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# A Bilevel Stochastic Programming Problem with Random Parameters in the Follower's Objective Function

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**Abstract**—Under study is a bilevel stochastic linear programming problem with quantile criterion. Bilevel programming problems can be considered as formalization of the process of interaction between two parties. The first party is a Leader making a decision first; the second is a Follower making a decision knowing the Leader's strategy and the realization of the random parameters. It is assumed that the Follower's problem is linear if the realization of the random parameters and the Leader's strategy are given. The aim of the Leader is the minimization of the quantile function of a loss function that depends on his own strategy and the optimal Follower's strategy. It is shown that the Follower's problem has a unique solution with probability 1 if the distribution of the random parameters is absolutely continuous. The lower-semicontinuity of the loss function is proved and some conditions are obtained of the solvability of the problem under consideration. Some example shows that the continuity of the quantile function cannot be provided. The sample average approximation of the problem is formulated. The conditions are given to provide that, as the sample size increases, the sample average approximation converges to the original problem with respect to the strategy and the objective value. It is shown that the convergence conditions hold for almost all values of the reliability level. A model example is given of determining the tax rate, and the numerical experiments are executed for this example.

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## INTRODUCTION

The approach of bilevel and multilevel optimization problems is used for modeling hierarchical systems in which the goals of all decision makers are open, and the decisions made at each level are known to the subjects at the lower levels. The bilevel problems are considered in detail in the monographs [1–3].

When modeling complex systems, as a rule, not all parameters are available at the modeling stage, and therefore there is a need to account for some random factors whose impact can be analyzed by using the stochastic programming approach. If we have to ensure the successful functioning of a system with given probability then the quality of its functioning can be described by using the quantile criterion. The quantile criterion represents the level of losses whose exceeding is forbidden with given probability [4]. Application of the quantile criterion is particularly appropriate in the case of complex hierarchical systems because, as a rule, high demands are made to the reliability of their functioning.

Stochastic statements of the bilevel problems are poorly understood. Let us mention some available results for bilevel problems of stochastic programming with the quantile criterion. In [5], a problem of transport network design was under study. As a random parameter there was considered the demand for transportation. To solve the problem, some genetic algorithm was proposed that was based on the stochastic modeling. In [6], under consideration was the bilevel problem with random parameters distributed according to the Gaussian law in the presence of the uncertainty described by fuzzy

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factors. The problem was reduced to a deterministic equivalent one. A stochastic bilevel problem of the competitive location of enterprises with a quantile criterion was considered in [7, 8]. As the random parameters of the problem were the profits of enterprises, and the random parameters are assumed discrete. In [7], the problem is reduced to a deterministic mixed integer bilevel problem, and the search for a local optimal solution is described. In [8], the procedure is presented for obtaining some upper estimates of the optimal profit that is based on the relaxation techniques. In [9], a bilevel problem with random parameters having discrete distribution is investigated, and the method of reduction to some equivalent deterministic mixed integer optimization problem is proposed.

In this article, under consideration is the bilevel problem in which the coefficients of the Follower's objective function are assumed random, and the set of feasible strategies depends on neither the strategy of the Leader or the realization of random parameters. A rather wide class of possible distributions of random parameters is looked over, which makes it difficult to use the methods designed for some special distributions. Some properties of the problem are proved for the absolutely continuous distributions of the random parameters.

The efficient approach to solving the problems of stochastic programming consists in construction of a sample average approximation of a stochastic problem. This approach rests on the replacement of the objective function of the problem by its sample estimate with optimization of the so-obtained estimate in the sequel. Such an approach is described and substantiated in [10] for the problems of stochastic programming with probabilistic constraints. In [11], this method is presented for the case of a general stochastic programming problem with the quantile criterion. In this article, we apply this approach to solve the bilevel problem under consideration.

The structure of the article is as follows: In Section 1, the statement is given of the bilevel stochastic programming problem with the quantile criterion and a linear Follower's problem with random coefficients of the objective function. In Section 2, the properties of the problem are under study, the semi-continuity of the quantile function and the solvability of the problem are proved. In Section 3, the construction of a sample average approximation to the problem as well as the conditions that guarantee the convergence of the sample estimates of the solution are presented. It is proved that the convergence holds for almost all reliability levels. In Section 4, a model problem is provided for determination of the rate of taxation. In Section 5, the results of numerical experiments are described.

## 1. STATEMENT OF THE PROBLEM

Let $u$ denote the Leader's strategy chosen from $U \subset \mathbb{R}^n$, and let $y$ stand for the Follower's strategy belonging to $\mathbb{R}^m$. Let $X$ be a random vector in the probability space $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), \mathbf{P})$, where $\mathcal{L}(\mathbb{R}^m)$ is a $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$. We assume that $X(x) = x$ for all $x \in \mathbb{R}^m$.

**Remark 1.** An arbitrary random vector of size $m$ given in abstract probability space induces a probability mass on the measurable space $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and, evidently, on the measurable space $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m))$, where $\mathcal{B}(\mathbb{R}^m)$ is the $\sigma$-algebra of the Borel measurable subsets $\mathbb{R}^m$. Therefore, without loss of generality, we can assume that a random vector $X$ is determined on the probability space $(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), \mathbf{P})$ and $X(x) = x$ for all $x \in \mathbb{R}^m$. Instead of the Borel $\sigma$-algebra we consider the wider $\sigma$-algebra of Lebesgue measurable sets to guarantee the completeness of the probability space which we need for studying the random quantities as the optimal solution for the optimization problem with random parameters. In more detail this issue is covered in [12]

Given $u$ and $x$, the Follower's problem is a linear programming problem that looks as follows:

$$(A_2 u + x)^\top y \to \min_y \tag{1}$$

under the constraints

$$B_2 y \leqslant b_2, \qquad y \geqslant 0,$$

where $A_2 \in \mathbb{R}^{m\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $b_2 \in \mathbb{R}^l$ are some given matrices and vector. Let $Y^*(u, x)$ denote the set of optimal strategies of the Follower's problem (1).

Let the Leader's loss function $\Phi(\cdot)\colon U \times \mathbb{R}^m \to [-\infty; +\infty]$ have the form

$$\Phi(u,x) \triangleq \begin{cases} \min\limits_{y \in Y^*(u,x)} (f + Cu)^\top y, & \text{if } Y^*(u,x) \neq \varnothing, \\ +\infty, & \text{if } Y^*(u,x) = \varnothing, \end{cases} \tag{2}$$

where $f \in \mathbb{R}^m$ and $C \in \mathbb{R}^{m\times n}$.

Consider the quantile function of the Leader's losses

$$u \mapsto [\Phi(u,X)]_\alpha \triangleq \min\{\varphi \mid P_\varphi(u) \geqslant \alpha\}, \qquad u \in U, \tag{3}$$

where

$$P_\varphi(u) \triangleq \mathbf{P}\{\Phi(u,X) \leqslant \varphi\},$$

and $\alpha \in (0,1)$ is the given reliability level. The function $u \mapsto P_\varphi(u)$ is called the *probability function*. We state the Leader's problem as

$$\psi^* \triangleq \min_{u \in U}\{c_1^\top u + [\Phi(u,X)]_\alpha\}, \tag{4}$$

where $c_1 \in \mathbb{R}^n$ is some vector. We assume that $U$ is given as $U \triangleq \{u \in \mathbb{R}^n \mid A_1 u \leqslant b_1\}$. The Leader's problem (4) is the bilevel stochastic problem with the quantile criterion. Note that this problem if formulated as an optimistic statement. This means that the Leader takes into account the Follower's optimal strategy as the best to himself (to the Leader).

## 2. PROPERTIES OF THE PROBLEM

Consider the Follower's problem (1). Give the proposition concerning the conditions that guarantee the unique solvability of the problem.

**Proposition 1.** *Let a random vector $X$ have an absolutely continuous distribution with respect to the Lebesgue measure, and let the set $\{y \in \mathbb{R}^m \mid B_2 y \leqslant b_2, y \geqslant 0\}$ of feasible strategies of the Follower's problem be nonempty and bounded. Then $Y^*(u,X)$ is always nonempty and consists of a single element with probability* 1.

*Proof.* We denote the set of feasible strategies of the Follower's problem by $Y$. By assumption that $Y$ is nonempty and bounded, the Follower's problem has an optimal solution for an arbitrary Leader's strategy $u$ and every realization $x$ of the random parameters of the problem.

Consider the case $m = 1$. Then the optimal solution of the Follower's problem can be nonunique if and only if $X = -A_2 u$, which is possible with zero probability.

Let $m \geq 2$. In this case, the optimal solution of the Follower's problem can be nonunique if and only if the vector $A_2 u + x$ is perpendicular to one of the faces of $Y$. By assumption on the absolute continuity of the distribution of random vector $X$, the probability of such an event is zero; which means that the solution of (1) will be unique with probability 1.

The proof of Proposition 1 is complete. □

Let us prove the assertion on the semicontinuity of the loss function $\Phi(\cdot)$:

**Proposition 2.** *The function $(u,x) \mapsto \Phi(u,x)$ is lower semicontinuous in $u$ at every fixed value of $x$ and measurable in $x$ for all $u$. Moreover, for each fixed $u \in U$ the range of $x \mapsto \Phi(u,x)$ is finite.*

*Proof.* Consider the Follower's problem in canonical form. Thereto, we introduce the vector of additional variables $y^+ \in \mathbb{R}^l$. We use the notation

$$c(u,x) \triangleq \begin{pmatrix} A_2 u + x \\ 0_l \end{pmatrix}, \qquad B \triangleq \begin{pmatrix} B_2 & I \end{pmatrix}, \qquad y' \triangleq \begin{pmatrix} y \\ y^+ \end{pmatrix},$$

where $0_l$ is the zero $l$-vector. Then the Follower's problem can be rewritten as

$$c^\top(u,x)y' \to \min_{y'} \tag{5}$$

under the constraints

$$By' = b_2, \qquad y' \geqslant 0.$$

From the theory of linear programming it is known that the basic solution $y_{\mathbf{B}}$ of (5) is optimal if and only if the corresponding fundamental matrix $\mathbf{B}$ is such that

$$c^\top(u,x) - c^\top_{\mathbf{B}}(u,x)\mathbf{B}^{-1}B \geqslant 0, \tag{6}$$

$$\mathbf{B}^{-1}b \geqslant 0, \tag{7}$$

where $c_{\mathbf{B}}(u,x)$ is the subvector of $c(u,x)$ composed of the entries of $c$ corresponding to the basis variables. We denote the set of fundamental matrices satisfying (6) and (7) by $\mathfrak{B}(u,x)$.

Introduce the function $f_{\mathbf{B}}(\cdot)\colon U \times \mathbb{R}^m \to (-\infty;+\infty]$ that coincides with the value of the loss function of the Leader (2) for all pairs $(u,x)$ for which the basis $\mathbf{B}$ is optimal in the Follower problem:

$$f_{\mathbf{B}}(u,x) = \begin{cases} ((f+Cu)^\top, 0_l^\top)^\top y_{\mathbf{B}}, & \text{if } \mathbf{B} \in \mathfrak{B}(u,x), \\ +\infty, & \text{otherwise.} \end{cases}$$

Since the function $(u,x) \mapsto f_{\mathbf{B}}(u,x)$ takes finite values on a closed set and is continuous on it, $f_{\mathbf{B}}(u,x)$ is lower semicontinuous.

Using the above notation, the Leader's loss function can be represented as

$$\Phi(u,x) = \min_{\mathbf{B}\in\mathfrak{B}} f_{\mathbf{B}}(u,x),$$

where $\mathfrak{B}$ is the set of all fundamental matrices composed of the columns of $B$. By Proposition 1, the value of $\Phi(u,x)$ is always finite. According to [12, Proposition 1.26] the pointwise minimum of finitely many lower semicontinuous functions is lower semicontinuous too. Therefore, the function $(u,x) \mapsto \Phi(u,x)$ is lower semicontinuous in the collection of all arguments, which means it is lower semicontinuous in $u$ at every fixed value of $x$, and also measurable on $x$ for all $u$.

From the fact that for given $u \in U$ and $\mathbf{B} \in \mathfrak{B}$ the function $x \mapsto f_{\mathbf{B}}(u,x)$ can take only two possible values and $\mathbf{B} \in \mathfrak{B}$ is finite; it follows that the set of possible values of $x \mapsto \Phi(u,x)$ for every $u \in U$ is finite.

The proof of Proposition 2 is complete. □

From Proposition 2 the existence of a solution of (4) follows:

**Corollary 1.** *If $U$ is nonempty and bounded then there is an optimal solution to* (4).

*Proof.* Since $U$ is defined by a system of linear inequalities, it is closed, and so $U$ is compact. As proven in [4], by the lower semicontinuity of $u \mapsto \Phi(u,x)$ for almost all $x$ and the measurability of $u \mapsto \Phi(u,x)$ for all $u \in U$, the quantile function $u \mapsto [\Phi(u,X)]_\alpha$ is lower semicontinuous on $U$. Therefore, the conditions of the Weierstrass Theorem are satisfied for $u \mapsto [\Phi(u,X)]_\alpha$ and $U$, which guarantees the existence of a minimum point. The proof of the corollary is complete. □

In the proof of Corollary 1 we essentially used the lower semicontinuity of $u \mapsto \Phi(u,x)$ which is guaranteed by the optimistic formulation of problem (4). In the case of a pessimistic statement, the lower semicontinuity of $u \mapsto \Phi(u,x)$ is not guaranteed; so in the general case a solution of (4) in the pessimistic statement can not exist. A solution of (4) in the pessimistic statement will exist and, moreover, coincide with the solution of (4) in an optimistic statement with an absolutely continuous random vector $X$, which follows from Proposition 1.

Let us give an example showing that the continuity of the quantile function is generally not guaranteed:

Let the Follower's problem be as follows:

$$(x+u)y \to \min_{y\in[0,1]}.$$

It is not hard to show that the set of optimal solutions to the Follower's problem has the form

$$Y^*(u,x) = \begin{cases} \{1\}, & \text{if } x+u<0,\\ \{0\}, & \text{if } x+u>0,\\ [0,1], & \text{if } x+u=0. \end{cases}$$

Let $f=1$ and $C=0$. Then

$$\Phi(u,x) = \min_{y\in Y^*(u,x)} fy = \begin{cases} 1, & \text{if } x+u<0,\\ 0, & \text{if } x+u\geqslant 0. \end{cases}$$

Write the quantile function as

$$\varphi_\alpha(u) = \begin{cases} 0, & \text{if } \mathbf{P}\{X+u\geqslant 0\}\geqslant \alpha,\\ 1, & \text{if } \mathbf{P}\{X+u\geqslant 0\}<\alpha. \end{cases}$$

For example, in the case of a uniform distribution on [0; 1] the quantile function looks as

$$\varphi_\alpha(u) = \begin{cases} 1, & \text{if } u<\alpha-1,\\ 0, & \text{if } u\geqslant \alpha-1. \end{cases}$$

Therefore, $u\mapsto\varphi_\alpha(u)$ is not a continuous function though it is lower semicontinuos.

For further studying the convergence of the sample approximations of the problem, the properties of $(u,\varphi)\mapsto P_\varphi(u)$ are of interest.

**Proposition 3.** *Let a random vector $X$ have an absolutely continuous distribution. Then the domain of $(u,\varphi)\mapsto P_\varphi(u)$ can be divided into finitely many polyhedra, on whose interiors the function is continuous, while it is upper semicontinuous at the boundary points.*

*Proof.* In [4, Theorem 2.2], it is proved that the upper semicontinuity of a probability function follows from the lower semicontinuity in $u\in U$ and measurability in $x$ of the loss function $(u,x)\mapsto\Phi(u,x)$.

Find the region of continuity of the probability function. Put

$$\varphi_{\mathbf{B}}(u) \triangleq \big((f+Cu)^\top, 0_l^\top\big)^\top y_{\mathbf{B}}, \qquad \mathbf{B}\in\mathfrak{B}.$$

With fixed $u\in U$ order the values $\varphi_{\mathbf{B}}(u)$ in increasing order. In result, we obtain the sequence $\{\varphi_{(k)}(u)\}_{k=1}^K$, where $K=|\mathfrak{B}|$, and

$$\varphi_{(1)}(u)\le\varphi_{(2)}(u)\le\dots\le\varphi_{(K)}(u).$$

Let $\mathbf{B}_{(k)}$ denote the fundamental matrix $\mathbf{B}$ corresponding to $\varphi_{(k)}(u)$. Using the above notation, we write

$$P_\varphi(u) = \begin{cases} 0, & \text{if } \varphi<\varphi_{(1)}(u),\\ \mathbf{P}\Big\{\bigcup_{i=1}^{k-1}\{\mathbf{B}_{(i)}\in\mathfrak{B}(u,X)\}\Big\}, & \text{if } \varphi_{(k-1)}(u)\le\varphi<\varphi_{(k)}(u),\\ 1, & \text{if } \varphi>\varphi_{(K)}(u). \end{cases}$$

The inequalities $\varphi_{(k-1)}(u)\le\varphi\le\varphi_{(k)}(u)$ define the polyhedra on each of whose interiors the function $(u,\varphi)\mapsto P_\varphi(u)$ is continuous because of the absolute continuity of the distribution of $X$.

The proof of Proposition 3 is complete. □

## 3. SAMPLE APPROXIMATION OF THE PROBLEM

Consider the sequence $\{X_N\}_{N=1}^{\infty}$ of independent random vectors distributed identically with $X$. We assume that this sequence is defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P}')$. Below, unless otherwise stated, almost sure convergence (a.s.c.) will be considered with respect to the probability measure $\mathbf{P}'$.

**Remark 2.** The procedure for constructing the probability space $(\Omega, \mathcal{F}, \mathbf{P}')$, where we can define a sequence of the independent identically distributed random vectors, is described in the proof of the Kolmogorov Theorem [13]. Note, that the requirement $X_N(x) = x$ similar to $X(x) = x$ cannot be satisfied due to the independence of random vectors forming the sequence $\{X_N\}_{N=1}^{\infty}$. Description of the probability space $(\Omega, \mathcal{F}, \mathbf{P}')$ is not given in the article since its structure is irrelevant for further discussion.

If the probability of the event $\{\varphi \mid \Phi(u, X) \leqslant \varphi\}$ in (3) is replaced with the sample estimate

$$P_{\varphi}^{(N)}(u) \triangleq \frac{1}{N} \sum_{k=1}^{N} \chi_{(-\infty,0]}(\Phi(u, X_k) - \varphi), \qquad N \in \mathbb{N},$$

where

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

then we obtain the next estimathe of the quantile function:

$$\varphi_{\alpha}^{(N)}(u) \triangleq \min\left\{\varphi \mid P_{\varphi}^{(N)}(u) \geqslant \alpha\right\}, \qquad N \in \mathbb{N}.$$

Thus, instead of the original problem (4) we can consider its sample approximation

$$\psi_N \triangleq \min_{u \in U} \left\{c_1^{\top} u + \varphi_{\alpha}^{(N)}(u)\right\}, \quad N \in \mathbb{N}, \tag{8}$$

$$u_N \triangleq \operatorname{Arg}\min_{u \in U} \left\{c_1^{\top} u + \varphi_{\alpha}^{(N)}(u)\right\}, \qquad N \in \mathbb{N}. \tag{9}$$

The resulting problem with fixed realizations of the random vectors is a deterministic bilevel problem; and to solve it, the appropriate methods can be applied [1–3].

In [11], the theorem is proved on convergence of the sample approximations of the general problem of stochastic programming with the quantile type criterion of the kind

$$\psi^* \triangleq \min_{u \in U} [\Psi(u, X)]_{\alpha}. \tag{10}$$

Note that problem (4) can be represented as (10) if $\Psi(u, x) = c_1^{\top} u + \Phi(u, x)$. In the theorem, $u_N$ and $\psi_N$ denote the optimal strategy and the optimal value of the objective function in the problem of minimizing the sample quantile estimate $[\Psi(u, X)]_{\alpha}$ corresponding to the sample size $N$. Also the next notation is used:

$$\widetilde{P}_{\psi}(u) \triangleq \mathbf{P}\{\Psi(u, X) \leq \psi\}.$$

**Theorem 1** [11]. *Let the following be fulfilled:*

(1) *$U$ is compact and nonempty;*

(2) *$\Psi(\cdot)\colon U \times \mathbb{R}^m \to (-\infty, +\infty]$ is measurable with respect to $\sigma$-algebra $\mathcal{B}(U) \times \mathcal{L}(\mathbb{R}^m)$ and lower semicontinuous in $u \in U$ for each fixed value of $x \in \mathbb{R}^m$;*

(3) *for all $\varepsilon > 0$, there is a pair $(\widetilde{u}, \widetilde{\psi}) \in U \times \mathbb{R}$ such that*

$$|\psi^* - \widetilde{\psi}| \leq \varepsilon, \quad \widetilde{P}_{\widetilde{\psi}}(\widetilde{u}) > \alpha. \tag{11}$$

*Then $\psi_N \to \psi^*$ as $N \to \infty$ (a.s.c.) and each limit point of the sequence $\{u_N\}_{N \in \mathbb{N}}$ is optimal in problem* (10) *(a.s.c.).*

From Proposition 2 the fulfillment of condition 2 of Theorem 1 follows.

**Proposition 4.** *Condition* 3 *of Theorem* 1 *is fulfilled for all but possible countably many $\alpha$ in* $(0, 1)$.

*Proof*. Assume that condition 3 of Theorem 1 is not false. Then there is $\varepsilon > 0$ such that for all $\psi \in [\psi^* - \varepsilon, \psi^* + \varepsilon]$ we have

$$\sup_{u \in U} \widetilde{P}_\psi(u) \leqslant \alpha.$$

By Corollary 1, problem (1) is solvable; therefore,

$$\sup_{\psi \in [\psi^* - \varepsilon, \psi^* + \varepsilon]} \sup_{u \in U} \widetilde{P}_\psi(u) = \max_{\substack{\psi \in [\psi^* - \varepsilon, \psi^* + \varepsilon], \\ u \in U}} \widetilde{P}_\psi(u) = \alpha.$$

Owing to monotonicity of the probability,

$$\psi \mapsto \sup_{u \in U} \widetilde{P}_\psi(u) \tag{12}$$

is monotonously nondecreasing; and so,

$$\sup_{u \in U} \widetilde{P}_\psi(u) = \alpha$$

for all $\psi \in [\psi^*, \psi^* + \varepsilon]$; consequently, the domain of (12) includes some regions on which the function is constant. To each of these regions, we can assign a rational number belonging to it; therefore, the set of constancy regions of (12) is at most countable. In this way, the set of values of $\alpha$ for which condition 3 of Theorem 1 is not satisfied is at most countable too.

The proof of Proposition 4 is complete. □

If $U$ is finite then Proposition 4 can be strengthened:

**Proposition 5.** *Let $U$ be finite. Then condition* 3 *of Theorem* 1 *is fulfilled for all but possibly finitely many $\alpha \in (0, 1)$.*

*Proof*. As it follows from the proof of Proposition 4, condition 3 of Theorem 1 can be unsatisfied only if

$$\widetilde{P}_\psi(u) = \mathbf{P}\{\Psi(u, X) \leq \psi\} = \alpha \tag{13}$$

for some pair $\psi \in \mathbb{R}$ and $u \in U$.

It follows from Proposition 2 that for every $u \in U$ the random variable $\Psi(u, X)$ has a discrete distribution with finitely many realizations. Thus,

$$A \triangleq \bigcup_{u \in U} \bigcup_{\psi \in \mathbb{R}} \{\widetilde{P}_\psi(u)\}$$

is finite. Therefore, (13) can fail only for $\alpha \in A$. This completes the proof of Proposition 5. □

To determine exactly at which points condition 3 of Theorem 1 can be false, we can use Proposition 3 about the properties of $(u, \varphi) \mapsto P_\varphi(u)$ and the equality $\widetilde{P}_\psi(u) = P_{\psi - c^\top u}(u)$.

To solve problem (4) we can propose the following algorithm that consists in increasing the sample size until an approximate solution of the problem be found:

1. Put $N := 0$. Select $\Delta \in \mathbb{N}$.
2. Find the implementation of a random vector $X$:

$$x^{N+1}, x^{N+2}, \ldots, x^{N+\Delta}.$$

3. Increase the sample size: $N := N + \Delta$.
4. Find $\psi_N$ and $u_N$ by solving (8), (9).
5. If $|\psi_N - \psi_{N-\Delta}| > \varepsilon$ where $\varepsilon > 0$ is the algorithm parameter defining accuracy then go to Step 1.

Also as an end criterion we can consider the stabilization of the objective function value over several iterations. By Theorem 1, if the corresponding conditions are met then the proposed algorithm converges in finitely many steps, which means a.s.c.

## 4. THE PROBLEM OF DETERMINING A TAX RATE

Consider some model problem that can be written as (4). As the Leader, we will consider the State and as the Follower, some Manufacturer. The Leader's task is to determine the level of the tax rate ensuring the maximum tax collection. The purpose of the Follower is to maximize their own profits after taxes. Let the Manufacturer have ability to produce several types of products.

The Leader's strategy is the level of the tax rate $u \in [0;1]$. The Follower strategy is the vector $y = (y_1, y_2, \dots, y_m)^\top$, where $y_j$ is the production volume of the product of type $j$, $j = \overline{1,m}$.

The Follower's problem is the standard task of production planning in presence of limited resources. It is believed that, for production of the unit product of type $j$, the Manufacturer needs $A_{2ij}$ units of the resource of type $i$, $j = \overline{1,m}$ and $i = \overline{1,l}$. The values of $A_{2ij}$ are determined by the technological matrix $A_2$. Let the price be $c_j$ for each type $j$ of the product, $j = \overline{1,m}$, and let the available resources be $b_{2i}$ of each type $i$, $i = \overline{1,l}$. Let $c$ and $b_2$ denote the vectors with the entries $c_j$ and $b_{2i}$ respectively.

Cost per unit of the production of type $j$ is equal to $X_j$, $j = \overline{1,m}$, and will be considered random. The random vector composed of these random variables will be denoted by $X$, and its realization, by $x$.

The Follower, making this decision, knows the realization of the random vector $X$ and the value of the tax rate. In this way, the Follower's problem is formulated as the task of maximizing the profits under resource constraints:

$$Y^*(u,x) = \operatorname{Arg}\min_{y \in Y}\{x^\top y - (1-u)c^\top y\},$$

where $Y$ is defined by the constraint system

$$A_2 y \leqslant b_2, \qquad y \geqslant 0.$$

The Leader's loss is given by the ratio

$$\Phi(u,x) = \min_{y \in Y^*(u,x)} \{-uc^\top y\}$$

which is equal to the amount of the tax collection taken with the opposite sign. We assume that the Leader aims to maximize the value of tax levies to be guaranteed with probability $\alpha$. Then, to find the Leader's optimal strategy, we can formulate the bilevel problem of stochastic programming with the quantile criterion:

$$[\Phi(u,X)]_\alpha \to \min_{u \in [0,1]}.$$

## 5. RESULTS OF NUMERICAL EXPERIMENTS

Consider the problem of determining the tax rate for the following data:

$$A_2 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad c = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

We will assume that the production costs $X_1$ and $X_2$ are independent and are described by a uniform distribution on the segment $[0,3]$.

The result of applying the algorithm to solve the problem for $\alpha = 0.8$ and $\alpha = 0.95$ is presented in the table. For convenience, the value of the objective function is given with the opposite sign.

To find the optimal value of the Leader's strategy $u_N$, the corresponding values in the approximation problem were varied with increment 0.005.

The table shows the stabilization of the values of the objective function and strategy with $N = 7000$. Moreover, the optimal tax rate for the reliability level of 0.8 is 63%; and for the level 0.95, the tax rate is 49.5%.

The plot of the guaranteed amount of tax collection for the level of reliability 0.8 is given in Fig. 1, and for the reliability level of 0.95, in Fig. 2.

As expected, the resulting plots show that the tax collection is low in both cases, of low and high tax rates. With a high tax rate it is unprofitable for the manufacturer to produce products, so it leaves the market. Note that with a slight deviation from the optimal state strategy the manufacturer will leave the market. For this reason, a strategy of the state obtained as a result of solving a model problem, while maximizing the tax collection, cannot be implemented in practice.

**Table 1.**

| | $\alpha = 0.8$ | | $\alpha = 0.95$ | | | $\alpha = 0.8$ | | $\alpha = 0.95$ | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | $u_N$ | $-\psi_N$ | $u_N$ | $-\psi_N$ | $N$ | $u_N$ | $-\psi_N$ | $u_N$ | $-\psi_N$ |
| 500 | 0.630 | 2.520 | 0.495 | 1.980 | 5000 | 0.630 | 2.520 | 0.500 | 2.000 |
| 1000 | 0.625 | 2.500 | 0.490 | 1.960 | 5500 | 0.625 | 2.500 | 0.495 | 1.980 |
| 1500 | 0.625 | 2.500 | 0.490 | 1.960 | 6000 | 0.625 | 2.500 | 0.495 | 1.980 |
| 2000 | 0.625 | 2.500 | 0.490 | 1.960 | 6500 | 0.630 | 2.520 | 0.500 | 2.000 |
| 2500 | 0.625 | 2.500 | 0.490 | 1.960 | 7000 | 0.630 | 2.520 | 0.495 | 1.980 |
| 3000 | 0.625 | 2.500 | 0.495 | 1.980 | 7500 | 0.630 | 2.520 | 0.495 | 1.980 |
| 3500 | 0.625 | 2.500 | 0.495 | 1.980 | ... | ... | ... | ... | ... |
| 4000 | 0.630 | 2.520 | 0.495 | 1.980 | 10000 | 0.630 | 2.520 | 0.495 | 1.980 |
| 4500 | 0.630 | 2.520 | 0.500 | 2.000 | | | | | |

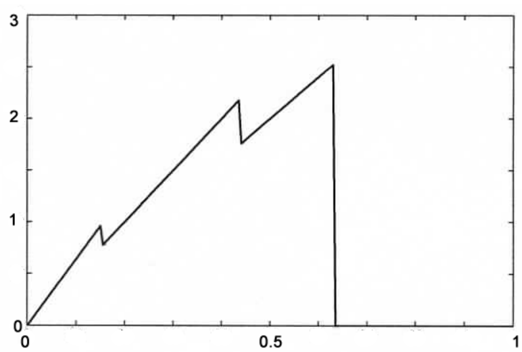


**Fig. 1.** Dependence of $-\varphi_{0.8}^{(10000)}(u)$ versus $u$.

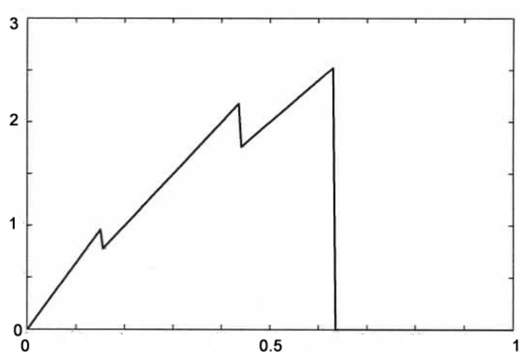


**Fig. 2.** Dependence of $-\varphi_{0.95}^{(10000)}(u)$ versus $u$.

## CONCLUSION

We study some bilevel stochastic linear programming problem with quantile criterion and random parameters in the Follower's objective function. Some assertions concerning the properties of the problem are proved. The convergence of sample approximations of the problem is shown and verified experimentally for a model problem.

In the future we plan to address a common bilevel stochastic programming problem with quantile criterion and obtain some generalization of the results of this article.

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