{X}$ includes at least one of the classes $\mathcal{X}^*_1$–$\mathcal{X}^*_9$ then $\mathcal{X}$ is 3-VC-hard.

Assume that $\mathcal{X}$ includes none of the classes $\mathcal{X}^*_1$–$\mathcal{X}^*_9$. It was proved in [16] that if $G_1 \in \mathcal{X}^*_1$ and $G_2 \in \mathcal{X}^*_2$ are arbitrary graphs with at most 5 vertices each and

$$\{G_1, G_2\} \neq \{K_{1,4}, \text{ the bull}\}, \qquad \{G_1, G_2\} \neq \{K_{1,4}, \text{ the butterfly}\};$$

then the class $Free(\{G_1, G_2\})$ is 3-VC-simple. But, it was proved (see the proof of Theorem 1 in [17]) that if G is a graph with at most 5 vertices and the class $Free(\{K_{1,4}, \text{the bull}, G\})$ includes none of the classes $\mathcal{X}^*_3$–$\mathcal{X}^*_6$ then $Free(\{K_{1,4}, \text{the bull}, G\})$ is 3-VC-simple.

Therefore, we may assume that

$$\mathcal{X} = Free(\{K_{1,4}, \text{the butterfly}, G_1, G_2\}),$$

where $\max(|V(G_1)|, |V(G_2)|) \leq 5$ and none of the graphs G_1 or G_2 belongs to any of the classes $\mathcal{X}_1^*$ and $\mathcal{X}_2^*$. Nevertheless, since $\mathcal{X} \not\supseteq \mathcal{X}_7^*$, we have $G_1 =$ cricket or $G_1 = C_5$.

Obviously, if $H \in Free(\{H' + K_1\})$ then

$$\text{either} \quad H \in Free(\{H'\}) \quad \text{or} \quad |V(H)| \leq |V(H')|(\Delta(H) + 1).$$

Problem 3-VC in the class $\mathcal{X}$ is polynomially reducible to the same problem for the set of irreducible graphs of this class; moreover, by the Ramsey Theorem, the maximal vertex degree of an irreducible graph in $\mathcal{X}$ is at most 8. Hence, if $G_2 = H' + K_1$ then Problem 3-VC in the class $\mathcal{X}$ is polynomially reducible to the same problem in the class $Free(\{K_{1,4}$, the butterfly, G_1, $H'\})$. Thus, if $G_1 = C_5$ then we may assume that

$$G_2 \in \{\text{the cricket, the kite, the diamond}\}$$

because $\mathcal{X} \not\supseteq \mathcal{X}_3^*$. This is impossible since $\mathcal{X} \not\supseteq \mathcal{X}_5^*$ and $\mathcal{X} \not\supseteq \mathcal{X}_8^*$. Assume that $G_1 =$ cricket. Then

$$G_2 \in \{\text{the kite, the diamond} + K_1, \text{the dart}, C_4, C_4 + K_1, W_4\}$$

since $\mathcal{X} \not\supseteq \mathcal{X}_5^*$ and $\mathcal{X} \not\supseteq \mathcal{X}_9^*$. The classes

$$Free(\{K_{1,4}, \text{ the butterfly, the cricket, the kite}\}),$$
$$Free(\{K_{1,4}, \text{ the butterfly, the cricket, the diamond}\})$$

are 3-VC-simple (see Lemmas 7 and 8 in [17]). By Lemma 32, the class

$$Free(\{K_{1,4}, \text{the butterfly, the cricket, the dart}\})$$

is 3-VC-simple. The cases when $G_2 \in \{C_4, C_4 + K_1, W_4\}$ are impossible.

Obviously, $\mathcal{X}_1' \subseteq \mathcal{X}_2'$ and $\mathcal{X}_1' \subseteq \mathcal{X}_3'$. Thus, Problem 3-VC in the class $\mathcal{X}_1'$ is polynomially reducible to the same problem in each of the classes $\mathcal{X}_2'$ and $\mathcal{X}_3'$. By the arguments of the previous paragraph, Problem 3-VC in the class $\mathcal{X}_2'$ is polynomially reducible to the same problem in the class $\mathcal{X}_2' \cap Free(\{C_4\})$, i.e., in $\mathcal{X}_1'$.

Therefore, the first two cases are polynomially equivalent and each of them is polynomially reducible to the third.

The proof of the theorem is complete. □

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