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On the Complexity of Minimizing Quasicyclic Boolean Functions

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Abstract—We investigate the Boolean functions that combine various properties: the extremal values of complexity characteristics of minimization, the inapplicability of local methods for reducing the complexity of the exhaustion, and the impossibility to efficiently use sufficient minimality conditions. Some quasicyclic functions are constructed that possess the properties of cyclic and zone functions, the dominance of vertex sets, and the validity of sufficient minimality conditions based on independent families of sets. For such functions, we obtain the exponential lower bounds for the extent and special sets and also a twice exponential lower bound for the number of shortest and minimal complexes of faces with distinct sets of proper vertices.

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INTRODUCTION

The two equivalent models are usually used for representing Boolean functions: analytical and geometric. The analytical model uses the notions of a Boolean function, implicants, and the diswjoint normal form (DNF) that depend on n variables. The equivalent notions of the geometric model are a subset of vertices, a face, and a complex of faces in the n-dimensional unit cube. In our exposition, we will use the following notions and notations for the n-dimensional unit cube B^n and the set P_n of Boolean functions of n variables:

A *face* in the unit cube B^n is a set of vertices

$$B^{n,\alpha_1,\dots,\alpha_k}_{i_1,\dots,i_k} = \{\widetilde{x} = (x_1,\dots,x_n) \in B^n \mid x_{i_1} = \alpha_1,\dots,x_{i_k} = \alpha_k\},$$

where $1 \le i_1 \le \dots \le i_k \le n$ and $\alpha_s \in \{0,1\}$ for $s = 1,\dots,k$. The set of indices $\{i_1,\dots,i_k\}$ is called the *direction* of the face. The *rank* and the *dimension* of a face are the numbers k and $n-k$ respectively. A vertex of the unit cube B^n is a face of rank n and dimension 0. Denote the vertices $(0,\dots,0) \in B^k$ and $(1,\dots,1) \in B^k$ by $\tilde{0}^k$ and $\tilde{1}^k$ respectively.

The *Cartesian product of faces* $g_1 = B^{r,\alpha_1,\dots,\alpha_s}_{i_1,\dots,i_s}$ and $g_2 = B^{n-r,\beta_1,\dots,\beta_t}_{j_1,\dots,j_t}$ is the face

$$g = g_1 \times g_2 = B^{n,\alpha_1,\dots,\alpha_s,\beta_1,\dots,\beta_t}_{i_1,\dots,i_s,j_1+r,\dots,j_t+r}.$$

Introduce the following notation in the unit cube B^n:

B^n_m is the layer of the cube with number m; i.e., the set of vertices for which the number of unit coordinates equals m, where $0 \le m \le n$;

$S^n_{m-h,m}$ is a zone of the cube; i.e., the set of layers of the cube with indexes $m-h,\dots,m$, where $0 \le h \le m \le n$;

G^n is the set of all different faces;

$N_M = \bigcup_{g\in M} g \subseteq B^n$ is the set of all vertices of a complex of faces $M \subseteq G^n$;

$N_f = \{\widetilde{\alpha} \in B^n \mid f(\widetilde{\alpha}) = 1\} \subseteq B^n$ is the set of unit vertices of a function $f \in P_n$.

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A face $g \subseteq N_f$ is called a *face of* f; if no face $g' \supset g$ is contained in N_f then g is called a *maximal face of* f. A complex of faces $M \subseteq G^n$ is called a *complex of* $f \in P_n$ if $N_M = N_f$.

Denote the sets of all faces and maximal faces of f by G_f and S_f respectively. Designate the set of maximal faces of f that contain $\widetilde{\alpha} \in N_f$ as $S_f(\widetilde{\alpha}) = \{g \in S_f \mid \widetilde{\alpha} \in g\}$. The minimal number of maximal faces containing a vertex of f will be denoted by

$$\sigma_f = \min_{\widetilde{\alpha} \in N_f} |S_f(\widetilde{\alpha})|.$$

The *complexity measure of complexes of faces* (or the *DNF*) is the functional $\mathcal{L}$ defined on the set of complexes of faces and satisfying the axioms of nonnegativity, monotonicity under multiplication, convexity under addition, and invariance under isomorphism [10]. A comlexity measure is called *additive* if the complexity of an arbitrary complex of faces is equal to the sum of the complexities of the faces.

The problem of minimizing a Boolean function f consists in finding a complex of faces of minimal complexity $\mathcal{L}(f)$ containing the set of unit vertices of the function. The sets of all complexes of maximal faces of a function f and of all such complexes minimal with resepct to a complexity measure $\mathcal{L}$ are denoted by $\mathcal{M}(f)$ and $\mathcal{M}_{\mathcal{L}}(f)$ respectively.

If the complexity of each face is 1 then the additive measure is denoted by l and called the *length*. If the complexity of every face is equal to the rank of the face then the complexity measure is denoted by L. Minimal complexes with respect to the complexity measures l and L are called *shortest* and *minimal* respectively. Denote the sets of shortest and minimal complexes of a function f by $\mathcal{M}_{l \cap L}(f)$.

The problem of minimizing Boolean functions for an additive complexity measure can be formulated as the problem of a minimal covering of a set of a generalized form [11].

The combinatorial statement of the problem of a minimal covering of a set is defined by a system of sets $\langle X, Y \rangle$, where X is a finite set of elements, $Y \subseteq 2^X \setminus \{\varnothing\}$ is a family of different sets, and by a nonnegative additive complexity functional $C\colon Y \to \mathbb{R}^+$. A family $S \subseteq Y$ is a *covering* of $X_S = \bigcup_{x \in S} y$, and the *complexity* of S is defined as $C(S) = \sum_{y \in S} C(y)$. The minimal covering problem $\langle X, Y, C \rangle$ consists in finding a family $S \subseteq Y$ of minimal complexity $C(X, Y)$ for which $X_S = X$.

Given an arbitrary subset $A \subset X$, define the two problems of generalized form $\langle A, X, Y, C \rangle$: Find a family $S \subseteq Y$ of minimal complexity such that (i) covers A, i.e., $A = X_S$; or (ii) contains A, i.e, $A \subseteq X_S$. Denote the complexity of a minimal covering for these problems by $C(A, X, Y)$ and $\widetilde{C}(A, X, Y)$ respectively. Obviously, $\widetilde{C}(A, X, Y) \le C(A, X, Y)$ for every $A \subset X$, and these problems are reducible to the standard statement of the minimal covering problem.

The *extent* of a function is the diameter of the greatest component of the *interval graph* whose vertices are the maximal faces of the function and whose edges are pairs of intersecting faces. The extent of a function f is denoted by $p(f)$.

The *dependence* between the unit vertices of a function is determined by the *dependence graph* whose vertices are the unit vertices of a function and whose edges are the pairs of vertices belonging to one face.

The *distances between vertices* $\widetilde{x}, \widetilde{y} \in B^n$ and *sets of vertices* $X, Y \subset B^n$ are

$$\rho(\widetilde{x}, \widetilde{y}) = |\{i \mid x_i \ne y_i,\ i = 1, \dots, n\}|, \qquad \rho(X, Y) = \min_{\widetilde{x} \in X,\ \widetilde{y} \in Y} \rho(\widetilde{x}, \widetilde{y})$$

respectively.

All undefined notions can be found in [1, 10].

The integer and upper integer parts of a real x are denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively. The symbol $o(1)$ stands for a quantity tending to zero as $n \to \infty$, and $\Theta(\varphi(n))$ for a function $\varphi(n) > 0$ means an aribtrary function $\psi(n) > 0$ for which there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\varphi(n) \le \psi(n) \le c_2\varphi(n)$ as $n \to \infty$.

Investigations of optimization problems for Boolean functions related to finding the minimal DNF and the minimal complexity of the transformations performed for simplifying the DNF show that such problems are natural optimization problems contained in the second level of the polynomial hierarchy [13]. Therefore, the main direction of research is the development of efficient methods for reducing the complexity of searching for an exact solution of the minimization problem.

Local approaches make it possible to postpone the use of exhaustion schemes and reduce their computational complexity. They are based on the study of the "geometric" structure of the set of maximal faces of a Boolean function. However, the local approaches prove to be inefficient in minimizing Boolean functions of large extent, for example, for cyclic functions [4, Section 2.2.14]. In this case, if the maximal number of elements covered by one set is equal to k then the minimal covering problem is polynomially solvable for $k = 2$ and is NP-hard for each fixed value $k \geq 3$ [2, Appendix A3].

The solution of the problem of minimizing a Boolean function for an unreducible set of maximal faces, which is called a *cyclic kernel* [12], is carried out with the use of exhaustion schemes. The local extrema, i.e., irredundant complexes of a function, are found quite efficiently in a certain amount. The search for a global extremum, i.e., for a minimal complex, is impossible without a complete search of all local extrema if there are no efficiently verifiable sufficient minimality conditions. Accordingly, the number of irredundant complexes and the number of minimal complexes are the generalized characteristics for the feasibility of the search while minimizing a particular Boolean function.

The results of investigations of different classes of Boolean functions with extremal and typical values of the parameters characterizing the complexity of different approaches to the minimization of Boolean functions are presented in the reviews [1, 5, 8]. For example, the "dense" functions have bounded extent, an exponential spread of lengths, and a twice exponential number of irredundant complexes (see [1, Section 3.2.7]).

1. DESCRIPTION OF THE CONSTRUCTION

For constructing and estimating the characteristics of quasicyclic Boolean functions, we use

(i) cyclic functions of exponential extent,

(ii) zone functions of bounded extent,

(iii) the domination of sets in the minimal covering problem,

(iv) sufficient conditions based on independent families of sets,

(v) transformations of sets that preserve the metric properties of sets of vertices in a unit cube.

A quasicyclic function is determined from a cyclic function and zone functions with the use of the operation of repetitionless product. The properties of cyclic, zone functions, and repetitionless product guarantee the possibility of applying sufficient minimality conditions for describing the shortest and minimal complexes for quasicyclic functions.

Definition 1. A Boolean function $f \in P_n$ is called *k-cyclic* if the interval graph of f is a cycle, the maximal dimension of faces is k, and each intersection of maximal faces contains one vertex. Denote the set of k-cyclic functions of n variables by $C_{n,k}$.

For a k-cyclic function f, under a certain numbering sequence $I_f = \{1, \dots, p+1\}$ of maximal faces $S_f = \{g_i,\, i \in I_f\}$, the edges of the interval graph constitute the set

$$\{(g_i, g_{i+1})\}_{i=1}^{p} \cup (g_{p+1}, g_1)$$

and the extent $p(f)$ is equal to $|S_f| - 1 = p$.

Introduce notations for a k-cyclic function f:

$k_f = \{k_i,\, i \in I_f\}$ are the dimensions of the maximal faces;

$A_f = \{\widetilde{\alpha}_i,\, i \in I_f\}$ are the vertices contained in the intersection of maximal faces, where $g_i \cap g_{i+1} = \{\widetilde{\alpha}_{i+1}\}$ for $i = 1, \dots, p$ and $g_{p+1} \cap g_1 = \{\widetilde{\alpha}_1\}$; i.e., $\widetilde{\alpha}_i, \widetilde{\alpha}_{i+1} \in g_i$ for $i = 1, \dots, p$ and $\widetilde{\alpha}_1, \widetilde{\alpha}_{p+1} \in g_{p+1}$;

$d_f = \{d_i,\, i \in I_f\}$ are the distances between vertices in A_f, where $d_i = \rho(\widetilde{\alpha}_i, \widetilde{\alpha}_{i+1})$ for $i = 1, \dots, p$ and $d_{p+1} = \rho(\widetilde{\alpha}_{p+1}, \widetilde{\alpha}_1)$;

$I_f^{\mathrm{ker}} = \{i \in I_f \mid k_i \geq 2\}$ are the numbers of the kernel faces, which are all faces of dimension greater than 1.

Denote the set of the proper vertices of the kernel faces by

$$W_f = N_f \setminus A_f = \bigcup_{i \in I_f^{\mathrm{ker}}} (g_i \setminus A_f).$$

Given a k-cyclic function f^*, the corresponding sequences will be denoted by I_{f^*}, $k_{f^*} = \{k_i^*,\ i \in I_{f^*}\}$, $A_{f^*} = \{\widetilde{\alpha}_i^*,\ i \in I_{f^*}\}$, etc.

Since $d_1 + \dots + d_{p+1} \equiv 0 \pmod 2$ for a k-quasicyclic function, a function has oddly many of maximal faces whenever the distance d_i is an odd number for an odd number of faces. A 1-cyclic function always has evenly many of maximal faces.

The properties of k-cyclic functions stem from the properties of a cycle graph with m vertices and m edges, for which the cardinality of the maximal independent set is equal to $\lfloor m/2 \rfloor$ and the length of a shortest edge covering is $\lceil m/2 \rceil$. Therefore, for m even, the shortest coverings consist of nonadjacent edges, and their number equals to 2; and for odd n, they consist of two adjacent and nonadjacent edges, and their number is equal to m.

For 1-cyclic functions, there are exactly two shortest coverings [4, Section 2.2.14]. The maximal value of the extent of a function $f \in P_n$ has order 2^n and is attained at 1-cyclic functions [3].

Constructing k-cyclic functions with certain properties is based on special transformations of cyclic functions.

Lemma 1. (i) *For a function $f \in C_{n-k,k}$, there exists a transformation into a function $f^* \in C_{n,k}$ for which there exists a subset of indices $\mathcal{I} \subset I_{f^*}$ such that $k_i = k_i^*$ and $d_i = d_i^*$ if $i \in I_{f^*} \setminus \mathcal{I}$, $k_i^* = d_i^* = k$ and the intersection vertices of the face g_i^* are the minimal and maximal vertices of the face if $i \in \mathcal{I}$, and*

$$p(f^*) \geq p(f) + |\mathcal{I}| \geq p(f) + 2\lfloor (p(f)+1)/4 \rfloor.$$

(ii) *For $f \in C_{n-3k,k}$ and $0 < \Delta k < k$, there exists a transformation to $f^* \in C_{n,k^*}$, such that $p(f^*) = p(f)$, $k_i \leq k_i^* \leq k_i + \Delta k$ for $i \in I_f = I_{f^*}$, and $k^* = \max\{k_i^*,\ i \in I_{f^*}\}$.*

Every face $g_i \in S_f$ of dimension $k_i \geq 2$ for which the minimal and maximal vertices are the intersection vertices can be replaced by a monotone chain of faces that are contained therein, intersect pairwise by the maximal and minimal vertices, and have total dimension k_i. Therefore, if one such face is replaced by two faces, the oddity of the number of maximal faces of the function changes.

Denote the vectors of $(x_1, \dots, x_n)$ and $(x_r, \dots, x_{r+k})$ by $\tilde{x}^n$ and $\tilde{x}^{r,r+k}$ respectively.

Definition 2. Refer to a Boolean function $f \in P_n$ for which $N_f = S^n_{m-h,m}$, where $0 \leq h \leq m \leq n$, as a *zone function* and denote it by $S^n_{m-h,m}(\tilde{x}^n)$.

The maximal faces of a zone function have dimension h, and the minimal and maximal vertices are contained in the cube layers B^n_{m-h} and B^n_m respectively. The value of a zone function does not change under any permutation of variables; i.e., a zone function is symmetric. For a zone function $f = S^n_{m-h,m}$, we have

$$p(f) = \lceil n/h \rceil, \qquad |N_f| = \sum_{i=m-h}^{m} \binom{n}{i}, \qquad |S_f| = \binom{n}{m}\binom{m}{h},$$

$$\sigma_f = \min\left\{ \binom{n-m+h}{h}, \binom{m}{h} \right\}, \qquad l(f) = \max\left\{ \binom{n}{m-h}, \binom{n}{m} \right\},$$

$$L(f) = l(f)(n-h);$$

the cube layers B^n_{m-h} and B^n_m are independent sets of vertices; each maximal face is contained in a shortest and minimal complex.

If the parameters of a zone function $f = S^n_{m-h,m}$ satisfy the conditions

$$1 \le h < m \le n/2, \qquad 0 \le n/2 - m = o(\sqrt{n}), \qquad 0 < \varepsilon \le h/m \le 1 - \varepsilon$$

then [9]

$$p(f) = \Theta(1), \qquad |N_f| \sim 2^{n-1}, \qquad |S_f| = 2^{n+m\mathcal{H}(h/m)}\Theta(n^{-1}) \ge 2^{n(1+1/2\cdot\mathcal{H}(\varepsilon))}\Theta(n^{-1}),$$
$$\log \mu_L(S^n_{m-h,m}) \sim \binom{n}{m} \log \binom{m}{h} \qquad \text{as } n \to \infty,$$

where $\mathcal{H}(z) = -z \log z - (1-z)\log(1-z)$ for $0 < z < 1$, and log is the logarithm with base 2. Such a function has bounded extent, a twice exponential number of minimal complexes, and the number of its maximal faces exponentially exceeds the number of unit vertices.

Definition 3. A system of sets $\langle X, Y\rangle$ and subsets A and B in a set X satisfies the *domination condition* $A \succ B$, if, for every family of sets $S \subseteq Y$, from $A \subseteq X_S$ it follows that $B \subseteq X_S$.

Obviously, from $A_1 \succ B_1$ and $A_2 \succ B_2$ it follows that $A_1 \cup A_2 \succ B_1 \cup B_2$. In the minimal covering problem, for reducing the dimension of the system of sets $\langle X, Y\rangle$, only the domination relation for the singletons $A = \{x_A\}$ and $B = \{x_B\}$ is used [12]. In this case, it suffices to consider sets $y \in Y$ rather than families of sets $S \subseteq Y$; i.e., $A \succ B$ whenever for every $y \in Y$ the relation $x_A \in y$ implies $x_B \in y$.

Definition 4. Given a system of sets $\langle X, Y\rangle$, an *independent family of sets* is a family $\mathcal{A} = \{A \mid A \subset X\}$ such that every $y \in Y$ intersects at most one set $A \in \mathcal{A}$.

Lemma 2. *Let $f = f_1 \vee f_2 \in P_{n-2}$ and put*

$$D_f(\tilde{x}^n) = \overline{x}_{n-1}\overline{x}_n f(\tilde{x}^{n-2}) \vee \overline{x}_{n-1}x_n f_1(\tilde{x}^{n-2}) \vee x_{n-1}\overline{x}_n f_2(\tilde{x}^{n-2})$$
$$= \overline{x}_{n-1} f_1(\tilde{x}^{n-2}) \vee \overline{x}_n f_2(\tilde{x}^{n-2}). \qquad (1)$$

(i) For a system $\langle N_{D_f}, S_{D_f}\rangle$, the sets of vertices $\{N^{0,1}_{D_f}, N^{1,0}_{D_f}\}$ constitute an independent set and $N^{0,1}_{D_f} \cup N^{1,0}_{D_f} \succ N^{0,0}_{D_f}$, where

$$N^{\sigma_{n-1},\sigma_n}_{D_f} = N_{D_f} \cap B^{n,\sigma_{n-1},\sigma_n}_{n-1,n} \qquad \text{for} \quad \sigma_{n-1}, \sigma_n \in \{0,1\}.$$

(ii) The minimal covering problem $\langle N_{D_f}, S_{D_f}, \mathcal{L}\rangle$ satisfies

$$\mathcal{L}(D_f) = \mathcal{L}(\overline{x}_{n-1} f_1) + \mathcal{L}(\overline{x}_n f_2),$$

and if $M_1 \in \mathcal{M}_{\mathcal{L}}(\overline{x}_{n-1} f_1)$ and $M_2 \in \mathcal{M}_{\mathcal{L}}(\overline{x}_n f_2)$ then $M_1 \cup M_2 \in \mathcal{M}_{\mathcal{L}}(D_f)$.

(iii) In the unit cube B^n, a face g belongs to S_{D_f} if and only if either $g = \widetilde{g} \times B^{2,0}_1$ and $\widetilde{g} \in S_{f_1}$, or $g = \widetilde{g} \times B^{2,0}_2$ and $\widetilde{g} \in S_{f_2}$, or $g = \widetilde{g} \times B^{2,0,0}_{1,2}$ and $\widetilde{g} \in S_f \setminus (S_{f_1} \cup S_{f_2})$.

For the covering problem for sets, the sufficient minimality conditions [11] use the notion of an independent family of sets.

Denote the family of sets that intersect $A \subseteq X$ by $Y_A \subseteq Y$. Note that every $A \subset X$ admits a unique representation as the union of pairwise disjoint sets:

$$X_{Y_A} = A \cup X^{\succ}_A \cup \bar{X}^{\succ}_A \subseteq X,$$

where $A \succ X^{\succ}_A$, $\bar{X}^{\succ}_A = X_{Y_A} \setminus (A \cup X^{\succ}_A)$, and some sets may be empty. Then an indepenedent family of sets $\mathcal{A}$ of the system $\langle X, Y\rangle$ satisfies [11, p. 96]

$$C(X,Y) \ge \sum_{A\in\mathcal{A}} \widetilde{C}(A, X, Y), \qquad (2)$$

$$C(X,Y) = \sum_{A\in\mathcal{A}} \widetilde{C}(A, X, Y) \quad \text{if } X = \bigcup_{A\in\mathcal{A}} (A \cup X^{\succ}_A). \qquad (3)$$

Definition 5. A product of functions that have no common variables is called *repetitionless*.

Introduce the following notation for $f_1 \in P_r$ and $f_2 \in P_{n-r}$:

$f_1 \times f_2 \in P_n$ is the repetitionless product of functions, where $N_{f_1 \times f_2}$ is the Cartesian product of the sets $N_{f_1} \subseteq B^r$ and $N_{f_2} \subseteq B^{n-r}$;

$M_1 \times M_2 = \{g = g_1 \times g_2 \in G^n \mid g_i \in M_i \subseteq \mathcal{M}(f_i),\ i = 1, 2\}$ is the product of complexes of faces;

$\mathcal{M}(f_1) \times \mathcal{M}(f_2) = \{M_1 \times M_2 \mid M_i \in \mathcal{M}(f_i),\ \ i = 1, 2\} \subseteq \mathcal{M}(f_1 \times f_2)$ is the product of the sets of the complexes of faces $\mathcal{M}(f_1)$ and $\mathcal{M}(f_2)$.

The set of all faces and maximal faces of functions satisfy

$$G_{f_1 \times f_2} = G_{f_1} \times G_{f_2}, \qquad S_{f_1 \times f_2} = S_{f_1} \times S_{f_2}.$$

Observe that

$$l(M_1 \times M_2) = l(M_1) l(M_2), \qquad L(M_1 \times M_2) = L(M_1) l(M_2) + l(M_1) L(M_2), \tag{4}$$

but possibly $l(f_1) l(f_2) > l(f_1 \times f_2)$ [1, Section 3.2.4].

Lemma 3. *For the repetitionless product of $f_1 \in P_r$ and $f_2 \in P_{n-r}$, a set of vertices $Q_1 \subset N_{f_1}$, and an independent set of vertices $Q_2 \subset N_{f_2}$ of f_2, the family of sets $\mathcal{A} = \{Q_1 \times \{\widetilde{x}\}, \widetilde{x} \in Q_2\}$ is an independent family of sets for the function $f_1 \times f_2 \in P_n$.*

Define the binary relation $\mathcal{R}_{f,L}$ on sets of vertices and complexes of maximal faces of a function f which is a sufficient condition for the membership of a complex in $\mathcal{M}_{l \cap L}(f)$.

Definition 6. A pair (Q, M) satisfies $(Q, M) \in \mathcal{R}_{f,L}$ if Q is an independent set of vertices of f, for each $\widetilde{\alpha} \in Q$ all $g \in S_f(\widetilde{\alpha})$ have the same complexity, $M \in \mathcal{M}(f)$, and $|M| = |Q|$.

Lemma 4. (i) *If $(Q, M) \in \mathcal{R}_{f,L}$ then $M \in \mathcal{M}_{l \cap L}(f)$.*

(ii) *If $(Q_1, M_1) \in \mathcal{R}_{f_1,L}$ and $(Q_2, M_2) \in \mathcal{R}_{f_2,L}$ then*

$$(Q_1 \times Q_2, M_1 \times M_2) \in \mathcal{R}_{f_1 \times f_2, L} \quad \textit{and} \quad M_1 \times M_2 \in \mathcal{M}_{l \cap L}(f_1 \times f_2).$$

(iii) *If for each face $g_i \in S_{f_i}$ there is a pair $(Q_i, M_i) \in \mathcal{R}_{f_i,L}$ and $g_i \in M_i$, where $i = 1, 2$, then each maximal face of $f_1 \times f_2$ is contained in some shortest and minimal complex.*

2. THE MAIN RESULTS

For a cyclic function $f \in C_{r,k}$, where $1 \le k < r$, let f_A and f_W denote the functions for which the unit vertices are given by the sets of vertices A_f and $W_f = N_f \setminus A_f$ respectively. Then

$$f_A(\tilde{x}^r) = 1 \qquad \text{for } \tilde{x}^r \in A_f = \{\widetilde{\alpha}_i,\ i \in I_f\},$$

$$f_W(\tilde{x}^r) = \bigvee_{i \in I_f} w_i(\tilde{x}^r),$$

where $w_i(\tilde{x}^r) = 1$ if $\tilde{x}^r \in g_i \setminus A_f$ and $g_i \in S_f$ for $i \in I_f = \{1, \dots, p+1\}$.

Obviously, $f(\tilde{x}^r) = f_A(\tilde{x}^r) \vee f_W(\tilde{x}^r)$ and $f_A(\tilde{x}^r) f_W(\tilde{x}^r) \equiv 0$.

The function $w_i(\tilde{x}^r)$, $i \in I_f$, is a connected component of the function $f_W(\tilde{x}^r)$ and is representable as a k_i-dimensional face of the cube B_r with two zero vertices from A_f at distance $d_i \le k_i$. Consequently, $w_i(\tilde{x}^r) \equiv 0$ if $k_i = 1$ and $f_W(\tilde{x}^r) \equiv 0$ for $f \in C_{r,1}$.

The set of quasicyclic functions $\mathcal{F}^n_{r,k}(m, h, h_1, h_2) \subset P_n$ is defined from the cyclic functions by the relation

$$F_f(\tilde{x}^n) = f(\tilde{x}^r) H(\tilde{x}^{r+1,n-2}) \overline{x}_{n-1} \overline{x}_n \vee f_W(\tilde{x}^r) \big(H_1(\tilde{x}^{r+1,n-2}) \overline{x}_{n-1} x_n \vee H_2(\tilde{x}^{r+1,n-2}) x_{n-1} \overline{x}_n \big),$$

where $f \in C_{r,k}$ and H, H_1, and H_2 are the zone functions for which

$$N_H = S^{n-r-2}_{m-h,m}, \qquad N_{H_1} = S^{n-r-2}_{m-h,m-h_1}, \qquad N_{H_2} = S^{n-r-2}_{m-h_2,m},$$

and the parameters satisfy the constraints

$$1 \le k < r < n, \qquad 1 \le h_1 - 1 \le h_2 < h < m \le (n-r-2)/2.$$

Denote the functions $H(\tilde{x}^{r+1,n-2})\overline{x}_{n-1}\overline{x}_n$, $H_1(\tilde{x}^{r+1,n-2})\overline{x}_{n-1}$, and $H_2(\tilde{x}^{r+1,n-2})\overline{x}_n$ by $\widehat{H}(\tilde{x}^{r+1,n})$, $\widehat{H}_1(\tilde{x}^{r+1,n})$, and $\widehat{H}_2(\tilde{x}^{r+1,n})$ respectively.

A quasicyclic function admits the representation

$$F_f(\tilde{x}^n) = F_A(\tilde{x}^n) \vee F_W(\tilde{x}^n),$$

where

$$F_A(\tilde{x}^n) = f_A(\tilde{x}^r)\widehat{H}(\tilde{x}^{r+1,n}),$$

$$F_W(\tilde{x}^n) = f_W(\tilde{x}^r)(\widehat{H}_1(\tilde{x}^{r+1,n}) \vee \widehat{H}_2(\tilde{x}^{r+1,n})) = F_{W,1}(\tilde{x}^n) \vee \dots \vee F_{W,p+1}(\tilde{x}^n), \tag{5}$$

$$\begin{aligned} F_{W,i}(\tilde{x}^n) &= w_i(\tilde{x}^r)(\widehat{H}_1(\tilde{x}^{r+1,n}) \vee \widehat{H}_2(\tilde{x}^{r+1,n})) \\ &= w_i(\tilde{x}^r)\widehat{H}(\tilde{x}^{r+1,n}) \vee w_i(\tilde{x}^r)(\widehat{H}_1(\tilde{x}^{r+1,n}) \vee \widehat{H}_2(\tilde{x}^{r+1,n})), \quad i \in I_f. \end{aligned} \tag{6}$$

Given functions φ and ψ with $N_\psi \subset N_\varphi$, let $\widetilde{\mathcal{L}}(\varphi, \psi)$ and $\widetilde{\mathcal{M}}_{\mathcal{L}}(\varphi, \psi)$ denote the complexity and the set of $\mathcal{L}$-minimal complexes of faces of the function φ that *include* $N_\psi \subset N_\varphi$.

Theorem 1. *Given $F_f \in \mathcal{F}^n_{r,k}(m, h, h_1, h_2)$, we have*

(i) $S_{F_f} = S_f \times S_{\widehat{H}} \cup S_{F_W}$ and $S_{F_W} = \bigcup_{i \in I_f} S_{F_{W,i}}$, where

$$S_{F_{W,i}} = \begin{cases} S_{w_i} \times S_{\widehat{H}_1} \cup S_{w_i} \times S_{\widehat{H}_2} & \text{for } i \in I_f^{\ker}, \\ \varnothing & \text{for } i \in I_f \setminus I_f^{\ker}. \end{cases}$$

(ii) $N^{0,1}_{F_{W,i}} \cup N^{1,0}_{F_{W,i}} \succ N^{0,0}_{F_{W,i}}$ for $i \in I_f^{\ker}$, and then $N^{0,1}_{F_{W,i}} \cup N^{1,0}_{F_W} \succ N^{0,0}_{F_W}$, where

$$N_F^{\sigma_1,\sigma_2} = N_F \cap B^{n,\sigma_1,\sigma_2}_{n-1,n} \quad \text{for } F \in P_n \text{ and } \sigma_1, \sigma_2 \in \{0,1\}.$$

(iii) $\mathcal{L}(F_f) = \widetilde{\mathcal{L}}(_f, F_A) + \mathcal{L}(F_W)$, and if $M_1 \in \widetilde{\mathcal{M}}_{\mathcal{L}}(F_f, F_A)$ and $M_2 \in \mathcal{M}_{\mathcal{L}}(F_W)$ then

$$M_1 \cup M_2 \in \mathcal{M}_{\mathcal{L}}(F_f).$$

(iv) If a function $f_{A,\widetilde{\alpha}} \in P_n$ is such that $N_{f_{A,\widetilde{\alpha}}} = A_f \times \{\widetilde{\alpha}\} \times \tilde{0}^2$, where $\widetilde{\alpha} \in B_m^{n-r-2}$, then $\mathcal{A} = \{N_{f_{A,\widetilde{\alpha}}},\ \widetilde{\alpha} \in B_m^{n-r-2}\}$ is an independent set of the system $\langle N_{F_f}, S_{F_f}\rangle$, and for every vertex $\widetilde{\alpha} \in B_m^{n-r-2}$ we have

$$\widetilde{\mathcal{L}}(F_f, F_A) \ge \widetilde{\mathcal{L}}(F_f, f_{A,\widetilde{\alpha}})\left|B_m^{n-r-2}\right|. \tag{7}$$

(v) The relations

$$\tilde{l}(F_f, F_A) = \tilde{l}(f, f_A)l(\widehat{H}), \qquad \widetilde{L}(F_f, F_A) \ge \widetilde{L}(f, f_A)l(\widehat{H}) + \tilde{l}(f, f_A)L(\widehat{H}), \tag{8}$$

hold; here equality is obtained if $\mathcal{M}_{l\cap L}(f, f_A) \neq \varnothing$; i.e., the complex $M_1 \times M_2 \in \widetilde{\mathcal{M}}_{l\cap L}(F_f, F_A)$ if

$$M_1 \in \widetilde{\mathcal{M}}_{l\cap L}(f, f_A), \qquad M_2 \in \mathcal{M}_L(\widehat{H}) = \mathcal{M}_l(\widehat{H}).$$

(vi) Each maximal face of F_f is contained in some complex $M \in \mathcal{M}_{l\cap L}(F_f)$ if every maximal face of f is contained in some complex $\widetilde{M} \in \widetilde{\mathcal{M}}_{l\cap L}(f, f_A)$.

Obviously, $\widetilde{\mathcal{M}}_{l\cap L}(f, f_A) \neq \varnothing$ when the maximal faces of $f \in C_{r,k}$ satisfy: either all faces have the same complexity or consecutive pairs of intersecting faces have the same complexity for an even number of faces (Lemma 4(i)).

Corollary 1. *Given $F_f \in \mathcal{F}^n_{r,k}(m, h, h_1, h_2)$ we have* (i) $p(F_f) \ge p(f)$, $|S_{F_f}| \ge |S_f| \times |S_H|$, and $\sigma_{F_f} \ge \min\{\sigma_H, \sigma_{H_1}, \sigma_{H_2}\}$.

(ii) The number and the cardinality of connected components of dominated and dominating sets are at most $\left|I_f^{\ker}\right|$ and $\min\{|N_H|, |N_{H_1}|, |N_{H_2}|\}$.

(iii) An independent family of sets $\left\{N_{f_{A,\widetilde{\alpha}}},\ \widetilde{\alpha} \in B_m^{n-r-2}\right\}$ for the system $\langle N_{F_f}, S_{F_f}\rangle$ consists of $\binom{n-r-2}{m}$ sets of cardinality $p(f)+1$.

Theorem 2. *For certain values of the parameters k, r, m, h, h_1, and h_2, the set $\mathcal{F}_{r,k}^n(m,h,h_1,h_2)$ contains functions possessing all of the following properties:*

- *their extents are exponential;*
- *their number of maximal faces exponentially exceeds the number of unit vertices, any unit vertex belongs to an exponential number of maximal faces, and each maximal face is contained in a shortest and minimal complex;*
- *their number of connected components of dominated and dominating sets of unit vertices that have exponential cardinality;*
- *the minimality of complexes of faces is justified by sufficient conditions that use an independent set of exponential cardinality or an independent family of sets in which each set has exponential cardinality;*
- *their number of shortest and minimal complexes of faces having different sets of proper vertices is twice exponential.*

The study of quasicyclic functions makes it possible to extend the understanding of problems of minimizing Boolean functions with the use of known approaches. The application of local methods for reducing the complexity of minimization for quasicyclic Boolean functions proved to be inefficient. Therefore, relevant is the study of classes of Boolean functions for which the application of local methods is efficient [7] of which reflect the specifics of applied problems [6].

3. PROOFS

Proof of Lemma 1. (i) Given $g_i \in S_f$ of $f \in C_{n-k,k}$, denote $g_i \times \tilde{0}^k$ and $g_i \times \tilde{1}^k$ in B^n by g_i^0 and g_i^1 respectively.

Define $f_0 \in C_{n,k}$ that coincides with f on $B_{n-k+1,\dots,n}^{n,0,\dots,0}$; i.e., f_0 has the set of maximal faces

$$S_{f_0} = \left\{g_i^0 \mid g_i \in S_f\right\}_{i=1}^{p+1} \subset B_{n-k+1,\dots,n}^{n,0,\dots,0}.$$

Obviously, $k_{f_0} = k_f$ and $d_{f_0} = d_f$.

Given $\widetilde{\alpha}_i \in A_f \subset B^{n-k}$, denote by $g^{\widetilde{\alpha}_i}$ the face of dimension k in B^n with minimal vertex $\widetilde{\alpha}_i^0 = \widetilde{\alpha}_i \times \tilde{0}^k$ and maximal vertex $\widetilde{\alpha}_i^1 = \widetilde{\alpha}_i \times \tilde{1}^k$. Addition of k-dimensional faces to a k-cyclic function $f_0 \in C_{n,k}$ is fulfilled for three or four consecutive maximal faces $\left\{g_i^0, g_{i+1}^0, g_{i+2}^0, g_{i+3}^0\right\}$ depending on the distance d_i, where $1 \le i < i+3 \le p+1$.

If $d_i = 1$ then the faces $\left\{g_i^0, g_{i+1}^0\right\}$ are replaced by $\left\{g_i^1, g_{i+1}^1\right\} \subset B_{n-k+1,\dots,n}^{n,1,\dots,1}$, the k-dimensional faces $\{g^{\widetilde{\alpha}_i}, g^{\widetilde{\alpha}_{i+2}}\}$ are added, and the faces $\left\{g_{i+2}^0, g_{i+3}^0\right\}$ remain unchanged.

If $d_i > 1$ then $\left\{g_i^0\right\}$ is replaced by $g_i^1 \subset B_{n-k+1,\dots,n}^{n,1,\dots,1}$, the k-dimensional faces $\{g^{\widetilde{\alpha}_i}, g^{\widetilde{\alpha}_{i+1}}\}$ are added, and the faces $\left\{g_{i+1}^0, g_{i+2}^0\right\}$ remain unchanged. The distance between these and subsequent added faces is at least 2.

Such a trasformation can be applied at least $\lfloor (p(f)+1)/4 \rfloor$ times, and each time two faces of dimension k are added in which the minimal and maximal vertices are the intersection vertices of the faces. In result, we obtain some k-cyclic function $f^* \in C_{n,k}$ for which $\mathcal{I}$ is the set of the indices of the added faces,

$$p(f^*) \ge p(f) + |\mathcal{I}|, \qquad |\mathcal{I}| \ge 2\lfloor (p(f)+1)/4 \rfloor.$$

(ii) For the sequence $\{k_i^* \mid k_i \le k_i^* \le k_i + \Delta k\}_{i=1}^{p+1}$, the function $f^* \in P_n$ is defined by the complex of faces $M = \{g_i^* \mid g_i^* = g_i \times \Delta g_i^*, g_i \in S_f\}_{i=1}^{p+1}$, where $\Delta g_i^* = \tilde{0}^{3k}$ for $k_i^* = k_i$ and $\Delta g_i^* = \Delta g_i \times \tilde{0}^{2k}$ for $i = 1 \pmod 3$, $\Delta g_i^* = \tilde{0}^k \times \Delta g_i \times \tilde{0}^k$ for $i = 2 \pmod 3$, and $\Delta g_i^* = \tilde{0}^{2k} \times \Delta g_i$ for $i = 0 \pmod 3$, where $\Delta g_i = B_{1,\dots,k_i^*-k_i}^{k,0,\dots,0} \subset B^k$, with $k_i^* > k_i$ for $i = 1,\dots,p+1$.

Obviously, the dimension of g_i^* is equal to k_i^*, for different faces $\rho(g_i^*, g_j^*) \ge \rho(g_i, g_j)$ for $i,j = 1,\dots,p+1$, $g_i^* \cap g_{i+1}^* = \{\widetilde{\alpha}_{i+1}^*\}$ if $i = 1,\dots,p$, while $g_{p+1}^* \cap g_1^* = \{\widetilde{\alpha}_1^*\}$, where $\widetilde{\alpha}_i^* = \widetilde{\alpha}_i \times \tilde{0}^{3k} \in B^n$ for $\widetilde{\alpha}_i \in A_f$.

Let us prove that all faces of the complex M are maximal and there are no other maximal faces of $f^* \in C_{n,k^*}$. If a maximal face g of f^* is not contained in M then g contains distinct vertices from different faces of M. Consequently, there exist vertices $\tilde{x}^*, \tilde{y}^* \in g$ such that $\rho(\tilde{x}^*, \tilde{y}^*) = 1$, $\tilde{x}^* \in g_i^*$, and $\tilde{y}^* \in g_j^*$, where $i,j \in 1,\dots,p+1$. Then $\rho(\tilde{x}^*, \tilde{y}^*) \ge \rho(g_i^*, g_j^*) \ge \rho(g_i, g_j)$ and for a k-cyclic function the distance between disjoint faces is at most 2 and can be 1 only for two faces intersecting with one face, and the intersection vertices are located at distance 1. Therefore, we may assume that $\tilde{x}^* \in g_1^*$ and $\tilde{y}^* \in g_2^*$ or $\tilde{y}^* \in g_3^*$.

Denote the vertices of B^{n-3k} coinciding with the vertices $\tilde{x}^*$ and $\tilde{y}^*$ at the coordinates $1,\dots,n-3k$ by $\widetilde{x}$ and $\widetilde{y}$ respectively. Obviously, $\rho(\widetilde{x}, \widetilde{y}) \le \rho(\tilde{x}^*, \tilde{y}^*) = 1$.

Consider the possible cases:

(1) In the case of $\tilde{x}^* \in g_1^*$, $\tilde{y}^* \in g_2^*$, and $\rho(\widetilde{x}, \widetilde{y}) = 0$; i.e., $\widetilde{x} = \widetilde{y} = \widetilde{\alpha}_2$, we infer that if $\tilde{x}^* = \widetilde{x}$ then $\tilde{x}^*, \tilde{y}^* \in g_2^*$; if $\tilde{y}^* = \widetilde{y}$ then $\tilde{x}^*, \tilde{y}^* \in g_1^*$; if $\tilde{x}^* \ne \widetilde{x}$ and $\tilde{y}^* \ne \widetilde{y}$ then $\rho(\tilde{x}^*, \tilde{y}^*) \ge 2$.

(2) In the case of $\tilde{x}^* \in g_1^*$, $\tilde{y}^* \in g_2^*$, and $\rho(\widetilde{x}, \widetilde{y}) = 1$, we conclude that if $\widetilde{x} = \widetilde{\alpha}_2$ and $\widetilde{y} \ne \widetilde{\alpha}_2$ then $\widetilde{x}, \widetilde{y} \in g_2 \subseteq g_2^*$ and $\tilde{x}^* \ne \widetilde{x}$ or $\tilde{y}^* \ne \widetilde{y}$; if $\widetilde{x} \ne \widetilde{\alpha}_2$ and $\widetilde{y} = \widetilde{\alpha}_2$ then $\widetilde{x}, \widetilde{y} \in g_1 \subseteq g_1^*$ and $\tilde{x}^* \ne \widetilde{x}$ or $\tilde{y}^* \ne \widetilde{y}$; if $\tilde{x}^* \ne \widetilde{x}$ or $\tilde{y}^* \ne \widetilde{y}$ then $\rho(\tilde{x}^*, \tilde{y}^*) > \rho(\widetilde{x}, \widetilde{y}) = 1$.

(3) In the case of $\tilde{x}^* \in g_1^*$, $\tilde{y}^* \in g_3^*$, and $\rho(\widetilde{x}, \widetilde{y}) \le 1$; i.e., $\widetilde{x} \in g_1$, $\widetilde{y} \in g_3$, and $1 \le d_2 = \rho(\widetilde{\alpha}_2, \widetilde{\alpha}_3) \le \rho(\widetilde{x}, \widetilde{y})$, which is possible only for $d_2 = 1$, $\widetilde{x} = \widetilde{\alpha}_2$, and $\widetilde{y} = \widetilde{\alpha}_3$; we conclude that if $\tilde{x}^* = \widetilde{\alpha}_2^*$ and $\tilde{y}^* = \widetilde{\alpha}_3^*$ then $\tilde{x}^*, \tilde{y}^* \in g_2^*$; if $\tilde{x}^* \ne \widetilde{\alpha}_2^*$ or $\tilde{y}^* \ne \widetilde{\alpha}_3^*$ then $\rho(\tilde{x}^*, \tilde{y}^*) > \rho(\widetilde{x}, \widetilde{y}) = 1$.

Thus, we have a contradiction: the vertices either belong to one face or are nonadjacent.

Lemma 1 is proved. □

Given the face $g = B_{j_1,\dots,j_t}^{n,\alpha_1,\dots,\alpha_t} \subseteq B^n$ and the set of indices $\{i_1,\dots,i_k\}$, where $1 \le i_1 < \dots < i_k \le n$, denote the face in the cube B^{n-k} obtained from g by removing the coordinates $i_1,\dots,i_k$ by $\widetilde{g}_{i_1,\dots,i_k}$.

Proof of Lemma 2. (i) Every complex of faces M of D_f is uniquely representable as the union of disjoint complexes $M = M_{0,0} \cup M_{0,1} \cup M_{1,0}$, where

$$M_{0,0} = \{g \in M \mid g \subseteq B_{n-1,n}^{n,0,0}\},$$

$$M_{0,1} = \{g \in M \mid g \cap B_{n-1,n}^{n,0,1} \ne \varnothing\}, \qquad M_{1,0} = \{g \in M \mid g \cap B_{n-1,n}^{n,1,0} \ne \varnothing\}$$

since $\{N_{D_f}^{0,1}, N_{D_f}^{1,0}\}$ is an independent family of sets for $\langle N_{D_f}, S_{D_f}\rangle$.

A complex of maximal faces M satisfies

$$N_{M_{0,1}} = N_{\overline{x}_{n-1} f_1}, \qquad N_{M_{1,0}} = N_{\overline{x}_n f_2}, \qquad N_{M_{0,1}} \cup N_{M_{1,0}} = N_{D_f}.$$

This means that

$$N_{D_f}^{0,1} = N_{M_{0,1}} \cap B_{n-1,n}^{n,0,1} \succ N_{M_{0,1}} \cap B_{n-1,n}^{n,0,0}, \qquad N_{D_f}^{1,0} = N_{M_{1,0}} \cap B_{n-1,n}^{n,1,0} \succ N_{M_{1,0}} \cap B_{n-1,n}^{n,0,0}$$

and hence

$$N_{D_f}^{0,1} \cup N_{D_f}^{1,0} \succ (N_{M_{0,1}} \cup N_{M_{1,0}}) \cap B_{n-1,n}^{n,0,0} = N_{D_f}^{0,0}.$$

(ii) The independence and domination properties of the family of sets $\{N^{0,1}_{D_f},\ N^{1,0}_{D_f}\}$ and relations (2) and (3) imply

$$\mathcal{L}(D_f) \geq \widetilde{\mathcal{L}}\big(N^{0,1}_{D_f}, N_{D_f}, S_{D_f}\big) + \widetilde{\mathcal{L}}\big(N^{1,0}_{D_f}, N_{D_f}, S_{D_f}\big) = \mathcal{L}(f_1\overline{x}_{n-1}) + \mathcal{L}(f_2\overline{x}_n),$$

where $\widetilde{\mathcal{L}}(A, N_{D_f}, S_{D_f})$ is the complexity $\mathcal{L}$ of the minimal complex $M \subseteq S_{D_f}$ that contains $A \subseteq N_{D_f}$. For complexes $M_1 \in \mathcal{M}_{\mathcal{L}}(\overline{x}_{n-1}f_1)$ and $M_2 \in \mathcal{M}_{\mathcal{L}}(\overline{x}_n f_2)$, we have

$$N_{M_1} \cup N_{M_2} = N_{D_f}, \qquad \mathcal{L}(M_1) + \mathcal{L}(M_2) = \mathcal{L}(\overline{x}_{n-1}f_1) + \mathcal{L}(\overline{x}_n f_2),$$

i.e., $M_1 \cup M_2 \in \mathcal{M}_{\mathcal{L}}(D_f)$.

(iii) The following cases are possible for $g \in S_{D_f}$ and the face $\widetilde{g}_{n-1,n}$, obtained from g by removing the coordinates $n-1$ and n:

If $g \cap B^{n,0,1}_{n-1,n} \neq \varnothing$ then

$$\widetilde{g}_{n-1,n} \subseteq N_{f_1}, \qquad g \subseteq \widetilde{g}_{n-1,n} \times B^{2,0}_1 \subseteq N_{f_1} \times B^{2,0}_1 \subseteq N_{D_f}.$$

Therefore, $g \in S_{D_f}$ if and only if $\widetilde{g}_{n-1,n} \in S_{f_1}$ and $g = \widetilde{g}_{n-1,n} \times B^{2,0}_1$.

If $g \cap B^{n,1,0}_{n-1,n} \neq \varnothing$ then

$$\widetilde{g}_{n-1,n} \subseteq N_{f_2}, \qquad g \subseteq \widetilde{g}_{n-1,n} \times B^{2,0}_2 \subseteq N_{f_2} \times B^{2,0}_2 \subseteq N_{D_f}.$$

In this case, $g \in S_{D_f}$ if and only if $\widetilde{g}_{n-1,n} \in S_{f_2}$ and $g = \widetilde{g}_{n-1,n} \times B^{2,0}_2$.

If $g \subset B^{n,0,0}_{n-1,n}$ then

$$\widetilde{g}_{n-1,n} \subseteq N_f = N_{f_1 \vee f_2}, \qquad g = \widetilde{g}_{n-1,n} \times B^{2,0,0}_{1,2} \subseteq N_f \times B^{2,0,0}_{1,2} \subseteq N_{D_f}.$$

Moreover, if $\widetilde{g}_{n-1,n} \in S_{f_1}$ then $g \subset \widetilde{g}_{n-1,n} \times B^{2,0}_1 \subseteq N_{D_f}$, and if $\widetilde{g}_{n-1,n} \in S_{f_2}$ then $g \subset \widetilde{g}_{n-1,n} \times B^{2,0}_2 \subseteq N_{D_f}$. Therefore, $g \in S_{D_f}$ if and only if $\widetilde{g}_{n-1,n} \in S_f \setminus (S_{f_1} \cup S_{f_2})$.

Lemma 2 is proved. □

Proof of Lemma 3. Suppose that $\mathcal{A}$ is not an independent family of sets; i.e., there are two sets $\{A_i, A_j\} \in \mathcal{A}$ and a face $g \in G_{f_1 \times f_2}$ for which $g \cap A_i \neq \varnothing$ and $g \cap A_j \neq \varnothing$. This means that

$$g \cap (Q_1 \times \{\widetilde{\alpha}_1\}) \neq \varnothing, \qquad g \cap (Q_1 \times \{\widetilde{\alpha}_2\}) \neq \varnothing$$

for some $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in Q_2$. But then the face $\widetilde{g}_{1,\dots,r} \subset N_{f_2}$ obtained from g by removing the coordinates $1, \dots, r$ contains $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$, which contradicts the independence of Q_2. Lemma 3 is proved. □

Proof of Lemma 4. (i) For a pair $(Q, M) \in \mathcal{R}_{f,L}$, the complex of faces $M \in \mathcal{M}(f)$ has the form $M = \{g_{\widetilde{\alpha}} \in S_f(\widetilde{\alpha}) \mid \widetilde{\alpha} \in Q\}$, and for M the lower bounds are attainable for the length and complexity of the minimal covering (2). Consequently, $M \in \mathcal{M}_l(f)$, $M \in \mathcal{M}_L(f)$, and $M \in \mathcal{M}_{l \cap L}(f)$.

(ii) Obviously, for the function $f_1 \times f_2$, we have

$$N_{f_1 \times f_2} = N_{f_1} \times N_{f_2}, \qquad S_{f_1 \times f_2} = S_{f_1} \times S_{f_2},$$
$$S_{f_1 \times f_2}(\widetilde{\alpha}_1 \times \widetilde{\alpha}_2) = S_{f_1}(\widetilde{\alpha}_1) \times S_{f_2}(\widetilde{\alpha}_2)$$

for every vertex $\widetilde{\alpha}_1 \times \widetilde{\alpha}_2 \in N_{f_1 \times f_2}$, The set of vertices $Q = Q_1 \times Q_2$ is independent and

$$M_1 \times M_2 = \{g_{\widetilde{\alpha}_1 \times \widetilde{\alpha}_2} = g_{\widetilde{\alpha}_1} \times g_{\widetilde{\alpha}_2} \mid \widetilde{\alpha}_1 \times \widetilde{\alpha}_2 \in Q = Q_1 \times Q_2,$$
$$\widetilde{\alpha}_i \in Q_i,\ g_{\widetilde{\alpha}_i} = M_i \cap S_{f_i}(\widetilde{\alpha}_i),\ i = 1, 2\} \in \mathcal{M}(f_1 \times f_2).$$

Moreover, each $g \in S_{f_1\times f_2}(\widetilde{\alpha}_1 \times \widetilde{\alpha}_2)$ is uniquely representable as $g = g_1 \times g_2$, where $g_i \in S_{f_i}(\widetilde{\alpha}_i)$ for $i = 1, 2$; i.e.,

$$L(g) = L(g_1) + L(g_2) = L_{\widetilde{\alpha}_1} + L_{\widetilde{\alpha}_2}.$$

Consequently, $(Q_1 \times Q_2, M_1 \times M_2) \in \mathcal{R}_{f_1\times f_2,L}$.

(iii) Each face $g \in S_{f_1\times f_2}$ admits a unique representation $g = g_1 \times g_2$, where $g_i \in S_{f_i}$ and $g_i \in M_i$ for some pair $(Q_i, M_i) \in \mathcal{R}_{f_i,L}$ for $i = 1, 2$. Then

$$g \in M_1 \times M_2 \in \mathcal{M}_{l\cap L}(f_1 \times f_2).$$

Lemma 4 is proved. □

Proof of Theorem 1. (i) Use the following representation for the function F_f:

$$F_f(\tilde{x}^n) = F(\tilde{x}^{n-2})\overline{x}_{n-1}\overline{x}_n \vee F_1(\tilde{x}^{n-2})\overline{x}_{n-1}x_n \vee F_2(\tilde{x}^{n-2})x_{n-1}\overline{x}_n,$$

where

$$F(\tilde{x}^{n-2}) = f(\tilde{x}^r)H(\tilde{x}^{r+1,n-2}),$$
$$F_j(\tilde{x}^{n-2}) = f_W(\tilde{x}^r)H_j(\tilde{x}^{r+1,n-2}), \qquad j = 1, 2.$$

Then $S_F = S_f \times S_H$,

$$S_{F_j} = S_{f_W} \times S_{H_j} = \bigcup_{i\in I_f^{\ker}} (S_{w_i} \times S_{H_j}) \qquad \text{for } j = 1, 2,$$

$S_F \cap (S_{F_1} \cup S_{F_2}) = \varnothing$; and Lemma 2 implies that

$$S_{F_f} = S_F \times B^{2,0,0}_{1,2} \cup S_{F_1} \times B^{2,0}_1 \cup S_{F_2} \times B^{2,0}_2.$$

Moreover,

$$S_{F_W} = S_{F_1} \times B^{2,0}_1 \cup S_{F_2} \times B^{2,0}_2 = \bigcup_{i\in I_f} S_{F_{W,i}},$$

where $S_{F_{W,i}} = S_{w_i} \times S_{\widehat{H}_1} \cup S_{w_i} \times S_{\widehat{H}_2}$, and for the function $w_i \in P_r$ the set of unit vertices N_{w_i} is representable by a face of dimension k_i in the cube B^r with two zero vertices the distance between which is equal to d_i.

Denote the function on B^k having exactly two zero vertices $\widetilde{\alpha}, \widetilde{\beta} \in B^k$ by $v_{\widetilde{\alpha},\widetilde{\beta}}$, and if $\widetilde{\alpha} = \tilde{0}^k$ and $\widetilde{\beta} = \tilde{1}^d\tilde{0}^{k-d}$ then denote the function by v_d.

If π is a permutation of the coordinates of B^k for which $\pi(\widetilde{\alpha} \oplus \widetilde{\beta}) = \tilde{1}^d\tilde{0}^{k-d}$ then $v_d(\pi(\widetilde{x} \oplus \widetilde{\alpha})) = v_{\widetilde{\alpha},\widetilde{\beta}}(\widetilde{x})$. In application to the vertices of B^k of the transformation $\pi(\widetilde{x} \oplus \widetilde{\alpha})$, faces go to faces; and the dimensions of faces and the membership relations for vertices, faces, and complexes of faces are preserved. Therefore, the properties of complexes of faces of the function $v_{\widetilde{\alpha},\widetilde{\beta}}$ are uniquely determined by the properties of complexes of faces of v_d.

In B^k, denote by $g_{i,j}$ and g_j the faces $B^{k,0,1}_{i,j}$ and $B^{k,1}_j$ respectively, where $i, j = 1, \dots, k$ and $i \neq j$.

For the function v_d, the sets of the shortest and minimal complexes of faces coincide, every maximal face is contained in some minimal complex, and the sufficient minimality conditions are based on an independent set of vertices of the layer B^k_1 because

$$v_1(\tilde{x}^k) = S^{k-1}_{1,k-1}(\tilde{x}^{2,k}), \qquad S_{v_1} = \mathcal{M}_L(v_1) = \mathrm{Ker}(v_1) = \{g_j\}^k_{j=2},$$
$$v_d(\tilde{x}^k) = S^d_{1,d-1}(\tilde{x}^d) \vee S^{k-d}_{1,r-d}(\tilde{x}^{d+1,k}), \qquad 1 < d < k,$$
$$S_{v_d} = \{g_{i,j},\ i \neq j\}^d_{i,j=1} \cup \mathrm{Ker}(v_d), \qquad \mathrm{Ker}(v_d) = \{g_j\}^k_{j=d+1},$$
$$\mathcal{M}_L(v_d) = \bigl\{M \times B^{k-d} \cup \mathrm{Ker}(v_d) \mid M \in \mathcal{M}_L\bigl(S^d_{1,d-1}\bigr)\bigr\},$$
$$v_k(\tilde{x}^k) = S^k_{1,k-1}(\tilde{x}^k), \qquad S_{v_k} = \{g_{i,j},\ i \neq j\}^k_{i,j=1}, \qquad \mathcal{M}_L(v_k) = \mathcal{M}_L\bigl(S^k_{1,k-1}\bigr),$$

where $\mathrm{Ker}(v_d)$ is the set of the kernel faces of v_d.

(ii) The domination relation $N^{0,1}_{F_{W,i}} \cup N^{1,0}_{F_{W,i}} \succ N^{0,0}_{F_{W,i}}$, where $i \in I^{\ker}_f$, follows from Lemma 2 since (6) holds.

(iii) Note that

$$N_{F_A} = N^{0,0}_{F_A}, \qquad N^{0,0}_{F_A} \cap N^{0,0}_{F_W} = \varnothing, \qquad g \cap \left(N^{0,1}_{F_W} \cup N^{1,0}_{F_W}\right) = \varnothing \qquad \text{for} \quad g \in S_{F_f} \setminus S_{F_W}$$

and $g \cap N_{F_A} = \varnothing$ for a face $g \in S_{F_W}$; i.e., the family $\left\{N_{F_A},\ N^{0,1}_{F_W},\ N^{1,0}_{F_W}\right\}$ is independent for the system $\langle N_{F_f},\ S_{F_f}\rangle$ and

$$\mathcal{L}(F_f) \geq \widetilde{\mathcal{L}}(F_f, F_A) + \widetilde{\mathcal{L}}(F_f, F_W\overline{x}_{n-1}x_n) + \widetilde{\mathcal{L}}(F_f, F_W x_{n-1}\overline{x}_n),$$
$$\widetilde{\mathcal{L}}(F_f, F_W\overline{x}_{n-1}x_n) = \widetilde{\mathcal{L}}(F_f, F_W\overline{x}_{n-1}), \qquad \widetilde{\mathcal{L}}(F_f, F_W x_{n-1}\overline{x}_n) = \widetilde{\mathcal{L}}(F_f, F_W\overline{x}_n).$$

Since the family $\left\{N^{0,1}_{F_W}, N^{1,0}_{F_W}\right\}$ is independent for the system $\langle N_{F_W}, S_{F_W}\rangle$, $N^{0,1}_{F_W} \cup N^{1,0}_{F_W} \succ N^{0,0}_{F_W}$ and $N^{0,1}_{F_W} \cup N^{1,0}_{F_W} \cup N^{0,0}_{F_W} = N_{F_W}$; it follows from (3) that

$$\widetilde{\mathcal{L}}(F_W, F_W\overline{x}_{n-1}x_n) + \widetilde{\mathcal{L}}(F_W, F_W x_{n-1}\overline{x}_n) = \mathcal{L}(F_W).$$

Therefore, $\mathcal{L}(F_f) \geq \widetilde{\mathcal{L}}(F_f, F_A) + \mathcal{L}(F_W)$ and equality is attained since the union of every two complexes $M_1 \in \widetilde{\mathcal{M}}_{\mathcal{L}}(F_f, F_A)$ and $M_2 \in \mathcal{M}_{\mathcal{L}}(F_W)$ is a complex of F_f.

(iv) Suppose that $\widetilde{\alpha}_1,\ \widetilde{\alpha}_2 \in B^{n-r-2}_m \times \tilde{0}^2$ and π is a permutation of the coordinates of the cube B^n for which $\pi(\widetilde{x} \times \widetilde{\alpha}_1) = \widetilde{x} \times \widetilde{\alpha}_2$ for every $\widetilde{x} \in B^r$. Then, for each complex $M \in \widetilde{\mathcal{M}}(F_f, f_{A,\widetilde{\alpha}_1})$, the complex $\pi(M) \in \widetilde{\mathcal{M}}(F_f, f_{A,\widetilde{\alpha}_2})$ has the same complexity $\mathcal{L}$ by the axiom of invariance under isomorphism. Therefore, $\widetilde{\mathcal{L}}(F_f, f_{A,\widetilde{\alpha}_1}) = \widetilde{\mathcal{L}}(F_f, f_{A,\widetilde{\alpha}_2})$.

Since $B^{n-r-2}_m \times \tilde{0}^2 \subset B^{n-r}$ is an independent vertex set for the function $\widehat{H} \in P_{n-r}$ and $A_f \subset N_f \subset B^r$; therefore, Lemma 3 implies that the family $\mathcal{A}$ is independent for the system $\langle N_{f\times\widehat{H}}, S_{f\times\widehat{H}}\rangle$. Since $N_{f_{A,\widetilde{\alpha}}} \subset N_{F_A}$ for $\widetilde{\alpha} \in B^{n-r-2}_m$, $N_{F_A} \cap N_{F_W} = \varnothing$, and $S_{F_f} = S_f \times S_{\widehat{H}} \cup S_{F_W}$; therefore, $\mathcal{A}$ is independent for the system

$$\langle N_{F_f}, S_{F_f}\rangle = \langle N_{f\times\widehat{H}} \cup N_{F_W},\ S_{f\times\widehat{H}} \cup S_{F_W}\rangle,$$

and estimate (7) stems from (2).

(v) Every complex of maximal faces $M \in \widetilde{\mathcal{M}}(F_f, f_{A,\widetilde{\alpha}})$ has the form

$$M = \{g_i = g_{i,A} \times g_{i,\widehat{H}} \in S_{F_f} \mid g_{i,A} \in S_f,\ g_{i,\widehat{H}} \in S_{\widehat{H}}\}^l_{i=1},$$

where $M_A = \{g_{i,A}\}^l_{i=1} \in \widetilde{\mathcal{M}}(f, f_A)$ and $l = l(M) = l(M_A)$. Consequently,

$$l(M_A) \geq \tilde{l}(f, f_A), \qquad L(M_A) \geq \widetilde{L}(f, f_A).$$

For every face $g_{i,\widehat{H}} \in S_{\widehat{H}}$, the rank of $R_{\widehat{H}}$ equals $L(g_{i,\widehat{H}}) = n - r - h$; therefore,

$$L(\widehat{H}) = l(\widehat{H})R_{\widehat{H}}, \qquad l(\widehat{H}) = l(H) = \left|B^{n-r-2}_m\right|.$$

Since $L(g_1 \times g_2) = L(g_1) + L(g_2)$ for every faces g_1 and g_2, we infer

$$\sum_{g_i \in M} (L(g_{i,A}) + L(g_{i,\widehat{H}})) = L(M_A) + l(M_A)R_{\widehat{H}} \geq \widetilde{L}(f, f_A) + \tilde{l}(f, f_A)R_{\widehat{H}}.$$

Hence, $\widetilde{L}(F_f, f_{A,\widetilde{\alpha}}) \geq \widetilde{L}(f, f_A) + \tilde{l}(f, f_A)R_{\widehat{H}}$; and from (7) we obtain

$$\tilde{l}(F_f, F_A) \geq \tilde{l}(f, f_A)l(\widehat{H}),$$
$$\widetilde{L}(F_f, F_A) \geq (\widetilde{L}(f, f_A) + \tilde{l}(f, f_A)R_{\widehat{H}})l(\widehat{H}) = \widetilde{L}(f, f_A)l(\widehat{H}) + \tilde{l}(f, f_A)L(\widehat{H}).$$

Using (4), we can show that these lower bounds are attainable. Note that $M_1 \times M_2 \in \mathcal{M}(F_f, F_A)$ for every complexes $M_1 \in \mathcal{M}(f, f_A)$ and $M_2 \in \mathcal{M}(\widehat{H})$. Let $M_2 \in \mathcal{M}_L(\widehat{H}) = \mathcal{M}_l(\widehat{H})$; i.e., let $L(M_2) = L(\widehat{H})$ and $l(M_2) = l(\widehat{H})$.

If $M_1 \in \mathcal{M}_l(f, f_A)$ then

$$l(M_1) = \tilde{l}(f, f, f_A), \qquad l(M_1 \times M_2) = l(f, f, f_A) l(\widehat{H});$$

i.e., $M_1 \times M_2 \in \widetilde{\mathcal{M}}_l(F_f, F_A)$.

If $M_1 \in \mathcal{M}_{l\cap L}(f, f_A)$ then

$$l(M_1) = \tilde{l}(f, f_A), \qquad L(M_1) = \widetilde{L}(f, f_A),$$
$$L(M_1 \times M_2) = L(f, f_A) l(\widehat{H}) + l(f, f_A) L(\widehat{H});$$

in other words, $M_1 \times M_2 \in \widetilde{\mathcal{M}}_{l\cap L}(F_f, F_A)$.

(vi) Items (i) and (iii) of Theorem 1 imply that $S_{F_f} = S_f \times S_{\widehat{H}} \cup S_{F_W}$ and $M_1 \cup M_2 \in \mathcal{M}_{\mathcal{L}}(F_f)$ for every complexes $M_1 \in \widetilde{\mathcal{M}}_{\mathcal{L}}(F_f, F_A)$ and $M_2 \in \mathcal{M}_{\mathcal{L}}(F_W)$. Therefore, it suffices to consider the following two cases for a face $g \in S_{F_f}$: $g \in S_f \times S_{\widehat{H}}$ or $g \in S_{F_W}$.

If $g \in S_f \times S_{\widehat{H}}$ then g is uniquely representable as $g = g_f \times g_{\widehat{H}}$, where $g_f \in S_f$ and $g_{\widehat{H}} \in S_{\widehat{H}}$. There always exists a complex of faces $M_{\widehat{H}}$ of the function $\widehat{H}$ for which $g_{\widehat{H}} \in M_{\widehat{H}} \in \mathcal{M}_L(\widehat{H}) = \mathcal{M}_l(\widehat{H})$. If there always exists a complex M_f of f for which $g_f \in M_f \in \widetilde{\mathcal{M}}_{l\cap L}(f, f_A)$ then Theorem 1(v) implies

$$g = g_f \times g_{\widehat{H}} \in M_f \times M_{\widehat{H}} \in \widetilde{\mathcal{M}}_{l\cap L}(F_f, F_A),$$

and g is contained in some complex in $\mathcal{M}_{l\cap L}(F_f)$.

If $g \in S_{F_W}$ then $g \in S_{F_{W,i}}$ for some $i \in I_f^{\ker}$ because the functions $F_{W,i}$ are connected components of F_W for $i \in I_f^{\ker}$. For $F_{W,i}$, the set of maximal faces is equal to $S_{F_{W,i}} = S_{w_i} \times S_{\widehat{H}_1} \cup S_{w_i} \times S_{\widehat{H}_2}$ (Theorem 1(i)). Therefore, every $g \in S_{F_{W,i}}$ is uniquely representable as $g = g_{w_i} \times g_1$ or $g = g_{w_i} \times g_2$, where $g_{w_i} \in S_{w_i}$ and $g_1 \in S_{\widehat{H}_1}$ or $g_2 \in S_{\widehat{H}_2}$. The functions w_i, $\widehat{H}_1$, and $\widehat{H}_2$ satisfy the hypotheses of Lemma 4(iii); i.e., for every faces $g_{w_i} \in S_{w_i}$, $g_1 \in S_{\widehat{H}_1}$, and $g_2 \in S_{\widehat{H}_2}$, there are complexes M_{w_i}, $M_{\widehat{H}_1}$, and $M_{\widehat{H}_2}$ containing these faces and independent sets of vertices Q_{w_i}, $Q_{\widehat{H}_1}$, and $Q_{\widehat{H}_2}$ identical for different maximal faces. Moreover, the maximal faces of each function that contain any vertex of the corresponding set have the same rank.

Consequently,

$$(Q_{w_i}, M_{w_i}) \in \mathcal{R}_{w_i,L}, \qquad (Q_{\widehat{H}_1}, M_{\widehat{H}_1}) \in \mathcal{R}_{\widehat{H}_1,L}, \qquad (Q_{\widehat{H}_2}, M_{\widehat{H}_2}) \in \mathcal{R}_{\widehat{H}_2,L}.$$

Obviously, for the function $F_{W,i}$, the set of vertices

$$Q = Q_{w_i} \times Q_{\widehat{H}_1} \cup Q_{w_i} \times Q_{\widehat{H}_2}$$

is an independent set and the complex of faces

$$M = M_{w_i} \times M_{\widehat{H}_1} \cup M_{w_i} \times M_{\widehat{H}_2}$$

is such that $|M| = |Q|$. In addition, all maximal faces of $F_{W,i}$ containing any vertex of Q have the same rank. Consequently, the hypotheses of Lemma 4(i) hold and $M \in \mathcal{M}_{l\cap L}(F_{W,i})$.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. The function $F_f \in \mathcal{F}^n_{r,k}(m, h, h_1, h_2)$ is obtained from $f \in C_{r,k}$ for the parameters

$$r = \lfloor (n-2)/2 \rfloor = n/2 - \Theta(1), \qquad r - 4k > n/4 - 4k = \Theta(n),$$
$$m = \lfloor (n-r-2)/2 \rfloor = n/4 - \Theta(1), \qquad h = \lfloor m/2 \rfloor = n/8 - \Theta(1),$$
$$h_1 = h_2 = \lfloor h/2 \rfloor = \frac{n}{16} - \Theta(1) \qquad \text{as } n \to \infty.$$

The construction of $f \in C_{r,k}$ is based on the transformations of Lemma 1. At the first step, from $f_0 \in C_{r-4k,1}$, the k-cyclic function $f_1 \in C_{r-3k,k}$ is obtained for which the number of maximal faces can be of a certain oddity, at least $2\lfloor \frac{1}{4}(p(f_0)+1)\rfloor$ faces have dimension k; moreover, the distance between the intersection vertices of these faces is equal to k, and the extent satisfies the estimate

$$p(f_1) \geq p(f_0) + 2\lfloor (p(f_0)+1)/4 \rfloor.$$

At the second step, from $f_1 \in C_{r-3k,k}$, we obtain a k-cyclic function $f \in C_{r,k}$ for which the dimensiothen of the faces can be increased to a value not exceeding k with preservation of other properties of f_1; i.e., $p(f) = p(f_1)$. From the function $f_0 \in C_{r-4k,1}$ for which $p(f_0) = 2^{\Theta(n)}$ for $r - 4k = \Theta(n)$, we can obtain $f \in C_{r,k}$ having the number of maximal faces of a certain oddity, exponential values of the extent and of the number of maximal faces of dimension k.

Then Theorem 1 and Corollary 1 yield the estimates:
the extent satisfies

$$p(F_f) \geq p(f) = \Theta(2^{r-4k}) = 2^{\Theta(n)};$$

for the ratio of the number of maximal faces and the number of unit vertices, from the fact that

$$|S_f| = p(f) + 1 \geq \Theta(2^{\frac{n}{2}-4k}), \qquad |S_H| \geq 2^{(n-r)+m\mathcal{H}(h/m)}\Theta(n^{-1}) = 2^{\frac{n}{2}+\frac{n}{4}}\Theta(n^{-1}),$$

it follows that

$$|S_{F_f}| \geq |S_f| \times |S_H| \geq 2^{n+\Theta(n)} \geq |N_{F_f}|2^{\Theta(n)};$$

the number of maximal faces σ_{F_f} that contain any vertex is at least

$$\min\{\sigma_H, \sigma_{H_1}, \sigma_{H_2}\} = \min\left\{\binom{m}{h}, \binom{m-\lfloor h/2\rfloor}{h-\lfloor h/2\rfloor}, \binom{m}{\lfloor h/2\rfloor}\right\} \geq 2^{\Theta(n)}.$$

For the connected components of dominated and dominating sets of the unit vertices of a function (Theorem 1(ii)), we have:
number of them is at least

$$\left|I_f^{\ker}\right| \geq 2\lfloor (p(f_0)+1)/4 \rfloor = \Theta(2^{\frac{n}{2}-4k}) = 2^{\Theta(n)},$$

and their size is at least

$$\min\left\{\left|N_{F_{W,i}}^{0,1}\right|, \left|N_{F_{W,i}}^{1,0}\right|, \left|N_{F_{W,i}}^{0,0}\right|\right\} \geq |N_{w_i}| \min\{|N_H|, |N_{H_1}|, |N_{H_2}|\} = 2^{\Theta(n)}.$$

For the independent family of sets that is used for justifying the minimality of complexes of faces (Theorem 1(iv)), the number of sets in the family is equal to

$$\left|B_m^{n-r-2}\right| = \binom{n-r-2}{m} = 2^{\Theta(n)},$$

and the size of each set in the family is $p(f) + 1 = 2^{\Theta(n)}$.

Theorem 1(vi) implies that every maximal face of F_f is contained in some complex $M \in \mathcal{M}_{l\cap L}(F_f)$ if any maximal face of f is contained in some complex $\widetilde{M} \in \widetilde{\mathcal{M}}_{l\cap L}(f, f_A)$.

For constructing complexes with different sets of proper vertices, we use the representation of a zone in the cube in the form

$$S_{m-h,m}^{n-r-2} = \bigcup_{i=0}^{h} C_i, \qquad m \leq (n-r-2)/2,$$

where C_i includes the monotone sequences of $i+1$ neighboring vertices with maximal vertex in the layer m and minimal vertex in the layer $m-i$ for $i = 0, \dots, h$ [9]. Every such a sequence of vertices is contained in one maximal face of the zone, and the set of such faces constitute a shortest and minimal complex of the zone function. Hence,

$$|C_i| = \binom{n-r-2}{m-i} - \binom{n-r-2}{m-i-1}, \quad i = 0, \dots, h-1, \qquad |C_h| = \binom{n-r-2}{m-h}.$$

The function $\widehat{H}$ for which the set of unit vertices coincides with $S^{n-r-2}_{m-h,m}$ in $B^{n-r-2} \times \tilde{0}^2$ possesses the same properties.

Represent the set of vertices $B^{n-r-2}_m \times \tilde{0}^2$ as the union of disjoint sets

$$Z_0 = C_0 \times \tilde{0}^2, \qquad \overline{Z}_0 = \left(B^{n-r-2}_m \times \tilde{0}^2\right) \setminus Z_0.$$

Denote the function for which the set of unit vertices is $N_{\widehat{H}} \setminus Z_0$ by H_Z. For the function H_Z, let $S_{H_Z} = \bigcup_{\widetilde{\alpha} \in \overline{Z}_0} S_{\widehat{H}}(\widetilde{\alpha})$ be the set of maximal faces, $\overline{Z}_0$ be an independent set of vertices, and let

$$M_Z = \{g_{\widetilde{\alpha}} \mid g_{\widetilde{\alpha}} \in S_{\widehat{H}}(\widetilde{\alpha}),\ \widetilde{\alpha} \in \overline{Z}_0\}$$

be a complex of maximal faces in which each face $g_{\widetilde{\alpha}}$ contains a monotone sequence of neighboring vertices with maximal vertex $\widetilde{\alpha} \in \overline{Z}_0$.

Then M_Z satisfies the conditions of Lemma 4 (i) and $M_Z \in \mathcal{M}_{l \cap L}(H_Z)$.

Represent the family $\mathcal{A} = \left\{N_{f_{A,\widetilde{\alpha}}},\ \widetilde{\alpha} \in B^{n-r-2}_m \times \tilde{0}^2\right\}$ (see Theorem 1) as $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \{N_{f_{A,\widetilde{\alpha}}} = A_f \times \{\widetilde{\alpha}\},\ \widetilde{\alpha} \in \overline{Z}_0\}, \qquad \mathcal{A}_2 = N_{F_A} \setminus \mathcal{A}_1.$$

Define the complex $M_1 = M_A \times M_Z$, where M_A is a complex in the set $\widetilde{\mathcal{M}}(f, f_A)$; i.e., $N_{M_1} \subset N_{F_f}$, and $N_{F_A} \setminus N_{M_1}$ contains only vertices of the family $\mathcal{A}_2$. Define the complex M_2 containing all vertices of $N_{F_A} \setminus N_{M_1}$ and representable in the form

$$M_2 = \bigcup_{\widetilde{\alpha} \in Z_0} M_{2,\widetilde{\alpha}}, \quad \text{where} \quad M_{2,\widetilde{\alpha}} = \{g_1 \times g_2 \mid g_1 \in M_{A,\widetilde{\alpha}},\ g_2 \in S_{\widehat{H}}(\widetilde{\alpha})\} \in \mathcal{M}(F_f, N_{f_{A,\widetilde{\alpha}}})$$

and the complexes $M_{A,\widetilde{\alpha}} \in \widetilde{\mathcal{M}}(f, f_A)$ can be distinct for distinct vertices $\widetilde{\alpha} \in Z_0$.

Then $M = M_1 \cup M_2 \in \widetilde{\mathcal{M}}(F_f, F_A)$ and, by analogy to Theorem 1(v), if $M_1, M_{A,\widetilde{\alpha}} \in \widetilde{\mathcal{M}}_{l \cap L}(f, f_A)$ for all vertices $\widetilde{\alpha} \in Z_0$ then

$$L(M) = \widetilde{L}(f, f_A) l(\widehat{H}) + \tilde{l}(f, f_A) L(\widehat{H}), \qquad M \in \widetilde{\mathcal{M}}_{l \cap L}(F_f, F_A).$$

Let the complexes $M_{A,\widetilde{\alpha}}$ be chosen in $\mathcal{M} \subseteq \widetilde{\mathcal{M}}_{l \cap L}(f, f_A)$ in which the complexes have different sets of proper vertices (for example, if $|S_f|$ is an odd number and all maximal faces are of one rank). If for defining the complex M_2 different collections of complexes $\{M_{A,\widetilde{\alpha}} \in \mathcal{M},\ \widetilde{\alpha} \in Z_0\}$ are used then the complexes $M_A \in \widetilde{\mathcal{M}}_A \subseteq \widetilde{\mathcal{M}}_{l \cap L}(F_f, F_A)$ appear with different sets of proper vertices.

Then the complexes of the form $M_A \cup M_W \in \mathcal{M}_{l \cap L}(F_f)$ if $M_A \in \widetilde{\mathcal{M}}_A$ and $M_W \in \mathcal{M}_{\mathcal{L}}(F_W)$ (Theorem 1(iii)) have different sets of proper vertices, and the number of such complexes is at least $|\mathcal{M}|^{|Z_0|}$, where $|\mathcal{M}| \geq 2$ and $|Z_0| = 2^{\Theta(n)}$.

The proof of Theorem 2 is complete. □

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